Vietnam Journal of Mathematics **29**:1 (2001) 85-89

Vietnam Journal
of
MATHEMATICS

Short Communication

Stability Radii of Linear Functional Differential Equations

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Received June 15, 2000

1. Introduction

In this paper, we present our new results on robust stability of linear retarded systems described by general linear functional differential equations (briefly, FDE) of the form

$$\dot{x}(t) = A_0 x(t) + \int_{-h}^{0} d[\eta(\theta)]x(t+\theta),$$

by using the state space approach based on the notion of stability radii. A formula for the complex stability radius of the system with respect to structured affine parameter perturbations is derived. Then, the class of positive linear retarded systems is studied in details. It is shown that for this class, real and complex stability radii coincide and can be computed by simple formulae expressed in terms of the system matrices. The results of this paper extend to general FDE the previous results for linear ordinary differential equations of the form $\dot{x}(t) = Ax(t)$ (see [4, 5]) and those for linear retarded systems $\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$ (see [6, 7]). Throughout the paper, the inequalities between real matrices (vectors) are understood elementwise. The set of all nonnegative matrices (nonnegative vectors) is denoted by $\mathbb{R}_+^{l\times q}$ ($\mathbb{R}_+^n$, respectively). For $P \in \mathbb{C}^{l\times q}$, $\|P\|$ will stand for operator norm of P associated with a given pair of *monotonic vector norms* on $\mathbb{C}^l$ and $\mathbb{C}^q$. We call $A \in \mathbb{R}^{n\times n}$ a *Metzler matrix* if all off-diagonal elements of A are nonnegative . We denote by $NBV([-h,0], \mathbb{C}^{m\times n})$ the set of all matrix functions $\eta(\cdot)$ which are of bounded variation and continuous from the left (c.f.l. for short) on $[-h, 0]$ satisfying $\eta(-h) = 0$. Then $NBV([-h,0], \mathbb{C}^{m\times n})$ being endowed with the norm $\|\eta\| = \mathrm{Var}(\eta; -h, 0)$ is a Banach space.

2. Complex Stability Radius

Consider a linear retarded system described by the following general functional differential equation

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \int_{-h}^{0} d[\eta(\theta)]x(t+\theta), \quad t \geq 0, \quad x(t) \in \mathbb{R}^n \\ x(\theta) &= \phi^0(\theta), \quad \theta \in [-h, 0], \end{aligned} \tag{1}$$

where $A_0 \in \mathbb{R}^{n\times n}$ and $\eta(\cdot) \in NBV([-h,0], \mathbb{R}^{n\times n})$ are given. The definition of η is assumed to be extended to $\mathbb{R}$ by setting $\eta(\theta) = \eta(-h) = 0$ for all $\theta \leq -h, \eta(\theta) = \eta(0)$ for all $\theta \geq 0$.

It is well-known (see, e.g. [3]) that, for any given $\phi^0 \in C := C([-h,0], \mathbb{R}^n)$, the system (1) has a unique function $x(\phi^0, \cdot)$ defined and continuous on $[-h, \infty)$. The system (1) is said to be exponentially asymptotically stable or, more simply, *Hurwitz stable*, if there are constants $c > 0, \alpha > 0$ such that for all $\phi \in C$, the solution $x(\phi, \cdot)$ of (1) satisfies

$$\|x(\phi, t)\| \leq ce^{-\alpha t}\|\phi\|, \quad t \geq 0.$$

The necessary and sufficient condition for the system (1) to be Hurwitz stable is that the set $\sigma(A_0, \eta)$ of all roots of its characteristic quasi-polynomial $H(s)$ is located in the open left half-plane $\mathbb{C}^- := \{s \in \mathbb{C} : \mathrm{Re} s < 0\}$, where

$$H(s) = sI - A_0 - \int_{-h}^{0} e^{s\theta} d[\eta(\theta)]. \tag{2}$$

Assume that the retarded system (1) is Hurwitz stable and subjected to affine parameter perturbations of the type

$$\begin{aligned} A_0 &\to A_{0\Delta} = A_0 + D_0 \Delta E_0, \quad \Delta \in \mathbb{C}^{l^0 \times q^0}, \\ \eta &\to \eta_\delta = \eta + D_1 \delta E_1, \quad \delta \in NBV([-h,0], \mathbb{C}^{l^1 \times q^1}). \end{aligned} \tag{3}$$

Here $D_i \in \mathbb{C}^{n\times l^i}$, $E_i \in \mathbb{C}^{q^i \times n}$, $i = 0, 1$ are given matrices determining the *structure* of perturbations, Δ and $\delta(\cdot)$ are unknown disturbances. We shall measure the size of perturbation $\tilde{\Delta} := [\Delta, \delta]$ by the norm $\|\tilde{\Delta}\| := \|\Delta\| + \|\delta\|, \quad \|\delta\| := \mathrm{Var}(\delta; -h, 0)$. Then the *complex stability radius* of the system with respect to perturbations of the form (3), is defined by

$$r_{\mathbb{C}} := r(A_0, \eta) := \inf\{\|\tilde{\Delta}\|; \tilde{\Delta} = [\Delta, \delta], \sigma(A_{0\Delta}, \eta_\delta) \not\subset \mathbb{C}^-\}. \tag{4}$$

If the disturbance matrices in (3) are restricted to the real spaces $\mathbb{R}^{l^0 \times q^0}$ and $NBV([-h,0], \mathbb{R}^{l^1 \times q^1})$, then we obtain the *real stability radius* $r_{\mathbb{R}}$. If $\sigma(A_{0\Delta}, \eta_\delta) \subset \mathbb{C}^-$ for all $\Delta \in \mathbb{C}^{l^0 \times q^0}, \delta \in NBV([-h,0], \mathbb{C}^{l^1 \times q^1})$ then we shall write $r_{\mathbb{C}} = +\infty$. To derive the formula for the complex stability radius, we define the associated *transfer functions* by setting

$$G_{ij}(s) = E_i H(s)^{-1} D_j \in \mathbb{C}^{q^i \times l^j}, \quad i, j \in M := \{0, 1\}.$$

If the system (1) is Hurwitz stable then $G_{ij}(s)$ are analytic on the closed half-plane $C^+ := \{s \in \mathbb{C} : \text{Re } s \geq 0\}$. The following theorem can be proved similarly as in [7], using the maximum modulus principle for the analytic functions $G_{ij}(s)$.

Theorem 2.1. *Let the retarded system (1) be Hurwitz stable and be subjected to structured perturbations of the form (3). Then*

$$\frac{1}{\max_{i,j\in M,\omega\in\mathbb{R}} \|G_{ij}(\imath\omega)\|} \leq r_{\mathbb{C}} \leq \frac{1}{\max_{i\in M,\omega\in\mathbb{R}} \|G_{ii}(\imath\omega)\|}. \tag{5}$$

In particular, if $D_0 = D_1$ or $E_0 = E_1$, then

$$r_{\mathbb{C}} = \frac{1}{\max_{i\in M,\omega\in\mathbb{R}} \|G_{ii}(\imath\omega)\|}. \tag{6}$$

Furthermore, it can be shown that if $D_0 = D_1$ or $E_0 = E_1$ then there always exists a destabilizing perturbation $\tilde{\Delta} \in \mathbb{C}^{l^0\times q^0} \times NBV([-h,0],\mathbb{C}^{l^1\times q^1})$ which is either of the form $[\Delta^*, 0]$ or $[0, \delta^*]$ such that $\|\tilde{\Delta}\| = \|\Delta^*\| = \|\delta^*\| = r_{\mathbb{C}}$ and, moreover, Δ^* is of *rank one* and δ^* is a step function of the following type:

$$\delta^*(\theta) = \begin{cases} 0 & \text{if } \theta = -h \\ \Delta_1 & \text{if } \theta \in (-h, 0], \end{cases} \tag{7}$$

where Δ_1 is also of rank one.

3. Real Stability Radius of Positive Systems

Consider a linear system described by the functional differential equation of the form

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \int_{-h}^{0} d[\eta(\theta)]x(t+\theta), \quad t \geq 0, \; x(t) \in \mathbb{R}^n \\ x(\theta) &= \phi^0(\theta), \quad \theta \in [-h, 0], \end{aligned} \tag{8}$$

where $A_0 \in \mathbb{R}^{n\times n}$ and $\eta \in NBV([-h,0],\mathbb{R}^{n\times n})$ are given. The solution of the system (8) will be denoted by $x(\phi^0, \cdot)$. System (8) is called *positive* if for every nonnegative initial function $\phi^0 \in C([-h,0],\mathbb{R}^n_+)$, the corresponding solution $x(\phi^0,\cdot)$ satisfies $x(\phi^0,t) \in \mathbb{R}^n_+$ for every $t \geq 0$. We have the following

Lemma 3.1. *The system (8) is positive if and only if A_0 is a Metzler matrix and $\eta(.)$ is an increasing matrix function.*

In particular, from Lemma 3.1 it follows that the linear retarded system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \quad t \geq 0$$

is positive iff A_0 is a Metzler and $A_1 \geq 0$ (see [2]).

Let the system (8) be positive and subject to perturbation of the form (3), where the structure matrices are nonnegative: $D_i \in \mathbb{R}_+^{n\times l^i}$, $E_i \in \mathbb{R}_+^{q^i\times n}$. In this case, it is easy to prove that the transfer functions G_{ij} defined by (5) have the following monotonicity property:

$$G_{ij}(t_1) \geq G_{ij}(t_2) \geq 0 \quad \text{for} \quad t_2 > t_1 > \mu_0, \quad i,j \in M := \{0,1\}, \tag{9}$$

where $\mu_0 := \max\{\mathrm{Re}s;\ s \in \sigma(A_0,\eta)\}$.
Define the stability radius of the system (8) subjected to *nonnegative perturbations* of the form (3) by setting

$$r_+ = \inf\{\|\tilde{\Delta} : \tilde{\Delta} = [\Delta,\delta] \in \mathcal{D}_+, \sigma(A_{0\Delta},\eta_\delta) \not\subset \mathbb{C}^-\}, \tag{10}$$

where

$$\mathcal{D}_+ = \{\tilde{\Delta} = [\Delta,\delta] : \Delta \in \mathbb{R}_+^{l^0\times q^0},\ \delta \in NBV([-h,0],\mathbb{R}^{l^1\times q^1}) \text{ and } \delta \text{ is increasing}\}.$$

Using the monotonicity property (9) and Perron–Frobenius theorem for Metzler matrices (see, e.g. [5]), we can prove the following main result of this paper.

Theorem 3.2. *Let the linear system* (8) *be positive and Hurwitz stable. Assume* A_0, η *are subjected to parameter affine perturbation of the form* (3) *where* $D_i \in \mathbb{R}_+^{n\times l^i}$, $E_i \in \mathbb{R}_+^{q^i\times n}$, $i \in M := \{0,1\}$. *If* $D_0 = D_1$ *or* $E_0 = E_1$ *then, we have*

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \frac{1}{\max_{i\in M} \|G_{ii}(0)\|}.$$

We illustrate the above result by the following simple example.

Example. Consider a positive linear time-delay system described by the following scalar equation

$$\dot{x}(t) = -x(t) + \int_{-1}^{0} e^{\theta} x(t+\theta)d\theta \quad t \geq 0,\ \ x(t) \in \mathbb{R}. \tag{11}$$

The characteristic equation of (11) is given by $(s^2+2s)e^s + e^{-1} = 0$. By Theorem 13.9 in [1], it is easy to verify that all roots of this equation have negative real parts and hence the system (11) is Hurwitz stable. Assume the system (11) is perturbed as follows

$$\dot{x}(t) = (-1+\delta)x(t) + \int_{-1}^{0} (e^{\theta} + \Delta(\theta))x(t+\theta)d\theta, \tag{12}$$

where $\delta \in \mathbb{R}$ is a unknown parameter and $\Delta(\theta)$ is a unknown integrable function on $[-h,0]$. By Theorem 3.2, we conclude that the perturbed system (12) is Hurwitz stable for all $\delta \in \mathbb{R}$, $\Delta(\cdot) \in L_1([-1,0],\mathbb{R})$ satisfying

$$|\delta| + \int_{-1}^{0} |\Delta(\theta)|d\theta < r_{\mathbb{R}} = \frac{1}{|H^{-1}(0)|} = e^{-1}.$$

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