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Short Communication

On Besov Smoothness and Non-Linear Approximations Using Wavelet Decompositions

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1. We are interested in non-linear n-term approximations with regard to the wavelet family $\mathbf{V}$ formed from the integer translates of the mixed dyadic scales of the tensor product multivariate de la Vallée Poussin kernel, and optimal continuous algorithms of n-term approximation in terms of non-linear n-widths, for multivariate periodic functions from of the Besov space of common mixed smoothness $\mathbf{B}^{\Omega}_{p,\theta}$. Its mixed smoothness is defined via mixed modulus of smoothness dominated by a function Ω of mixed modulus of smoothness type. For a given Ω of a special form, we give the asymptotic order of these quantities.

2. Let X be a quasi-normed linear space and $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ a family of elements in X. Denote by $\mathbf{M}_n(\Phi)$ the non-linear manifold of all linear combinations of n free terms from Φ of the form $\varphi = \sum_{k\in Q} a_k\varphi_k$, where Q is a set of natural numbers with $|Q| = n$. Here and later $|Q|$ denotes the cardinality of Q. Let W be a subset in X. The best n-term approximation $\sigma_n(W, \Phi, X)$ by the family Φ is given by

$$\sigma_n(W, \Phi, X) := \sup_{f\in W} \inf_{\varphi\in \mathbf{M}_n(\Phi)} \|f - \varphi\|.$$

An (continuous) algorithm of n-term approximation by Φ of the elements from W, is represented as a (continuous) mapping S from W into $\mathbf{M}_n(\Phi)$. Notions of non-linear n-widths $\alpha_n(W, X), \tau_n(W, X), \tau'_n(W, X)$ based on optimality of continuous algorithms of n-term approximation, have been introduced in [4].

There are other notions of non-linear n-widths which are based on continuous algorithms of non-linear approximations different from n-term approximation, and related to problems discussed in the present paper. They are the Alexandroff non-linear n-width $a_n(W, X)$ and the non-linear manifold n-width $\delta_n(W, X)$ (see definitions, e.g., in [4, 6]), and the non-linear n-width $\beta_n(W, X)$ (see [4]).

3. Let us define mixed smoothness Besov spaces of functions on the d-torus $\mathbf{T}^d = [-\pi,\pi]^d$. For a positive integer l, the univariate symmetric difference operator $\Delta_h^l, h \in \mathbf{T}$, is defined inductively by $\Delta_h^l := \Delta_h^1 \Delta_h^{l-1}$, starting from $\Delta_h^1 f := f(\cdot + h/2) - f(\cdot - h/2)$. Further, for $l \in \mathbf{N}^d$, we let the multivariate mixed lth difference operator $\Delta_h^l, h \in \mathbf{T}^d$, be defined by

$$\Delta_h^l f := \Delta_{h_1}^{l_1} \Delta_{h_2}^{l_2} \cdots \Delta_{h_d}^{l_d} f,$$

where the univariate operator $\Delta_{h_j}^l$, is applied to the variable x_j. Let

$$\Omega_l(f,t)_p := \sup_{|h_j|<t_j, j=1,\dots,d} \|\Delta_h^l f\|_p, \quad t \in \mathbf{R}_+^d,$$

be the lth mixed modulus of smoothness of f.

For $l \in \mathbf{N}^d$, we introduce the class $\mathbf{MS}_l$ of functions of mixed modulus of smoothness type as follows. It consists of all non-negative functions Ω on $\mathbf{R}_+^d$ such that:

(i) $\Omega(t) = 0$ if $t_1 t_2 \dots t_d = 0$,
(ii) $\Omega(t) \le \Omega(t')$ if $t \le t'$,
(iii) $\Omega(k_1 t_1, \dots, k_d t_d) \le (k_1 k_2 \cdots k_d)^l \Omega(t)$, for any $k \in \mathbf{N}^d$,
(iv) there exist positive numbers $a_j < l_j$, $j = 1, 2, \dots, d$, such that for any $h > 0$

$$\int_h^\infty \Omega(t) t_j^{-a_j-1} dt_j \le C(l) t^{-a_j} \Omega(t_1, \dots, t_{j-1}, h, t_{j+1}, \dots, t_d), \; j = 1, \dots, d,$$

(v) there exists a positive number b such that for any $h > 0$

$$\int_0^h \Omega(t) t_j^{-b-1} dt_j \le C'(l) t^{-b} \Omega(t_1, \dots, t_{j-1}, h, t_{j+1}, \dots, t_d), \; j = 1, \dots, d.$$

Notice that $\mathbf{MS}_l \subset \mathbf{MS}_{l'}$ if $l \le l'$. For $\Omega \in \mathbf{MS}_l$ and $0 < p, \theta \le \infty$, let $\mathbf{B}_{p,\theta}^\Omega$ denote the Besov space of all functions on $\mathbf{T}^d$, for which the quasi-norm

$$\|f\|_{\mathbf{B}_{p,\theta}^\Omega} := \|f\|_p + |f|_{\mathbf{B}_{p,\theta}^\Omega} \tag{1}$$

is finite, where $\|\cdot\|_p$ is the usual p-integral norm in $L_p := L_p(\mathbf{T}^d)$ and

$$|f|_{\mathbf{B}_{p,\theta}^\Omega} := \left(\int_{\mathbf{R}_+^d} \{\Omega_l(f,t)_p / \Omega(t)\}^\theta \prod_{j=1}^d t_j^{-1} dt \right)^{1/\theta}, \theta < \infty, \tag{2}$$

(the integral changed to the supremum for $\theta = \infty$). For $1 \le p \le \infty$, the definition of $\mathbf{B}_{p,\theta}^\Omega$ does not depend on l, i. e., for a given Ω, (1)-(2) determine equivalent quasi-norms for all l such that $\Omega \in \mathbf{MS}_l$.

Let A be a given compact subset of $\mathbf{R}_+^d$ such that

$$\rho(A) := \min\{\max\{\rho_j : \rho_j e^j \in A, \; j = 1, \dots, d\}\},$$

where $\{e^j\}_{j=1}^d$ is the canonical basis in $\mathbf{R}^d$. We define the function Ω_A on $\mathbf{R}_+^d$ by

$$\Omega_A(t) := \inf_{\alpha \in A} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_d^{\alpha_d}.$$

It is easy to check that $\Omega_A \in \mathbf{MS}_l$ for all l such that

$$l_j > \max\{\alpha_j : \alpha \in A\},\ j = 1, 2, ..., d. \tag{3}$$

Therefore, (1)–(2) define the Besov space $\mathbf{B}_{p,\theta}^A := \mathbf{B}_{p,\theta}^{\Omega_A}$ for any l satisfying the condition (3). We say that the Besov space $\mathbf{B}_{p,\theta}^A$ has the mixed smoothness A. For $r > 0$, let $A_r := \{r(e) : e \subset \{1, 2, \ldots, d\}\}$ where $r(e)$ denotes the element of $\mathbf{R}_+^d$ such that $r(e)_j = r$ for $j \in e$, and $r(e)_j = 0$ for $j \notin e$. We use the notation: $\mathbf{B}_{p,\theta}^r := \mathbf{B}_{p,\theta}^{A_r}$.

4. Under certain conditions a function on $\mathbf{T}^d$ is decomposed into a series

$$f = \sum_{k,s} \lambda_{k,s} \varphi_{k,s}, \tag{4}$$

of wavelets which are the integer translates

$$\varphi_{k,s}(x) := \varphi_k(x - 2\pi s/2^k),\ s \in \mathbf{Z}_+^d, s_j = 0, \ldots, 2^k - 1,\ k \in \mathbf{Z}_+$$

of scaling functions φ_k. Notice that for periodic wavelet decompositions (4), φ_k are different for $k \in \mathbf{Z}_+$. The coefficients $\lambda_{k,s} := \lambda_{k,s}(f)$ are certain functionals of f. Since we investigate non-linear approximations of periodic functions, we are restricted to consider periodic wavelets only. The interested reader can consult [1, 7] for basic ideas and knowledge on wavelets.

Wavelet decompositions are quite appropriate to non-linear approximations, in particular, n-term approximation because of their good approximation properties. Firstly, they provide a simultaneous time and frequency localization. This allows us to select different numbers of terms $\varphi_{k,s}$ at each kth dyadic scale for n-term approximation, depending on a given target function. Secondly, they give discrete descriptions of equivalent norms and semi-norms for Sobolev and Besov spaces in terms of coefficients functionals $\lambda_{k,s}(f)$. Using these discrete characterizations, we can process a quantization or discretization of our approximation problems. In the discrete form, they are more convenient for study and numerical computation. The interested reader can consult [2] for a survey on non-linear approximation using wavelet decompositions.

Our purpose is to investigate the n-term approximation of the Besov class $\mathbf{SB}_{p,\theta}^A$ with regard to multivariate wavelets generated from de la Vallée Poussin kernels. The Besov class of the mixed smoothness A

$$\mathbf{SB}_{p,\theta}^A := \{f \in \mathbf{B}_{p,\theta}^A : \|f\|_{\mathbf{B}_{p,\theta}^A} \leq 1\}$$

is defined as the unit ball of the space $\mathbf{B}_{p,\theta}^A$. Periodic wavelets and corresponding wavelet decompositions (4), do not appropriate to the mixed structure of smoothness A. For n-term approximations of functions from $\mathbf{SB}_{p,\theta}^A$, we have to construct so-called mixed wavelets and mixed wavelet decompositions . Let

$$V_m(t) := \frac{1}{3m^2} \sum_{k=m}^{2m-1} D_k(t) = \frac{\sin(mt/2)\sin(3mt/2)}{3m^2 \sin^2(t/2)}$$

be the de la Vallée Poussin kernel of order m, where $D_m(t) := \sum_{|k|\le m} e^{ikt}$ is the univariate Dirichlet kernel of order m. Next we define

$$\varphi_0 := 1, \ \varphi_k := V_{2^{k-1}}, \ k = 1, 2, \dots$$

For $k \in \mathbf{Z}^d_+ = \{s \in \mathbf{Z}^d : s_j \ge 0, \ j = 1, \dots, d\}$, we let the mixed dyadic scaling functions

$$\varphi_k(x) := \varphi_{k_1}(x_1)\varphi_{k_2}(x_2) \cdots \varphi_{k_d}(x_d)$$

be defined as the tensor product of the univariate scaling functions $\varphi_{k_j}(x_j)$ and the mixed wavelets

$$\varphi_{k,s} := \varphi_k(\cdot - 2\pi s/2^k), \ s \in Q_k,$$

be defined as the integer translates of φ_k, where

$$Q_k := \{s \in \mathbf{Z}^d : 0 \le s_j < 2^{k_j}, \ j = 1, \dots, d\}$$

and $2\pi s/2^k := 2\pi(s_1/2^{k_1}, s_2/2^{k_2}, \dots, s_d/2^{k_d})$.

For n-term approximation of functions from $\mathbf{SB}^A_{p,\theta}$, we take the family of mixed wavelets:

$$\mathbf{V} := \{\varphi_{k,s} : k \in \mathbf{Z}^d_+, \ s \in Q_k\}.$$

5. We use the notation $F \asymp F'$ if $F \ll F'$ and $F' \ll F$, and $F \ll F'$ if $F \le CF'$ with C an absolute constant. Denote by γ_n any one of $\alpha_n, \tau_n, \tau'_n, \beta_n, a_n$ and δ_n.

For given A and p, θ, q, we established the asymptotic orders of $\gamma_n(\mathbf{SB}^A_{p,\theta}, L_q)$ and $\sigma_n(\mathbf{SB}^A_{p,\theta}, \mathbf{V}, L_q)$. It turns out that they are closely related to the following convex problem in $\mathbf{R}^d$

$$(\mathbf{1}, x) \to \sup, \ x \in A^o_+, \tag{5}$$

where $A^o_+ := \{x \in \mathbf{R}^d : (\alpha, x) \le 1, \ \alpha \in A, \ x_j \ge 0, \ j = 1, \dots, d\}$, $\mathbf{1} := (1, 1, \dots, 1) \in \mathbf{R}^d$. Let $1/r = 1/r(A)$ be the optimal value of of this problem, i.e.,

$$1/r := \sup\{(\mathbf{1}, x) : x \in A^o_+\}.$$

Further, let $\nu = \nu(A)$ be the linear dimensions of the set of solutions of (5), i.e.,

$$\nu := \dim\{x \in A^o_+ : (\mathbf{1}, x) = 1/r\},$$

$\mu = \mu(A) = d - 1 - \nu(A)$ and

$$v(h) := \mathrm{Vol}_{d-1}\{x \in A^o_+ : (\mathbf{1}, x) = 1/r - h\},$$

where $\mathrm{Vol}_m G$ denotes the m-dimensional volume of $G \subset \mathbf{R}^d$. One can explicitly construct from the set A a function w so that $w = w(A, \cdot)$ is a concave modulus of continuity if $\nu < d - 1$, and $w = 1$ if $\nu = d - 1$ and

$$v(h) \asymp w^\mu(h) \text{ as } h \to 0$$

(see [3]). Notice that $r(A) = \min\{t > 0 : t\mathbf{1} \in \mathrm{co}A\}$ and μ is the linear dimension of the minimal extreme subset of $\mathrm{co}A$ containing the point $r\mathbf{1}$, where $\mathrm{co}A$ denotes the convex hull of A. If $A = A_r$, then $r(A) = r$, and if the set A is finite, then we have [3]

$$w(A,h) = h \text{ if } \nu(A) < d-1.$$

We proved the following

Theorem 1. *Let $1 < p, q < \infty$, $0 < \theta \le \infty$. Assume that either $\rho(A) > 1/p$ and $\theta \ge p$ or the condition $\rho(A) > \max\{0, 1/p - 1/q\}$ and $\theta \ge \min\{q, 2\}$. Then we have*

$$\gamma_n(\mathbf{SB}^A_{p,\theta}, L_q) \asymp \sigma_n(\mathbf{SB}^A_{p,\theta}, \mathbf{V}, L_q) \asymp n^{-r}(w^\mu(1/\log n)\log^{d-1} n)^{r+1/2-1/\theta},$$

where $r = r(A)$, $\mu = \mu(A)$ and $w = w(A,\cdot)$. In addition, we can explicitly construct an asymptotically optimal (continuous) algorithm S of n-term approximation with regard to $\mathbf{V}$ such that

$$\sup_{f\in \mathbf{SB}^A_{p,\theta}} \|f - S(f)\|_q \ll n^{-r}(w^\mu(1/\log n)\log^{d-1} n)^{r+1/2-1/\theta}.$$

If the mixed smoothness A in Theorem 1 is finite, then we have

$$\gamma_n(\mathbf{SB}^A_{p,\theta}, L_q) \asymp \sigma_n(\mathbf{SB}^A_{p,\theta}, \mathbf{V}, L_q) \asymp n^{-r}(\log n)^{\nu(r+1/2-1/\theta)},$$

where $\nu = \nu(A)$. The last asymptotic order was proved in [4] for $\theta \ge 2$. Theorem 1 has been proved in [5] for the Besov class $\mathbf{SB}^r_{p,\theta}$. The asymptotic order of the n-term approximation $\sigma_n(\mathbf{SB}^r_{p,\infty}, U^d, L_q)$ with regard to the family U^d formed from the integer translates of the mixed dyadic scales of the tensor product multivariate Dirichlet kernel [8].

To prove Theorem 1 we essentially developed methods of [5] and used the following special decompositions for $\mathbf{B}^A_{p,\theta}$. Under the condition $\rho(A) > \max\{0, 1/p - 1/q\}$, every $f \in \mathbf{B}^A_{p,\theta}$ has a wavelet decomposition into the series

$$f = \sum_{k\in \mathbf{Z}^d_+} \sum_{s\in Q_k} \lambda_{k,s}\varphi_{k,s} \tag{6}$$

with the convergence in the space $\mathbf{B}^\Omega_{p,\theta}$, where $\lambda_{k,s} = \lambda_{k,s}(f)$ are linear (continuous) coefficients functionals of f. Moreover, there holds the following quasi-norms equivalence

$$\|f\|_{\mathbf{B}^A_{p,\theta}} \asymp \Big(\sum_{k=0}^{\infty}\Big\{2^{S(A,k)-|k|/p}\Big(\sum_{s\in Q_k}|\lambda_{k,s}|^p\Big)^{1/p}\Big\}^\theta\Big)^{1/\theta}, \tag{7}$$

where $S(A,x) := \sup\{(\alpha, x) : \alpha \in A\}$ is the support function of A and $|k| := k_1 + k_2 + \dots + k_d$. To construct an asymptotically optimal (continuous) algorithm S of n-term approximation with regard to $\mathbf{V}$ and establish the upper bound of Theorem 1, for a natural number ξ, we constructed a sequense of subsets of $\mathbf{Z}^d_+$ $\Delta(\xi,\eta), \eta = \xi, \xi+1, \dots$ such that $\mathbf{Z}^d_+ = \cup_{\eta=\xi}^\infty \Delta(\xi,\eta)$ and

$$S(A,k) \le r\eta,\ k \in \Delta(\xi,\eta),$$

and for any pair ξ, η satisfying the condition $\eta \le (2\lambda - 1)\xi$ with $\lambda > 1$ a fixed number,

$$|\Delta(\xi,\xi)| \ll \xi^{d-1} w^{\mu}(1/\xi), \quad |\Delta(\xi,\eta)| \ll \xi^{d-1} w^{\mu}(1/\xi)(\eta-\xi)^{\mu}, \ \eta > \xi.$$

From the wavelet decomposition (6) - (7) we deduced that a function $f \in \mathbf{SB}^{A}_{p,\theta}$ is decomposed into the series

$$f = \sum_{\eta=\xi}^{\infty} f_{\xi,\eta}, \quad f_{\xi,\eta} = \sum_{k \in \Delta(\xi,\eta)} \sum_{s \in Q_k} \lambda_{k,s} \varphi_{k,s},$$

converging in the norm of L_q, with the following property

$$\left(\sum_{k \in \Delta(\xi,\eta)} \left(\sum_{s \in Q_k} |\lambda_{k,s}|^p \right)^{\theta/p} \right)^{1/\theta} \ll 2^{-(r-1/p)\eta}$$

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