Vietnam Journal of Mathematics **29**:2 (2001) 131–158

Vietnam Journal
of
MATHEMATICS

Quantum Co-Adjoint Orbits of MD_4-Groups*

Nguyen Viet Hai

Hai Phong Teacher's Training College, Hai Phong city, Vietnam

Received March 24, 2000

Revised August 9, 2000

Abstract. Using $\star$-product on co-adjoint orbits (K-orbits) of the MD_4- groups we obtain quantum co-adjoint orbits via Fedosov deformation quantization. From this we obtain the full list of corresponding irreducible unitary representations of the exponential MD_4-groups. For groups $G_{4,2,3(\frac{\pi}{2})}$; $G_{4,2,4}$; $G_{4,3,4(\frac{\pi}{2})}$ and $G_{4,4,1}$, which are neither nilpotent nor exponential, we obtain also the explicit formulas.

1. Introduction

In the early 70's Berezin treated the general mathematical definition of quantization as a kind of functors from the category of classical mechanics to a certain category of associative algebras. At about the same time, Flato, Fronsdal, Bayen, Lichnerowicz and Sternheimer considered quantization as a deforming process of the commutative product of classical observables into a family of non-commutative $\star$-products which are parameterized by the Planck constant $\hbar$ and satisfy the *correspondence principle.* They systematically developed the notion of deformation quantization as a theory of $\star$-products and gave an independent formulation of quantum mechanics based on this notion (see [4]).

It was proved by Gerstenhaber that a formal deformation quantization exists on an arbitrary symplectic manifold, see for example [8] for a detailed explanation. It is however formal and quite complicated in general. We would like to simplify it in some particular cases. From the orbit method, it is well-known that co-adjoint orbits are homogeneous symplectic manifolds with a flat action. A natural question is to associate in a reasonable way to these orbits some quantum

* This work was supported in part by the Vietnam National Foundation for Fundamental Science Research.

objects, what could be called *quantum co-adjoint orbits*.

Let us denote by G a connected and simply connected Lie group, its Lie algebra $\mathfrak{g} = T_e\mathrm{G}$ as the tangent space at the neutral element e. To each element $g \in \mathrm{G}$, one associates a map

$$A(g) : \mathrm{G} \longrightarrow \mathrm{G}, \quad h \mapsto ghg^{-1}.$$

The corresponding tangent map is

$$A(g)_* : \mathfrak{g} = T_e\mathrm{G} \longrightarrow \mathfrak{g} = T_e\mathrm{G},$$

$$X \in \mathfrak{g} \mapsto \frac{d}{dt} g\exp(tX)g^{-1}|_{t=0} \in \mathfrak{g}.$$

This defines an action, called adjoint action and denoted as usual by Ad, of group G in its Lie algebra $\mathfrak{g}$. We define the co-adjoint action of group G in the dual vector space $\mathfrak{g}^*$ by the formula

$$\langle K(g)F, A\rangle := \langle F, Ad(g^{-1})A\rangle$$

for all $F \in \mathfrak{g}^*$, $A \in \mathfrak{g}$, $g \in \mathrm{G}$.

Group G acts by symplectomorphisms on Ω, where $\Omega = \Omega_F = \{K(g)F \mid g \in \mathrm{G}\} \subseteq \mathfrak{g}^*$ is the co-adjoint orbit passing through F, and each element $A \in \mathfrak{g}$ appears as a C^∞-function $\tilde{A}$ on Ω: $\tilde{A}(F) = \langle F, A\rangle$, $F \in \Omega$.

Let ξ_A be the Hamiltonian field defined by

$$(\xi_A f)(F) = \frac{d}{dt} f(F.\exp(tA))|_{t=0}, \quad \forall f \in \mathrm{C}^\infty(\Omega).$$

$\tilde{A}$ is then the Hamiltonian function associated to Hamiltonian vector field ξ_A, i.e.: $\xi_A(f) = \{\tilde{A}, f\}$, $f \in \mathrm{C}^\infty(\Omega)$.

In 1980 Do Ngoc Diep introduced the notion of the MD-groups, MD-algebras and then Le Anh Vu gave a complete classification of the MD_4-groups (see [5]). In this paper, applying the procedure of deformation quantization we shall obtain quantum co-adjoint orbits of all MD_4-groups. It is to emphasize here that there is a general theory for exponential and compact groups. However it is difficult to calculate explicitly the $\star$-product and the corresponding representations in concrete cases. Our consideration here concerns not only with these groups but also with non-exponential and noncompact Lie group G. Our main result gives explicit $\star$-product formulas and then, all representations of the exponential and non-exponential groups of the type MD_4.

The paper is organized as follows. In Sec. 2 we recall basic definitions, preliminary results. The full list of irreducible unitary representations of the real diamond Lie algebra is constructed in Sec. 3, and that of the exponential MD_4-groups is introduced in Sec. 4. Section 5 is devoted to the following non-exponential groups

$$\mathrm{G}_{4,2,3(\frac{\pi}{2})}; \mathrm{G}_{4,2,4}; \mathrm{G}_{4,3,4(\frac{\pi}{2})}; \mathrm{G}_{4,4,1}.$$

By direct computations and by exponentiating we obtain corresponding representations of the MD_4-groups.

2. Basic Definitions and Preliminary Results

2.1. MD$_4$-algebras and MD$_4$-groups

Definition 2.1. [5] *We say that a solvable Lie group* G *belongs to the class* MD *if and only if every its K-orbit has dimension* 0 *or is maximal. A Lie algebra is of class* MD *if and only if its corresponding Lie group is of the same class.*

We also recall the following results for MD$_4$-algebras (i.e. $\dim \mathfrak{g} = 4$).

Theorem 2.2. *Assume* $\mathfrak{g}$ *is an* MD$_4$*-algebra with generators* X, Y, Z, T.

I. *If* $\mathfrak{g}$ *is decomposable then it is of the form* $\mathfrak{g} = \mathbf{R}^n \oplus \tilde{\mathfrak{g}}$ *for* $n = 1, 2, 3, 4$ *and some indecomposable ideal* $\tilde{\mathfrak{g}}$.

II. *If* $\mathfrak{g}$ *is indecomposable then* $\mathfrak{g}$ *is of class* MD$_4$ *if and only if it is generated by the generators* X, Y, Z, T *with only non-trivial commutation relation which is one of following relations defined in each case:*

1. $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \mathbf{R}Z \cong \mathbf{R}$, *and*

   1.1. $[T, X] = Z$ $\quad (\mathfrak{g}_{4,1,1})$

   1.2. $[T, Z] = Z$ $\quad (\mathfrak{g}_{4,1,2})$

2. $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \mathbf{R}Y + \mathbf{R}Z \cong \mathbf{R}^2$, *and*

   2.1. $[T, Y] = \lambda Y, [T, Z] = Z; \lambda \in \mathbf{R}^* = \mathbf{R}\backslash\{0\}$ $\quad (\mathfrak{g}_{4,2,1(\lambda)})$

   2.2. $[T, Y] = Y; [T, Z] = Y + Z$ $\quad (\mathfrak{g}_{4,2,2})$

   2.3. $\mathrm{ad}_T = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix},$ $\quad (\mathfrak{g}_{4,2,3(\varphi)})$

   2.4. $\mathrm{ad}_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathrm{ad}_X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad (\mathfrak{g}_{4,2,4}) = \mathrm{Lie}(\mathrm{Aff}(\mathbf{C}))$

3. $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \mathbf{R}X + \mathbf{R}Y + \mathbf{R}Z$, *and*

   3.1. $\mathrm{ad}_T = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda_1, \lambda_2 \in \mathbf{R}^*$ $\quad (\mathfrak{g}_{4,3,1(\lambda_1,\lambda_2)})$

   3.2. $\mathrm{ad}_T = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda \in \mathbf{R}^*$ $\quad (\mathfrak{g}_{4,3,2(\lambda)})$

   3.3. $\mathrm{ad}_T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad (\mathfrak{g}_{4,3,3})$

   3.4. $\mathrm{ad}_T = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \lambda \in \mathbf{R}^*, \varphi \in (0, \pi)$ $\quad (\mathfrak{g}_{4,3,4(\lambda)})$

4. $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \mathbf{R}X + \mathbf{R}Y + \mathbf{R}Z \cong \mathfrak{h}_3$ *- the 3-dimensional Heisenberg Lie algebra and*

4.1. $\mathrm{ad}_T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $[X,Y] = Z$ $\qquad (\mathfrak{g}_{4,4,1} = \mathrm{Lie}(\mathbf{R} \ltimes_j \mathbf{H}_3))$

4.2. $\mathrm{ad}_T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $[X,Y] = Z$ $\qquad (\mathfrak{g}_{4,4,2} = \mathrm{Lie}(\mathbf{R} \ltimes \mathbf{H}_3))$

(In this case the group is called the real diamond Lie group, $\mathrm{G} = \mathbf{R} \ltimes \mathbf{H}_3$*).*

From now on we shall denote by $\mathfrak{g}$ an MD_4- algebra with the standard basis X, Y, Z, T over $\mathbf{R}$. It is isomorphic to $\mathbf{R}^4$ as vector space. The coordinates in this standard basis is denoted by (a, b, c, d). We identify its dual vector space $\mathfrak{g}^*$ with $\mathbf{R}^4$ with the help of the dual basis X^*, Y^*, Z^*, T^* and with the local coordinates $(\alpha, \beta, \gamma, \delta)$. Thus, for all $U \in \mathfrak{g}$ we have $U = aX + bY + cZ + dT$ and for all $F \in \mathfrak{g}^*, F = \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^*$. Finally,
Ω_F is the co-adjoint orbit passing through $F \in \mathfrak{g}^*$.

Theorem 2.3. (The picture of co-adjoint orbit) [5]

1.1. *Case* $\mathrm{G} = \mathrm{G}_{4,1,1}$.

i. *Each point F with the coordinate $\gamma = 0$ is a 0-dimensional co-adjoint orbit*

$$\Omega_F = \Omega_{(\alpha,\beta,0,\delta)}.$$

ii. *The subset $\gamma \neq 0$ is decomposed into a family of 2-dimensional co-adjoint orbits*

$$\Omega_F = \Omega_F^{(1,1)} = \{(\alpha + \gamma d, \beta, \gamma, -\gamma a + \delta)\} = \{(x, \beta, \gamma, t) | x, t \in \mathbf{R}\}, \quad (1)$$

which are **planes**.

1.2. *Case* $\mathrm{G} = \mathrm{G}_{4,1,2}$.

i. *Each point F with the coordinate $\gamma = 0$ is a 0-dimensional co-adjoint orbit*

$$\Omega_F = \Omega_{(\alpha,\beta,0,\delta)}.$$

ii. *The subset $\gamma \neq 0$ is decomposed into a family of 2-dimensional co-adjoint orbits:*

$$\Omega_F^{(1,2)} = \left\{\alpha, \beta, \gamma e^d, -\gamma c \sum_1^{\infty} \frac{d^{n-1}}{n!} + \delta\right\}$$
$$= \{(\alpha, \beta, z, t) | z, t \in \mathbf{R}, \gamma z > 0\}, \quad (2)$$

which are **half-planes**, *parameterized by the coordinates $\alpha, \beta \in \mathbf{R}$.*

2.1. *Case* $\mathrm{G} = \mathrm{G}_{4,2,1(\lambda)}$, $\lambda \in \mathbf{R}^*$.

i. *Each point on the plane $\beta = \gamma = 0$ is a 0-dimensional co-adjoint orbit*

$$\Omega_F = \Omega_{(\alpha,0,0,\delta)}.$$

ii. *The open set $\beta^2 + \gamma^2 \neq 0$ is decomposed into the union of 2-dimensional* **cylinders**

$$\Omega_F = \Omega_F^{(2,1)} = \{(\alpha, \beta e^{s\lambda}, \gamma e^s, t) | s, t \in \mathbf{R}\}. \quad (3)$$

2.2. *Case* $\mathrm{G} = \mathrm{G}_{4,2,2}$.

i. *Each point on the plane $\beta = \gamma = 0$ is a 0-dimensional co-adjoint orbit*

$$\Omega_F = \Omega_{(\alpha,0,0,\delta)}.$$

ii. *The open set $\beta^2+\gamma^2 \neq 0$ is decomposed into the union of 2-dimensional* **cylinders**

$$\Omega_F = \Omega_F^{(2,2)} = \{(\alpha, \beta e^s, \beta s e^s + \gamma e^s, t) | s, t \in \mathbf{R}\}. \tag{4}$$

2.3. *Case* $G = G_{4,2,3(\varphi)}$ *with* $\varphi \in (0,\pi)$. *We identify* $\mathfrak{g}^*_{4,2,3(\varphi)}$ *with* $\mathbf{R}\times\mathbf{C}\times\mathbf{R}$ *and* $F = (\alpha,\beta,\gamma,\delta)$ *with* $(\alpha, \beta+i\gamma, \delta)$. *Then,*

i. *Each point* $(\alpha, 0, \delta)$ *is a 0-dimensional co-adjoint orbit*

$$\Omega_F = \Omega_{(\alpha, 0+i0, \delta)}.$$

ii. *The open set $\beta + i\gamma \neq 0$ is decomposed into the union of 2-dimensional co-adjoint orbits*

$$\Omega_F = \Omega_F^{(2,3)} = \{(\alpha, (\beta+i\gamma)e^{se^{i\varphi}}, t) | s, t \in \mathbf{R}\}, \tag{5}$$

which are also **cylinders.**

2.4. *Case* $G = G_{4,2,4} = \widetilde{\mathrm{Aff}(\mathbf{C})}$

i. *Each point* $(\alpha, 0, 0, \delta)$ *is a 0-dimensional co-adjoint orbit*

$$\Omega_F = \Omega_{(\alpha,0,0,\delta)}.$$

ii. *The open set $\beta^2+\gamma^2 \neq 0$ is the single 4-dimensional co-adjoint orbit*

$$\Omega_F = \Omega_F^{(2,4)} = \{(x,y,z,t) | y^2+z^2 \neq 0\} = \mathbf{R}\times(\mathbf{R}^2)^*\times\mathbf{R}. \tag{6}$$

3.1. *Case G is one of the groups* $G_{4,3,1(\lambda_1,\lambda_2)}$, $G_{4,3,2(\lambda)}$ *or* $G_{4,3,3}$

i. *Each point $F = \delta T^*$ on the line $\alpha=\beta=\gamma=0$ is a 0-dimensional co-adjoint orbit.*

ii. *The open set $\alpha^2+\beta^2+\gamma^2 \neq 0$ is decomposed into a family of co-adjoint orbits, which are* **cylinders**, *coressponding to the groups* $G_{4,3,1(\lambda_1,\lambda_2)}$, $G_{4,3,2(\lambda)}$, $G_{4,3,3}$

$$\Omega_F^{(3,1)} = \{(\alpha e^{s\lambda_1}, \beta e^{s\lambda_2}, \gamma e^s, t) | s, t \in \mathbf{R}\},$$

tag7

$$\Omega_F^{(3,2)} = \{(\alpha e^{s\lambda}, \alpha s e^{s\lambda} + \beta e^{s\lambda}, \gamma e^s, t) | s, t \in \mathbf{R}\}, \tag{8}$$

$$\Omega_F^{(3,3)} = \{(\alpha e^{s\lambda}, \alpha s e^s + \beta e^s, \frac{1}{2}\alpha s^2 e^s + \beta s e^s + \gamma e^s, t) | s, t \in \mathbf{R}\}. \tag{9}$$

3.2. *Case* $G = G_{4,3,4(\lambda,\varphi)}$ *for* $\lambda \in \mathbf{R}^*, \varphi \in (0,\pi)$. *We identify* $\mathfrak{g}^*_{4,3,4(\lambda,\varphi)}$ *with* $\mathbf{C}\times\mathbf{R}^2$ *and* $F = (\alpha,\beta,\gamma,\delta)$ *with* $(\alpha+i\beta, \gamma, \delta)$. *Then,*

i. *Each point of the line defined by the condition $\alpha=\beta=\gamma=0$ is a 0-dimensional co-adjoint orbit*

$$\Omega_F = \Omega_{(0,0,\delta)} = \{(0+i.0, 0, \delta)\}.$$

ii. *The open set $|\alpha+i\beta|^2+\gamma^2 \neq 0$ is decomposed into an union of co-adjoint orbits, which are* **cylinders**

$$\Omega_F = \Omega_F^{(3,4)} = \{((\alpha+i\beta)e^{se^{i\varphi}}, \gamma e^{s\lambda}, t) | s, t \in \mathbf{R}\}. \tag{10}$$

4.1. *Case* $G = G_{4,4,1} = \mathbf{R} \ltimes_j \mathfrak{h}_3$

i. *Each point of the line defined by the conditions $\alpha=\beta=\gamma=0$ is a 0-dimensional co-adjoint orbit*

$$\Omega_F = \Omega_{(0,0,0,\delta)} = \{(0,0,0,\delta)\}.$$

ii. *The set* $\alpha^2 + \beta^2 \neq 0, \gamma = 0$ *is the union of 2-dimensional co-adjoint orbits, which are* **rotation cylinders**

$$\Omega_F^{(4,1,a)} = \{(\alpha cos\theta - \beta sin\theta, \alpha sin\theta + \beta cos\theta, 0, t)|\theta, t \in \mathbf{R}\} \quad (11)$$

i.e.

$$\Omega_F^{(4,1,a)} = \{(x, y, 0, t)|x^2 + y^2 = \alpha^2 + \beta^2; \quad x, y, t \in \mathbf{R}\}.$$

iii. *The open set* $\gamma \neq 0$ *is decomposed into a union of 2-dimensional co-adjoint orbits*

$$\Omega_F^{(4,1,b)} = \{(x, y, \gamma, t)|x^2 + y^2 - 2\gamma t = \alpha^2 + \beta^2 - 2\gamma\delta; \quad x, y, t \in \mathbf{R}\}, \quad (12)$$

which are **rotation paraboloids**.

4.2. *Case* $G = G_{4,4,2} = \mathbf{R} \ltimes \mathbf{H}_3$, *the real diamond group.*

i. *Each point of the line* $\alpha = \beta = \gamma = 0$ *is a 0-dimensional co-adjoint orbit*

$$\Omega_F = \Omega_{(0,0,0,\delta)}.$$

ii. *The set* $\alpha \neq 0, \beta = \gamma = 0$ *is union of 2-dimensional co-adjoint orbits, which are just* **half-planes**

$$\Omega_F^{(4,2,a)} = \{(x, 0, 0, t) \mid x, t \in \mathbf{R}, \alpha x > 0\}. \quad (13)$$

iii. *The set* $\alpha = \gamma = 0, \beta \neq 0$ *is union of 2-dimensional co-adjoint orbits, which are just* **half-planes**

$$\Omega_F^{(4,2,b)} = \{(0, y, 0, t) \mid y, t \in \mathbf{R}, \beta y > 0\}. \quad (14)$$

iv. *The set* $\alpha\beta \neq 0, \gamma = 0$ *is decomposed into a family of 2-dimensional co-adjoint orbits, which are just* **hyperbolic-cylinders**

$$\Omega_F^{(4,2,c)} = \{(x, y, 0, t) \mid x, y, t \in \mathbf{R} \ \& \ \alpha x > 0, \ \beta y > 0, \ xy = \alpha\beta\}. \quad (15)$$

v. *The open set* $\gamma \neq 0$ *is decomposed into a family of 2-dimensional co-adjoint orbits , which are just* **hyperbolic-paraboloids**

$$\Omega_F^{(4,2,d)} = \{(x, y, \gamma, t) \mid x, y, t \in \mathbf{R} \ \& \ xy - \alpha\beta = \gamma(t - \delta)\}. \quad (16)$$

Thus, we have 15 families of 2-dimensional co-adjoint orbits and a single 4-dimensional co-adjoint orbit $\Omega_F^{(2,4)} \cong \mathbf{C} \times \mathbf{C}^*$. They are strictly homogeneous symplectic manifolds with a flat action (see [12]).

2.2. Moyal $\star$-product and quantum co-adjoint orbits

Let us denote by Λ the 2-tensor associated with the standard form $\omega = dp \wedge dq = \sum_{j=1}^n dp_j \wedge dq_j$ of the symplectic space $\mathbf{R}^{2n}$. We consider the well-known Moyal $\star$-product of two smooth functions $u, v \in C^\infty(\mathbf{R}^{2n})$, defined by

$$u \star v = u.v + \sum_{r \geq 1} \frac{1}{r!} \left(\frac{1}{2i}\right)^r P^r(u, v),$$

where

$$P^1(u, v) = \{u, v\}; \quad P^r(u, v) := \Lambda^{i_1 j_1} \Lambda^{i_2 j_2} \ldots \Lambda^{i_r j_r} \partial^r_{i_1 i_2 \ldots i_r} u \partial^r_{j_1 j_2 \ldots j_r} v,$$

with

$$\partial^r_{i_1 i_2 \ldots i_r} := \frac{\partial^r}{\partial x^{i_1} \ldots \partial x^{i_r}}; \quad x := (p,q) = (p_1, \ldots, p_n, q^1, \ldots, q^n).$$

It is well-known that this series converges in the Schwartz distribution spaces $\mathcal{S}(\mathbf{R}^{2n})$. Furthermore, the following results are known (see e.g. [1]): If $u, v \in \mathcal{S}(\mathbf{R}^{2n})$, then

- $\overline{u} \star \overline{v} = \overline{v \star u}$,
- $\int (u \star v)(\xi) d\xi = \int uv d\xi$,
- $\ell_u : \mathcal{S}(\mathbf{R}^{2n}) \longrightarrow \mathcal{S}(\mathbf{R}^{2n})$, defined by $\ell_u(v) = u \star v$ is continuous in $L^2(\mathbf{R}^{2n}, d\xi)$ and then can be extended to a bounded linear operator (still denoted by ℓ_u) on $L^2(\mathbf{R}^{2n}, d\xi)$.

Let now G be an MD$_4$-group, $\mathfrak{g} = \mathrm{LieG}$. For each $A \in \mathfrak{g}$, the corresponding Hamiltonian function is $\tilde{A}$ and we can put $\ell_A(u) = i\tilde{A} \star u$, $u \in \mathrm{L}^2(\mathbf{R}^2, \frac{dpdq}{2\pi})$. It is then continuated to the whole space $\mathrm{L}^2(\mathbf{R}^2, \frac{dpdq}{2\pi})$. Let us denote by $\mathcal{F}_p(f)$ the partial Fourier transform of the function f from the variable p to the variable x, i.e.

$$\mathcal{F}_p(f)(x,q) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ipx} f(p,q) dp$$

and by $\mathcal{F}_p^{-1}(f)$ the inverse Fourier transform.

Now we put $\hat{\ell}_A := \mathcal{F}_p \circ \ell_A \circ \mathcal{F}_p^{-1}$ with $A \in \mathfrak{g}$.

Definition 2.4. *Let Ω_F be the K-orbit of the co-adjoint representations of Lie group G passing through a point $F \in \mathfrak{g}^* = Lie(\mathrm{G})^*$. With A running over the Lie algebra $\mathfrak{g} = \mathrm{LieG}$, $[\Omega_F, \hat{\ell}_A]$ is called a quantum co-adjoint orbit of Lie group* G.

2.3. Some known results for special classes of groups

We recall some results which will be used frequently in this paper.

Theorem A. (B. Kostant, L. Auslander), (see [12, p. 241]) *Let* G *be a connected and simply connected solvable Lie group. The following assertions hold.*

1. *The group* G *belongs to type I if and only if the space $\mathcal{O}(\mathrm{G})$ is T_0 and all forms B_Ω (Kirillov forms) are exact.*
2. *If* G *is of type I, all irreducible representations of* G *are obtained by the orbit method. To every orbit Ω there corresponds a family of irreducible representations, parametrized by the characters of the group $\pi_1(\Omega)$.*
3. *Representations that correspond to different orbits or different characters of the fundamental group of the orbit are necessarily inequivalent.*

We note that if the group G is exponential then all G-orbits in $\mathfrak{g}^*$ are homeomorphic to euclidean space. In particular, it follows from the above theorem that exponential groups are of type I and that for these groups, there is an one-to-one correspondence between the sets $\hat{\mathrm{G}}$ and $\mathcal{O}(\mathrm{G})$.

Theorem B. [2, Proposition 2.6] *Let G be an exponential connected and simply connected Lie group. The operator $\hat{\ell}_A := \mathcal{F}_p \circ \ell_A \circ \mathcal{F}_p^{-1}$, $A \in \mathfrak{g}$, is the differential form of the unitary irreducible representation π^{Ω} of the group G corresponding to an orbit $\Omega \in \mathcal{O}(G)$ by the Kirillov-Kostant method.*

In other words, the unitary irreducible representations of the exponential Lie group obtained from $\star$-product are isomorphic to the unitary irreducible representations obtained from the Kirillov-Kostant method. It is to emphasize that if $\hat{\ell}_A$ is a representation of the Lie algebra $\mathfrak{g}$ of a connected and simply connected Lie group G, then the representation of G (obtained from $\star$-product) is constructed by the formula

$$T(\exp A) = \exp \hat{\ell}_A,$$

where exp is the canonical mapping of the Lie algebra into the group.

3. Representations of the Real Diamond Group

First, consider the real diamond group $G = \mathbf{R} \ltimes \mathbf{H}_3$. This group has a lot of non-trivial 2-dimensional co-adjoint orbits, which are the half-planes, the hyperbolic cylinders and the hyperbolic paraboloids.

3.2. Quantum co-adjoint orbits of the real diamond group

Each element $A = aX + bY + cZ + dT \in \mathfrak{g}_{4,4,2} = \text{Lie}(\mathbf{R} \ltimes \mathbf{H}_3)$ can be considered as the restriction of the corresponding linear functional $\tilde{A}$ onto co-adjoint orbits $(\subset \mathfrak{g}^*)$, $\tilde{A}(F) = \langle F, A \rangle$. We have

Proposition 3.1. *There is a diffeomorphism ψ from $\mathbf{R}^2$ onto $\Omega_F, (p,q) \mapsto \psi(p,q)$ such that*

1. *Hamiltonian function $\tilde{A}$ in canonical coordinates (p,q) of the orbit Ω_F is of the form*

$$\tilde{A} \circ \psi(p,q) = \begin{cases} dp + a\alpha e^{-q} & \text{if } \Omega_F = \Omega_F^{(4,1,a)} \\ dp + b\beta e^{q} & \text{if } \Omega_F = \Omega_F^{(4,1,b)} \\ dp + a\alpha e^{-q} + b\beta e^{q} & \text{if } \Omega_F = \Omega_F^{(4,1,c)} \\ (d \pm b\gamma e^{q})p \pm ae^{-q} \pm b(\alpha\beta - \gamma\delta)e^{q} + c\gamma & \text{if } \Omega_F = \Omega_F^{(4,1,d)} \end{cases}$$

2. *In the canonical coordinates (p,q) of the orbit Ω_F, the Kirillov form ω coincides with the standard form $dp \wedge dq$.*

Proof. 1. We adapt the diffeomorphism ψ to each of the following cases (for 2-dimensional co-adjoint orbits, only).

- With $\alpha \neq 0, \beta = \gamma = 0$, set

$$\psi: \quad (p,q) \in \mathbf{R}^2 \mapsto \psi(p,q) = (\alpha e^{-q}, 0, 0, p) \in \Omega_F^{(4,2,a)}.$$

Element $F \in \mathfrak{g}^*$ is of the form $F = \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^*$, hence for $A = aX + bY + cZ + dT \in \mathfrak{g}$, $\tilde{A}(F) = \langle F, A\rangle = \langle \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^*, aX + bY + cZ + dT\rangle = \alpha a + \beta b + \gamma c + \delta d.$

It follows that $F' = \alpha e^{-q} X^* + pT^* \in \Omega_F^{(4,2,a)}$,

$$\tilde{A} \circ \psi(p,q) = \langle F', A\rangle = a\alpha e^{-q} + dp. \tag{17}$$

- With $\alpha = \gamma = 0, \beta \neq 0$, set

$$\psi : \quad (p,q) \in \mathbf{R}^2 \mapsto \psi(p,q) = (0, \beta e^q, 0, p) \in \Omega_F^{(4,2,b)}.$$

$\tilde{A}(F) = \langle F, A\rangle = \alpha a + \beta b + \gamma c + \delta d$. From this, $F' = \beta e^q Y^* + pT^* \in \Omega_F^{(4,2,b)}$,

$$\tilde{A} \circ \psi(p,q) = \langle F', A\rangle = b\beta e^q + dp. \tag{18}$$

- With $\alpha\beta \neq 0, \gamma = 0$, set

$$\psi : \quad (p,q) \in \mathbf{R}^2 \mapsto \psi(p,q) = (\alpha e^{-q}, \beta e^q, 0, p) \in \Omega^{(4,2,c)}.$$

$$\tilde{A} \circ \psi(p,q) = a\alpha e^{-q} + b\beta e^q + dp. \tag{19}$$

- At last, if $\gamma \neq 0$, we consider the orbit with the first coordinate $x > 0$, set

$$\psi : \quad (p,q) \in \mathbf{R}^2 \mapsto \psi(p,q) = (e^{-q}, (\alpha\beta + \gamma p - \gamma\delta)e^q, \gamma, p) \in \Omega_F^{(4,2,d)}.$$

We have

$$\begin{aligned}\tilde{A} \circ \psi(p,q) &= ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q + c\gamma + dp \\ &= (d + b\gamma e^q)p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma. \end{aligned}\tag{20}$$

The case $x < 0$ is similarly treated: set

$$\psi : \quad (p,q) \in \mathbf{R}^2 \mapsto \psi(p,q) = (-e^{-q}, -(\alpha\beta + \gamma p - \gamma\delta)e^q, \gamma, p) \in \Omega_F^{(4,2,d)},$$

$$\begin{aligned}\tilde{A} \circ \psi(p,q) &= -ae^{-q} - b(\alpha\beta + \gamma p - \gamma\delta)e^q + c\gamma + dp \\ &= (d - b\gamma e^q)p - ae^{-q} - b(\alpha\beta - \gamma\delta)e^q + c\gamma. \end{aligned}\tag{21}$$

2. We consider only the following case (the rest are similar): set

$$\psi : \quad (p,q) \in \mathbf{R}^2 \mapsto \psi(p,q) = (e^{-q}, (\alpha\beta + \gamma p - \gamma\delta)e^q, \gamma, p) \in \Omega_F^{(4,2,d)},$$

$$\tilde{A} \circ \psi(p,q) = (d + b\gamma e^q)p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma.$$

In canonical Darboux coordinates (p,q),

$$F' = e^{-q}X^* + (\alpha\beta + \gamma p - \gamma\delta)e^q Y^* + \gamma Z^* + pT^* \quad \in \Omega_F^{(4,2,d)},$$

and for $A = aX + bY + cZ + dT$, $\ B = a'X + b'Y + c'Z + d'T$, we have

$$\xi_A(f) = \{\tilde{A}, f\} = (d + b\gamma e^q)\frac{\partial f}{\partial q} - \Big[- ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q\Big]\frac{\partial f}{\partial p},$$

$$\xi_B(f) = \{\tilde{B}, f\} = (d' + b'\gamma e^q)\frac{\partial f}{\partial q} - \Big[- a'e^{-q} + b'(\alpha\beta + \gamma p - \gamma\delta)e^q\Big]\frac{\partial f}{\partial p}.$$

From this, consider two vector fields

$$\xi_A = (d + b\gamma e^q)\frac{\partial}{\partial q} - \big[- ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q\big]\frac{\partial}{\partial p},$$

$$\xi_B = (d' + b'\gamma e^q)\frac{\partial}{\partial q} - \left[- a'e^{-q} + b'(\alpha\beta + \gamma p - \gamma\delta)e^q\right]\frac{\partial}{\partial p}.$$

On one hand, (see [12]) $\omega_{F'}(\xi_A, \xi_B) = \langle F', [A, B]\rangle = \langle e^{-q}X^* + (\alpha\beta + \gamma p - \gamma\delta) e^q Y^* + \gamma Z^* + pT^*, (ad' - da')X + (db' - bd')Y + (ab' - ba')Z\rangle$.

It follows therefore that

$$\omega_{F'}(\xi_A, \xi_B) = (ad' - da')e^{-q} + (db' - bd')(\alpha\beta + \gamma p - \gamma\delta)e^q + \gamma(ab' - ba'). \quad (22)$$

On the other hand,

$$\begin{aligned} dp \wedge dq(\xi_A, \xi_B) &= dp(\xi_A)dq(\xi_B) - dp(\xi_B)dq(\xi_A) \\ &= -[-ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q](d' + b'\gamma e^q) \\ &\quad + [-a'e^{-q} + b'(\alpha\beta + \gamma p - \gamma\delta)e^q](d + b\gamma e^q) \\ &= [(ad' - da')e^{-q} + (db' - d'b)(\alpha\beta + \gamma p - \gamma\delta)e^q + \gamma(ab' - a'b)]. \end{aligned} \quad (23)$$

From (22) and (23) we conclude that in the canonical coordinates the Kirillov form is just the standard symplectic form $\omega = dp \wedge dq$. ■

Definition 3.2. *The chart ψ^{-1} on Ω_F, given in Proposition 3.1, is called the adapted chart on Ω_F.*

In the next subsection we shall see that each adapted chart carries the Moyal $\star$-product from $\mathbf{R}^2$ onto Ω_F.

3.2. Irreducible unitary representations of $G = \mathbf{R} \ltimes \mathbf{H}_3$

Proposition 3.3. *In the above mentioned canonical Darboux coordinates (p, q) on the orbit Ω_F, the Moyal $\star$-product satisfies the relation*

$$i\tilde{A} \star i\tilde{B} - i\tilde{B} \star i\tilde{A} = i\widetilde{[A, B]}, \forall A, B \in \mathfrak{g} = \mathrm{Lie}(\mathbf{R} \ltimes \mathbf{H}_3).$$

Proof. We prove the proposition for the K-orbit $\Omega_F = \Omega_F^{(4,2,d)}$, $\tilde{A} = (d + b\gamma e^q)p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma$ (the other cases are proved similarly). Consider the elements $A = aX + bY + cZ + dT, B = a'X + b'Y + c'Z + d'T \in \mathfrak{g}$. Then as said above, the corresponding Hamiltonian functions are

$$\tilde{A} = (d + b\gamma e^q)p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma,$$
$$\tilde{B} = (d' + b'\gamma e^q)p + a'e^{-q} + b'(\alpha\beta - \gamma\delta)e^q + c'\gamma.$$

It is easy then to see that

$$P^0(\tilde{A}, \tilde{B}) = \tilde{A}.\tilde{B},$$

$$
\begin{aligned}
P^1(\tilde{A},\tilde{B}) &= \{\tilde{A},\tilde{B}\} = \partial_p\tilde{A}\partial_q\tilde{B} - \partial_q\tilde{A}\partial_p\tilde{B}\\
&= (d+b\gamma e^q)[-a'e^{-q} + b'(\alpha\beta+\gamma p-\gamma\delta)e^q]\\
&\quad -(d'+b'\gamma e^q)[-ae^{-q} + b(\alpha\beta+\gamma p-\gamma\delta)e^q]\\
&= [(ad'-da')e^{-q} + (db'-d'b)(\alpha\beta+\gamma p-\gamma\delta)e^q + (ab'-ba')\gamma],\\
P^2(\tilde{A},\tilde{B}) &= \Lambda^{12}\Lambda^{12}\partial^2_{pp}\tilde{A}\partial^2_{qq}\tilde{B} + \Lambda^{12}\Lambda^{21}\partial^2_{pq}\tilde{A}\partial^2_{qp}\tilde{B} + \Lambda^{21}\Lambda^{12}\partial^2_{qp}\tilde{A}\partial^2_{pq}\tilde{B}\\
&\quad + \Lambda^{21}\Lambda^{21}\partial^2_{qq}\tilde{A}\partial^2_{pp}\tilde{B} = -2bb'\gamma^2e^{2q},\\
P^3(\tilde{A},\tilde{B}) &= \Lambda^{12}\Lambda^{12}\Lambda^{12}\partial^3_{ppp}\tilde{A}\partial^3_{qqq}\tilde{B} + \Lambda^{12}\Lambda^{12}\Lambda^{21}\partial^3_{ppq}\tilde{A}\partial^3_{qqp}\tilde{B}\\
&\quad + \Lambda^{12}\Lambda^{21}\Lambda^{12}\partial^3_{pqp}\tilde{A}\partial^3_{qpq}\tilde{B} + \Lambda^{21}\Lambda^{12}\Lambda^{12}\partial^3_{qpp}\tilde{A}\partial^3_{pqq}\tilde{B}\\
&\quad + \Lambda^{21}\Lambda^{21}\Lambda^{12}\partial^3_{qqp}\tilde{A}\partial^3_{ppq}\tilde{B} + \Lambda^{21}\Lambda^{12}\Lambda^{21}\partial^3_{qpq}\tilde{A}\partial^3_{pqp}\tilde{B}\\
&\quad + \Lambda^{12}\Lambda^{21}\Lambda^{21}\partial^3_{pqq}\tilde{A}\partial^3_{qpp}\tilde{B} + \Lambda^{21}\Lambda^{21}\Lambda^{21}\partial^3_{qqq}\tilde{A}\partial^3_{ppp}\tilde{B} = 0.
\end{aligned}
$$

By induction on k we have $P^k(\tilde{A},\tilde{B}) = 0,\ \forall k \geq 3.$

Thus,

$$
\begin{aligned}
i\tilde{A}\star i\tilde{B} - i\tilde{B}\star i\tilde{A} &= \tfrac{1}{2i}\left[P^1(i\tilde{A},i\tilde{B}) - P^1(i\tilde{B},i\tilde{A})\right]\\
&= i\left[(ad'-da')e^{-q} + (db'-d'b)(\alpha\beta+\gamma p-\gamma\delta)e^q + (ab'-a'b)\gamma\right].
\end{aligned}
$$

On the other hand, as

$$
\begin{aligned}
[A,B] &= [aX+bY+cZ+dT, a'X+b'Y+c'Z+d'T]\\
&= (ad'-da')X + (db'-d'b)Y + (ab'-a'b)Z,
\end{aligned}
$$

we obtain $i\widetilde{[A,B]} = i[(ad'-da')e^{-q} + (db'-d'b)(\alpha\beta+\gamma p-\gamma\delta)e^q + (ab'-a'b)\gamma] = i\tilde{A}\star i\tilde{B} - i\tilde{B}\star i\tilde{A}$. The proposition is hence proved. ∎

Consequently, to each adapted chart, we associate a G-covariant $\star$-product. Then there exists a representation τ of G in $\mathrm{Aut}(\mathbf{C}^\infty(\Omega,\mathbf{R})[[\nu]])$, such that (see [9], here $\nu = i/2$)

$$\tau(g)(u\star v) = \tau(g)u\star\tau(g)v.$$

Because of the relation in Proposition 3.3, we have

Corollary 3.4.

$$\ell_{[A,B]} = \ell_A\circ\ell_B - \ell_B\circ\ell_A =: [\ell_A,\ell_B]. \tag{24}$$

This implies that the correspondence

$$A \mapsto \ell_A = i\tilde{A}\star$$

is a representation of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathbf{R}\ltimes\mathbf{H}_3)$ on the space $\mathrm{C}^\infty(\Omega_F,\mathbf{R})[[\frac{i}{2}]]$ of formal power series in the parameter $\nu = i/2$ with coefficients in $\mathrm{C}^\infty(\Omega_F,\mathbf{R})$.

Lemma 3.5. *We have*

1. $\partial_p\mathcal{F}_p^{-1}(f) = i\mathcal{F}_p^{-1}(x.f)$,
2. $\mathcal{F}_p(p.v) = i\partial_x\mathcal{F}_p(v)$,

3. *For all $k \geq 2$, then* $P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) =$

$$= \begin{cases} a\alpha e^{-q}\partial^k_{p\ldots p}\mathcal{F}_p^{-1}(f) & \text{if } \tilde{A} \text{ is defined by } (17) \\ (-1)^k b\beta e^q \partial^k_{p\ldots p}\mathcal{F}_p^{-1}(f) & \text{if } \tilde{A} \text{ is defined by } (18) \\ \left[a\alpha e^{-q} + (-1)^k b\beta e^q\right]\partial^k_{p\ldots p}\mathcal{F}_p^{-1}(f) & \text{if } \tilde{A} \text{ is defined by } (19) \\ (-1)^{k-1}k.b\gamma e^q \partial^k_{qp\ldots p}\mathcal{F}_p^{-1}(f) + & \\ +[ae^{-q} + (-1)^k b(\alpha\beta + \gamma p - \gamma\delta)e^q]\partial^k_{p\ldots p}\mathcal{F}_p^{-1}(f) & \text{if } \tilde{A} \text{ is defined by } (20) \end{cases}$$

Proof. The first two formulas are well-known from the theory of Fourier transforms. We shall prove the third. Remark that $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the standard symplectic Darboux coordinates (p, q) on the orbit Ω_F, we have

- If $\tilde{A} = a\alpha e^{-q} + dp$ then

$$\begin{aligned} P^2(\tilde{A}, \mathcal{F}_p^{-1}(f)) &= \Lambda^{12}\Lambda^{12}\partial^2_{pp}\tilde{A}\partial^2_{qq}\mathcal{F}_p^{-1}(f)) + \Lambda^{12}\Lambda^{21}\partial^2_{pq}\tilde{A}\partial^2_{qp}\mathcal{F}_p^{-1}(f)) \\ &\quad + \Lambda^{21}\Lambda^{12}\partial^2_{qp}\tilde{A}\partial^2_{pq}\mathcal{F}_p^{-1}(f)) + \Lambda^{21}\Lambda^{21}\partial^2_{qq}\tilde{A}\partial^2_{pp}\mathcal{F}_p^{-1}(f)) \\ &= a\alpha e^{-q}\partial^2_{pp}\mathcal{F}_p^{-1}(f), \\ P^3(\tilde{A}, \mathcal{F}_p^{-1}(f)) &= (-1)^6 a\alpha e^{-q}\partial^3_{ppp}\mathcal{F}_p^{-1}(f) = a\alpha e^{-q}\partial^3_{ppp}\mathcal{F}_p^{-1}(f), \\ P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) &= a\alpha e^{-q}\partial^k_{p\ldots p}\mathcal{F}_p^{-1}(f), \quad \forall k \geq 2, \end{aligned}$$

- If $\tilde{A} = b\beta e^q + dp$ then

$$P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) = (-1)^k b\beta e^q \partial^k_{p\ldots p}\mathcal{F}_p^{-1}(f), \quad \forall k \geq 2.$$

- If $\tilde{A} = a\alpha e^{-q} + b\beta e^q + dp$ then

$$\begin{aligned} P^2(\tilde{A}, \mathcal{F}_p^{-1}(f)) &= \Lambda^{12}\Lambda^{12}\partial^2_{pp}\tilde{A}\partial^2_{qq}\mathcal{F}_p^{-1}(f)) + \Lambda^{12}\Lambda^{21}\partial^2_{pq}\tilde{A}\partial^2_{qp}\mathcal{F}_p^{-1}(f)) \\ &\quad + \Lambda^{21}\Lambda^{12}\partial^2_{qp}\tilde{A}\partial^2_{pq}\mathcal{F}_p^{-1}(f)) + \Lambda^{21}\Lambda^{21}\partial^2_{qq}\tilde{A}\partial^2_{pp}\mathcal{F}_p^{-1}(f)) \\ &= \left[a\alpha e^{-q} + (-1)^2 b\beta e^q\right]\partial^2_{pp}\mathcal{F}_p^{-1}(f). \\ P^3(\tilde{A}, \mathcal{F}_p^{-1}(f)) &= [a\alpha e^{-q} + (-1)^3 b\beta e^q]\partial^3_{ppp}\mathcal{F}_p^{-1}(f). \end{aligned}$$

By induction on k we have

$$P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) = \left[a\alpha e^{-q} + (-1)^k b\beta e^q\right]\partial^k_{p\ldots p}\mathcal{F}_p^{-1}(f)), \quad \forall k \geq 2.$$

- If $\tilde{A} = (d + b\gamma e^q)p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma$ then

$$\begin{aligned} P^2(\tilde{A}, \mathcal{F}_p^{-1}(f)) &= \Lambda^{12}\Lambda^{12}\partial^2_{pp}\tilde{A}\partial^2_{qq}\mathcal{F}_p^{-1}(f)) + \Lambda^{12}\Lambda^{21}\partial^2_{pq}\tilde{A}\partial^2_{qp}\mathcal{F}_p^{-1}(f)) \\ &\quad + \Lambda^{21}\Lambda^{12}\partial^2_{qp}\tilde{A}\partial^2_{pq}\mathcal{F}_p^{-1}(f)) + \Lambda^{21}\Lambda^{21}\partial^2_{qq}\tilde{A}\partial^2_{pp}\mathcal{F}_p^{-1}(f)) \\ &= (-1)2.b\gamma e^q \partial_{qp}\mathcal{F}_p^{-1}(f) \\ &\quad + \left[ae^{-q} + (-1)^2 b(\alpha\beta + \gamma p - \gamma\delta)e^q\right]\partial^2_{pp}\mathcal{F}_p^{-1}(f). \\ P^3(\tilde{A}, \mathcal{F}_p^{-1}(f)) &= (-1)^2.3b\gamma e^q \partial_{qpp}\mathcal{F}_p^{-1}(f) \\ &\quad + \left[ae^{-q} + (-1)^3 b(\alpha\beta + \gamma p - \gamma\delta)e^q\right]\partial^3_{ppp}\mathcal{F}_p^{-1}(f). \end{aligned}$$

From this we also obtain :

$$P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) = (-1)^{k-1}.k.b\gamma e^q \partial^k_{qp\dots p}\mathcal{F}_p^{-1}(f) + \big[ae^{-q} + (-1)^k b(\alpha\beta + \gamma p - \gamma\delta)e^q\big]\partial^k_{p\dots p}\mathcal{F}_p^{-1}(f) \;\; \forall k \geq 3.$$

The lemma is proved. ∎

We study now the convergence of the formal power series. In order to do this, we look at the $\star$-product of $i\tilde{A}$ as the $\star$-product of symbols and define the differential operators corresponding to $i\tilde{A}$.

Theorem 3.6. *For each $A \in \mathrm{Lie}(\mathbf{R} \ltimes \mathbf{H}_3)$ and for each compactly supported C^∞-function $f \in C_0^\infty(\mathbf{R}^2)$ we have*

$$\hat{\ell}_A(f) = \begin{cases} \big[d(\tfrac{1}{2}\partial_q - \partial_x) + ia\alpha e^{-(q-\frac{x}{2})}\big]f & \text{if } \tilde{A} \text{ is defined by (17)} \\ \big[d(\tfrac{1}{2}\partial_q - \partial_x) + ib\beta e^{(q-\frac{x}{2})}\big]f & \text{if } \tilde{A} \text{ is defined by (18)} \\ \big[d(\tfrac{1}{2}\partial_q - \partial_x) + i(a\alpha e^{-(q-\frac{x}{2})} + b\beta e^{(q-\frac{x}{2})})\big]f & \text{if } \tilde{A} \text{ is defined by (19)} \\ \big[(d + b\gamma e^{q-\frac{x}{2}})(\tfrac{1}{2}\partial_q - \partial_x)\big]f & \\ +i\big[ae^{-(q-\frac{x}{2})} + b(\alpha\beta - \gamma\delta)e^{q-\frac{x}{2}} + c\gamma\big]f & \text{if } \tilde{A} \text{ is defined by (20)} \\ \big[(d - b\gamma e^{q-\frac{x}{2}})(\tfrac{1}{2}\partial_q - \partial_x)\big]f & \\ +i\big[-ae^{-(q-\frac{x}{2})} - b(\alpha\beta - \gamma\delta)e^{q-\frac{x}{2}} + c\gamma\big]f & \text{if } \tilde{A} \text{ is defined by (21).} \end{cases}$$

Proof. Applying Lemma 3.5 we obtain

1. If $\tilde{A} = a\alpha e^{-q} + dp$ then

$$\hat{\ell}_A(f) = \mathcal{F}_p(i\tilde{A} \star \mathcal{F}_p^{-1}(f)) = i\mathcal{F}_p\Big(\sum_{r\geq 0}\Big(\frac{1}{2i}\Big)^r\frac{1}{r!}P^r(\tilde{A}, \mathcal{F}_p^{-1}(f))\Big) =$$

$$= i\mathcal{F}_p\Big\{(a\alpha e^{-q} + dp)\mathcal{F}_p^{-1}(f) + \frac{1}{1!}\frac{1}{2i}[d\partial_q\mathcal{F}_p^{-1}(f) + a\alpha e^{-q}\partial_p\mathcal{F}_p^{-1}(f)] + \frac{1}{2!}\Big(\frac{1}{2i}\Big)^2.a\alpha e^{-q}\partial_p p^2\mathcal{F}_p^{-1}(f) + \dots + \frac{1}{r!}(\frac{1}{2i})^r a\alpha e^{-q}\partial^r_{p\dots p}\mathcal{F}_p^{-1}(f) + \dots\Big\}$$

$$= i\Big\{a\alpha e^{-q}f + d\mathcal{F}_p(p.\mathcal{F}_p^{-1}(f)) + \frac{1}{1!}\frac{1}{2i}[d\partial_q f + a\alpha e^{-q}\mathcal{F}_p(\partial_p\mathcal{F}_p^{-1}(f))] + \frac{1}{2!}\Big(\frac{1}{2i}\Big)^2.a\alpha e^{-q}\mathcal{F}_p(\partial^2_{pp}\mathcal{F}_p^{-1}(f)) + +\frac{1}{3!}(\frac{1}{2i})^3.a\alpha e^{-q}\mathcal{F}_p(\partial^3_{ppp}\mathcal{F}_p^{-1}(f)) + \dots + \frac{1}{r!}\Big(\frac{1}{2i}\Big)^r.a\alpha e^{-q}\mathcal{F}_p(\partial^r_{p\dots p}\mathcal{F}_p^{-1}(f)) + \dots\Big\}$$

$$= d\Big(\frac{1}{2}\partial_q - \partial_x\Big)f + ia\alpha e^{-q}\Big[1 + \frac{x}{2} + \frac{1}{2!}\Big(\frac{x}{2}\Big)^2 + \dots + \frac{1}{r!}\Big(\frac{x}{2}\Big)^r + \dots\Big]f$$

$$= d\Big(\frac{1}{2}\partial_q - \partial_x\Big)f + ia\alpha e^{-q}e^{\frac{x}{2}}f$$

$$= d\Big(\frac{1}{2}\partial_q - \partial_x\Big)f + ia\alpha e^{-(q-\frac{x}{2})}f.$$

2. If $\tilde{A} = b\beta e^q + dp$ then $\hat{\ell}_A(f) = d(\frac{1}{2}\partial_q - \partial_x)f + ib\beta e^{q-\frac{x}{2}}f$.

3. If $\tilde{A} = a\alpha e^{-q} + b\beta e^{q} + dp$ then

$$\begin{aligned}
\hat{\ell}_A(f) &= i\mathcal{F}_p\Big\{(a\alpha e^{-q} + b\beta e^{q} + dp)\mathcal{F}_p^{-1}(f) \\
&\quad + \frac{1}{2i}\big[d\partial_q\mathcal{F}_p^{-1}(f) - (-a\alpha e^{-q} + b\beta e^{q})\partial_p\mathcal{F}_p^{-1}(f)\big] \\
&\quad + \frac{1}{2!}\Big(\frac{1}{2i}\Big)^2\big[a\alpha e^{-q} + (-1)^2 b\beta e^{q}\big]\partial^2_{pp}\mathcal{F}_p^{-1}(f) + \dots \\
&\quad + \frac{1}{r!}\Big(\frac{1}{2i}\Big)^r\big[a\alpha e^{-q} + (-1)^r b\beta e^{q}\big]\partial^r_{p\dots p}\mathcal{F}_p^{-1}(f) + \dots\Big\} \\
&= ia\alpha e^{-q}.f + id\mathcal{F}_p\big(p.\mathcal{F}_p^{-1}(f)\big) + ib\beta e^{q} f + \frac{1}{2}d\partial_q f \\
&\quad + \frac{1}{2}a\alpha e^{-q}\mathcal{F}_p\big(\partial_p\mathcal{F}_p^{-1}(f)\big) - \frac{1}{2}b\beta e^{q}\mathcal{F}_p\big(\partial_p\mathcal{F}_p^{-1}(f)\big) \\
&\quad + \dots i\frac{1}{r!}\Big(\frac{1}{2i}\Big)^r a\alpha e^{-q}\mathcal{F}_p\big(\partial^r_{p\dots p}\mathcal{F}_p^{-1}(f)\big) \\
&\quad + i\frac{1}{r!}\Big(\frac{-1}{2i}\Big)^r b\beta e^{q}\mathcal{F}_p\big(\partial^r_{p\dots p}\mathcal{F}_p^{-1}(f)\big) + \dots \\
&= d\Big(\frac{1}{2}\partial_q - \partial_x\Big)f + ia\alpha e^{-q}\Big[1 + \frac{x}{2} + \dots + \frac{1}{r!}\Big(\frac{x}{2}\Big)^r + \dots\Big]f \\
&\quad + ib\beta e^{q}\Big[1 + \Big(\frac{-x}{2}\Big) + \dots + \frac{1}{r!}\Big(\frac{-x}{2}\Big)^r + \dots\Big]f \\
&= d\Big(\frac{1}{2}\partial_q - \partial_x\Big)f + i\Big[a\alpha e^{-\left(q-\frac{x}{2}\right)} + b\beta e^{q-\frac{x}{2}}\Big]f.
\end{aligned}$$

4. For each $\tilde{A}$ is as in (20), remark that

$$\begin{aligned}
P^0(\tilde{A}, \mathcal{F}_p^{-1}(f)) &= \tilde{A}.\mathcal{F}_p^{-1}(f); P^1(\tilde{A}, \mathcal{F}_p^{-1}(f)) = \{\tilde{A}, \mathcal{F}_p^{-1}(f)\} \\
&= (d + b\gamma e^{q})\partial_q\mathcal{F}_p^{-1}(f) - [-ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^{q}]\partial_p\mathcal{F}_p^{-1}(f).
\end{aligned}$$

Applying Lemma 3.5 we obtain

$$\begin{aligned}
\hat{\ell}_A(f) = i\Big\{&\mathcal{F}_p\big([dp + ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^{q} + c\gamma]\mathcal{F}_p^{-1}(f)\big) \\
&+ \frac{1}{2i}\frac{1}{1!}\mathcal{F}_p\Big(\big[d + b\gamma e^{q}\big]\partial_q\mathcal{F}_p^{-1}(f) \\
&- \big[-ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^{q}\big]\partial_p\mathcal{F}_p^{-1}(f)\Big) \\
&+ \Big(\frac{1}{2i}\Big)^2\frac{1}{2!}\mathcal{F}_p\Big(-2b\gamma e^{q}\partial^2_{pq}\mathcal{F}_p^{-1}(f) \\
&+ \big[ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^{q}\big]\partial^2_{pp}(\mathcal{F}_p^{-1}(f)\Big) \\
&+ \dots + \Big(\frac{1}{2i}\Big)^r\frac{1}{r!}\mathcal{F}_p\Big((-1)^{r-1} rb\gamma e^{q}\partial^r_{p\dots pq}\mathcal{F}_p^{-1}(f) \\
&+ (-1)^r[(-1)^r ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^{q}]\partial^r_{p\dots p}\mathcal{F}_p^{-1}(f)\Big) + \dots\Big\}
\end{aligned}$$

$$= i\Big\{ae^{-q}f + b(\alpha\beta - \gamma\delta)e^q f + d\mathcal{F}_p\big(p\mathcal{F}_p^{-1}(f)\big)$$

$$+ b\gamma e^q \mathcal{F}_p\big(p\mathcal{F}_p^{-1}(f)\big) + \frac{1}{2i}\frac{1}{1!}\big(d + b\gamma e^q\big)\partial_q f$$

$$- \frac{1}{2i}\big[-ae^{-q}ixf + b(\alpha\beta - \gamma\delta)e^q ixf + b\gamma e^q\mathcal{F}_p\big(p\mathcal{F}_p^{-1}(xf)\big)\big]$$

$$+ \Big(\frac{1}{2i}\Big)^2\frac{1}{2!}(-2b\gamma e^q)\mathcal{F}_p\big(\partial^2_{pq}\mathcal{F}_p^{-1}(f)\big)$$

$$+ \Big(\frac{1}{2i}\Big)^2\frac{1}{2!}\big[ae^{-q}(ix)^2 f + b(\alpha\beta - \gamma\delta)e^q(ix)^2 f + b\gamma e^q\mathcal{F}_p\big(pi^2\mathcal{F}_p^{-1}(x^2f)\big)\big]$$

$$+ \cdots + \Big(\frac{1}{2i}\Big)^r\frac{1}{r!}(-1)^{r-1}rb\gamma e^q\partial^r_{p\ldots pq}\mathcal{F}_p^{-1}(f) + \Big(\frac{1}{2i}\Big)^r\frac{1}{r!}\big[ae^{-q}(ix)^r f$$

$$+ (-1)^r b(\alpha\beta - \gamma\delta)e^q(ix)^r f + b\gamma e^q\mathcal{F}_p\big(p(ix)^r\mathcal{F}_p^{-1}(f)\big)\big] + \ldots\Big\}$$

$$= i\Big[ae^{-q}\Big(1 + \frac{1}{2!}\frac{x}{2} + \cdots + \frac{1}{r!}\Big(\frac{x}{2}\Big)^r\ldots\Big)f\Big]$$

$$+ i\Big[b(\alpha\beta - \gamma\delta)e^q\Big(1 - \frac{1}{2!}\frac{x}{2} + \cdots + (-1)^r\frac{1}{r!}\Big(\frac{x}{2}\Big)^r\ldots\Big)f\Big]$$

$$+ ic\gamma f + i^2 d\partial_x f + \frac{1}{2}d\partial_q f + ib\gamma e^q \times$$

$$\Big[i\partial_x f - \frac{1}{2i}\mathcal{F}_p\big(pi\mathcal{F}_p^{-1}(xf)\big) + \cdots + \Big(\frac{1}{2i}\Big)^r\frac{1}{r!}(-1)^r\mathcal{F}_p\big(pi^r\mathcal{F}_p^{-1}(x^r f)\big) + \ldots\Big]$$

$$= d\Big(\frac{1}{2}\partial_q - \partial_x\Big)f + \Big[iae^{-\left(q-\frac{x}{2}\right)} + ib(\alpha\beta - \gamma\delta)e^{q-\frac{x}{2}}\Big]f$$

$$+ ic\gamma f + \frac{1}{2}e^{-\frac{x}{2}}b\gamma e^q\partial_q f - b\gamma e^q e^{-\frac{x}{2}}\partial_x f$$

$$= \Big(d + b\gamma e^{q-\frac{x}{2}}\Big)\Big(\frac{1}{2}\partial_q - \partial_x\Big)f + \Big[iae^{-\left(q-\frac{x}{2}\right)} + ib(\alpha\beta - \gamma\delta)e^{q-\frac{x}{2}} + ic\gamma\Big]f.$$

5. At last, if $\tilde{A}$ is defined by (21) then

$$\hat{\ell}_A(f) = \Big(d - b\gamma e^{q-\frac{x}{2}}\Big)\Big(\frac{1}{2}\partial_q - \partial_x\Big)f + \big[-iae^{-(q-\frac{x}{2})} - ib(\alpha\beta - \gamma\delta)e^{q-\frac{x}{2}} + ic\gamma\big]f.$$

The theorem is proved. ■

Remark. Setting new variables $s = q - \frac{x}{2}$, $t = q + \frac{x}{2}$, we have

$$\hat{\ell}_A(f) = \begin{cases} (d\partial_s + ia\alpha e^{-s})f|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (17)} \\ (d\partial_s + ib\beta e^{s})f|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (18)} \\ (d\partial_s + i[a\alpha e^{-s} + b\beta e^s])f|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (19)} \\ \Big((d + b\gamma e^s)\partial_s & \\ \quad + i[ae^{-s} + b(\alpha\beta - \gamma\delta)e^s + c\gamma]\Big)f|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (20)} \\ \Big((d - b\gamma e^s)\partial_s & \\ \quad + i[-ae^{-s} - b(\alpha\beta - \gamma\delta)e^s + c\gamma]\Big)f|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (21)} \end{cases}$$

Theorem 3.7.

1. *With above notations we obtain the operators*

$$\hat{\ell}_A = \begin{cases} \hat{\ell}_A^{(a)} = (d\partial_s + ia\alpha e^{-s})|_{(s,t)}, \\ \hat{\ell}_A^{(b)} = (d\partial_s + ib\beta e^{s})|_{(s,t)}, \\ \hat{\ell}_A^{(c)} = (d\partial_s + i[a\alpha e^{-s} + b\beta e^{s}])|_{(s,t)}, \\ \hat{\ell}_A^{(d)} = \Big((d + b\gamma e^{s})\partial_s + i[ae^{-s} + b(\alpha\beta - \gamma\delta)e^{s} + c\gamma]\Big)|_{(s,t)}, \\ \hat{\ell}_A^{(d')} = \Big((d - b\gamma e^{s})\partial_s + i[-ae^{-s} - b(\alpha\beta - \gamma\delta)e^{s} + c\gamma]\Big)|_{(s,t)}, \end{cases}$$

which provide the representations of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathbf{R} \ltimes \mathbf{H}_3)$.

2. *For all* $A, B \in \mathfrak{g}$,
$$\hat{\ell}_A \circ \hat{\ell}_B - \hat{\ell}_B \circ \hat{\ell}_A = \hat{\ell}_{[A,B]}.$$

We call

- $[\Omega_F^{(4,2,a)}, \hat{\ell}_A^{(a)}]$; $[\Omega_F^{(4,2,b)}, \hat{\ell}_A^{(b)}]$ the quantum half-planes,
- $[\Omega_F^{(4,2,c)}, \hat{\ell}_A^{(c)}]$ the quantum hyperbolic cylinder,
- $[\Omega_F^{(d)}; \hat{\ell}_A^{(d)}, \hat{\ell}_A^{(d')}]$ the quantum hyperbolic paraboloid,

with respect to the co-adjoint action of real diamond group G.

As $G = \mathbf{R} \ltimes \mathbf{H}_3$ is an exponential, connected, simply connected Lie group, we obtain all irreducible unitary representations T of G (see Theorems A, B) defined by the following formula

$$T(\exp A) := \exp(\hat{\ell}_A), \quad \forall A \in \mathfrak{g}.$$

More precisely,

$$\exp(\hat{\ell}_A) = \begin{cases} \exp(d\partial_s + ia\alpha e^{-s})|_{(s,t)} & \text{if } \tilde{A} \text{ defined by (17)} \\ \exp(d\partial_s + ib\beta e^{s})|_{(s,t)} & \text{if } \tilde{A} \text{ defined by (18)} \\ \exp(d\partial_s + i[a\alpha e^{-s} + b\beta e^{s}])|_{(s,t)} & \text{if } \tilde{A} \text{ defined by (19)} \\ \exp((d + b\gamma e^{s})\partial_s & \\ \quad + i[ae^{-s} + b(\alpha\beta - \gamma\delta)e^{s} + c\gamma])|_{(s,t)} & \text{if } \tilde{A} \text{ defined by (20)} \\ \exp((d - b\gamma e^{s})\partial_s & \\ \quad + i[-ae^{-s} - b(\alpha\beta - \gamma\delta)e^{s} + c\gamma])|_{(s,t)} & \text{if } \tilde{A} \text{ defined by (21)} \end{cases}$$

This means that we find out all representations $T(\exp A)$ of the real diamond Lie group $\mathbf{R} \ltimes \mathbf{H}_3$, those could be implicitly obtained using the orbit method. However, our result here gives more precise analytical formulas than those obtained by the orbit method.

4. Quantum Co-Adjoint Orbits of the Exponential MD_4-Groups

Throughout this section, we denote by G one of the following groups: (with $\varphi \neq \pi/2$)

$G_{4,1,1}$; $G_{4,1,2}$; $G_{4,2,1(\lambda)}$; $G_{4,2,2}$; $G_{4,2,3(\varphi)}$; $G_{4,3,1(\lambda_1,\lambda_2)}$; $G_{4,3,2(\lambda)}$; $G_{4,3,3}$; $G_{4,3,4(\lambda,\varphi)}$. These are exponential groups (see [5]).

4.1. Hamiltonian functions in canonical coordinates of Ω_F

Denote by Ω_F the co-adjoint orbit at $F \in \mathfrak{g}^*, \mathfrak{g} = \text{LieG}$, $A = aX + bY + cZ + dT \in \mathfrak{g}$.

Proposition 4.1. *Each nontrivial orbit $\Omega_F \subset \mathfrak{g}^*$ of the co-adjoint representation of G admits a global diffeomorphism ψ*

$$\psi: \quad (p,q) \in \mathbf{R}^2 \longmapsto \psi(p,q) \in \Omega_F, \quad \textit{such that}$$

(i) *Hamiltonian function $\widetilde{A} = \langle F', A \rangle$, $(F' \in \Omega_F)$ is of the form*

$$\widetilde{A} \circ \psi(p,q) = \Phi(\alpha,\beta,\gamma,\delta,q).p + \Psi(\alpha,\beta,\gamma,\delta,q),$$

where $\Phi(\alpha,\beta,\gamma,\delta,q), \Psi(\alpha,\beta,\gamma,\delta,q)$ are C^∞-functions on $\mathbf{R}$.

(ii) *The Kirillov form ω is*

$$\omega = dp \wedge dq. \tag{25}$$

Proof.

(i) The diffeomorphism ψ will be chosen case by case.

1. Case $G_{4,1,1}$ and $\Omega_F = \Omega_F^{(1,1)}$. We chose the diffeomorphism:

$$\psi: \quad \mathbf{R}^2 \longrightarrow \Omega_F^{(1,1)}; \ (p,q) \longmapsto (q,\beta,\gamma,p). \quad \text{Then,}$$

$$\widetilde{A} \circ \psi(p,q) = d.p + (aq + b\beta + c\gamma). \tag{26}$$

2. Case $G_{4,1,2}$ and $\Omega_F = \Omega_F^{(1,2)}$. Then we take

$$\psi: \quad \mathbf{R}^2 \longrightarrow \Omega_F^{(1,2)}; \ (p,q) \longmapsto (\alpha,\beta,\gamma e^q,p),$$

$$\widetilde{A} \circ \psi(p,q) = d.p + (c\gamma e^q + a\alpha + b\beta). \tag{27}$$

3. Case $G_{4,2,1(\lambda)}$ and $\Omega_F = \Omega_F^{(2,1)}$.

$$\psi: \quad \mathbf{R}^2 \longrightarrow \Omega_F^{(2,1)}; \ (p,q) \longmapsto (\alpha,\beta e^{q\lambda},\gamma e^q,p),$$

$$\widetilde{A} \circ \psi(p,q) = dp + (c\gamma e^q + a\alpha + b\beta e^{q\lambda}). \tag{28}$$

4. Case $G_{4,2,2}$ and $\Omega_F = \Omega_F^{(2,2)}$.

$$\psi: \quad \mathbf{R}^2 \longrightarrow \Omega_F^{(2,2)}; \ (p,q) \longmapsto (\alpha,\beta e^q,\beta q e^q + \gamma e^q,p),$$

$$\widetilde{A} \circ \psi(p,q) = d.p + c\beta q e^q + (b\beta + c\gamma)e^q + a\alpha. \tag{29}$$

5. Case $G_{4,2,3(\varphi)}$, $\varphi \neq \pi/2$ and $\Omega_F = \Omega_F^{(2,3)}$

$$\psi: \quad \mathbf{R}^2 \longrightarrow \Omega_F^{(2,3)}; \ (p,q) \longmapsto (\alpha,(\beta + i\gamma)e^{qe^{i\varphi}},p),$$

$$\widetilde{A} \circ \psi(p,q) = d.p + (b + ic)(\beta + i\gamma)e^{qe^{i\varphi}} + a\alpha. \tag{30}$$

6. Case $G_{4,3,1(\lambda_1,\lambda_2)}$ and $\Omega_F = \Omega_F^{(3,1)}$.

$$\psi: \quad \mathbf{R}^2 \longrightarrow \Omega_F^{(3,1)}; \quad (p,q) \longmapsto (\alpha e^{q\lambda_1}, \beta e^{q\lambda_2}, \gamma e^q, p),$$

$$\widetilde{A} \circ \psi(p,q) = d.p + a\alpha e^{q\lambda_1} + b\beta e^{q\lambda_2} + c\gamma e^q. \tag{31}$$

7. Case $G_{4,3,2(\lambda)}$ and $\Omega_F = \Omega_F^{(3,2)}$.

$$\psi: \quad \mathbf{R}^2 \longrightarrow \Omega_F^{(3,2)}; \quad (p,q) \longmapsto (\alpha e^{q\lambda}, \alpha q e^{q\lambda} + \beta e^{q\lambda}, \gamma e^q, p),$$

$$\widetilde{A} \circ \psi(p,q) = d.p + (a\alpha + bq\alpha + b\beta)e^{q\lambda} + c\gamma e^q. \tag{32}$$

8. Case $G_{4,3,3}$ and $\Omega_F = \Omega_F^{(3,3)}$.

$$\psi: \quad \mathbf{R}^2 \longrightarrow \Omega_F; \ (p,q) \longmapsto (\alpha e^q, \alpha q e^q + \beta e^q, \tfrac{1}{2}\alpha q^2 e^q + \beta q e^q + \gamma e^q, p),$$

$$\widetilde{A} \circ \psi(p,q) = d.p + (a\alpha + b\alpha + b\beta + \frac{1}{2}c\alpha q^2 + c\beta q + c\gamma)e^q. \tag{33}$$

9. Case $G_{4,3,4(\lambda,\varphi)}, \varphi \neq \frac{\pi}{2}$ and $\Omega_F^{(3,4)}$

$$\psi: \quad \mathbf{R}^2 \longrightarrow \Omega_F^{(3,4)}; \quad (p,q) \longmapsto ((\alpha + i\beta)e^{qe^{i\varphi}}, \gamma e^{q\lambda}, p),$$

$$\widetilde{A} \circ \psi(p,q) = d.p + (a+ib)(\alpha + i\beta)e^{qe^{i\varphi}} + c\gamma e^{q\lambda}. \tag{34}$$

(ii) We prove for the case $G_{4,2,3(\varphi)}, \varphi \neq \frac{\pi}{2}$, that the Kirillov form on Ω_F is $dp \wedge dq$.

From the Hamiltonian function $\tilde{A} \circ \psi(p,q)$ we have

$$\xi_A(f) = \{\tilde{A}, f\} = d\frac{\partial f}{\partial q} - (b+ic)(\beta + i\gamma)e^{i\varphi}e^{qe^{i\varphi}}\frac{\partial f}{\partial p},$$

with $A = aX + bY + cZ + dT \in \mathfrak{g}_{4,2,3(\varphi)}$

$$\xi_B(f) = \{\tilde{B}, f\} = d'\frac{\partial f}{\partial q} - (b' + ic')(\beta + i\gamma)e^{i\varphi}e^{qe^{i\varphi}}\frac{\partial f}{\partial p},$$

with $B = a'X + b'Y + c'Z + d'T \in \mathfrak{g}_{4,2,3(\varphi)}$. Thus,

$$\begin{aligned} dp \wedge dq(\xi_A, \xi_B) &= dp(\xi_A)dq(\xi_B) - dp(\xi_B)dq(\xi_A) \\ &= [(db' - d'b) + i(dc' - d'c)](\beta + i\gamma)e^{i\varphi}e^{qe^{i\varphi}} \end{aligned}$$

On the other hand, (see [12])

$$\omega_{F'}(\xi_A, \xi_B) = \langle F', [A,B] \rangle = [(db' - d'b) + i(dc' - d'c)](\beta + i\gamma)e^{i\varphi}e^{qe^{i\varphi}}$$

This implies (25).

The other cases can be proved similarly.

The proposition is hence completely proved. ■

4.2. Computation of operators $\hat{\ell}_A$

Since the Hamiltonian function is of the form $\Phi(\alpha,\beta,\gamma,\delta,q).p + \Psi(\alpha,\beta,\gamma,\delta,q)$, one can prove that

$$P^r(\tilde{A}, \tilde{B}) = 0 \ \forall r \geq 3, \ \forall A, B \in \mathfrak{g}.$$

From this we have the following proposition.

Proposition 4.2. *With $A, B \in \mathfrak{g}$, the Moyal $\star$-product satisfies the relation*

$$i\tilde{A} \star i\tilde{B} - i\tilde{B} \star i\tilde{A} = i\widetilde{[A,B]}. \tag{35}$$

This implies that the correspondence

$$A \mapsto \ell_A = i\tilde{A}\star.$$

is a representation of the Lie algebra $\mathfrak{g}$ on the space $C^\infty(\Omega_F)[[\frac{i}{2}]]$ of formal power series.

Putting $\Phi(q) = \Phi(\alpha,\beta,\gamma,\delta,q); \Psi(q) = \Psi(\alpha,\beta,\gamma,\delta,q)$, we have

Lemma 4.3.

$$P^r(\tilde{A}, \mathcal{F}_p^{-1}(f)) = (-1)^r \partial_q^r(\Psi)\partial_p^r \mathcal{F}_p^{-1}(f) \ \forall f \in \Lambda^2(\mathbf{R}^2, dpdq/2\pi) \ \forall r \geq 2.$$

Proof: The proof is straightforward. ■

Theorem 4.4. *For each $A \in \mathfrak{g}$ and for each compactly supported C^∞-function $f \in C_0^\infty(\mathbf{R}^2)$, we have*

$$\hat{\ell}_A(f) = \Phi(q - \tfrac{x}{2})(\tfrac{1}{2}\partial_q - \partial_x)f + i\Psi(q - \tfrac{x}{2})f.$$

Setting new variables $s = q - \frac{x}{2}, t = q + \frac{x}{2}$, then

$$\hat{\ell}_A(f) = \Phi(s)\frac{\partial f}{\partial s} + i\Psi(s)f|_{(s,t)}, \ \text{ i.e. } \hat{\ell}_A = [\Phi(s)\frac{\partial}{\partial s} + i\Psi(s)]|_{(s,t)}. \tag{36}$$

Proof.

$$P^0(\tilde{A}, \mathcal{F}_p^{-1}(f)) = [\Phi(q).p + \Psi(q)]\mathcal{F}_p^{-1}(f).$$

$$P^1(\tilde{A}, \mathcal{F}_p^{-1}(f)) = \Phi(q)\partial_q \mathcal{F}_p^{-1}(f) - [p\partial_q\Phi(q) + \partial_q\Psi(q)]\mathcal{F}_p^{-1}(f).$$

$$P^r(\tilde{A}, \mathcal{F}_p^{-1}(f)) = (-1)^r \partial_q^r \Psi \partial_p^r \mathcal{F}_p^{-1}(f) \ \forall r \geq 2.$$

From this and Lemmas 3.5 and 4.3, we have:

$$\begin{aligned}
\hat{\ell}_A(f) &= \mathcal{F}_p \circ \ell_A \circ \mathcal{F}_p^{-1}(f) = i\mathcal{F}_p(\tilde{A} \star \mathcal{F}_p^{-1}(f)) \\
&= i\mathcal{F}_p\Big(\sum_{r\geq 0}\Big(\frac{1}{2i}\Big)^r \frac{1}{r!} P^r(\tilde{A}, \mathcal{F}_p^{-1}(f))\Big) \\
&= i\mathcal{F}_p\Big\{[\Phi(q).p + \Psi(q)]\mathcal{F}_p^{-1}(f) \\
&\quad + \frac{1}{1!}\frac{1}{2i}(\Phi(q)\partial_q \mathcal{F}_p^{-1}(f) - [p\partial_q\Phi(q) + \partial_q\Psi(q)]\mathcal{F}_p^{-1}(f)) \\
&\quad + \dots + \frac{1}{r!}\Big(\frac{1}{2i}\Big)^r((-1)^r \partial_q^r \Psi \partial_p^r \mathcal{F}_p^{-1}(f)) + \dots\Big\} \\
&= \Phi\Big(q - \frac{x}{2}\Big)\Big(\frac{1}{2}\partial_q - \partial_x\Big)f + i\Psi\Big(q - \frac{x}{2}\Big)f.
\end{aligned}$$

The theorem is proved. ■

As a direct consequence of the definition of $\hat{\ell}_A$, we have

Corollary 4.5. *For all* $A, B \in \mathfrak{g}$,

$$\hat{\ell}_A \circ \hat{\ell}_B - \hat{\ell}_B \circ \hat{\ell}_A = \hat{\ell}_{[A,B]}.$$

From Theorems 3.7 and 4.4 we obtain the quantum half planes, the quantum planes, the quantum hyperbolic cylinders, quantum hyperbolic paraboloids ... of the corresponding groups. At the same time, we have all irreducible unitary representations of these groups (see Theorems A, B)

$$T(\exp A) = \exp(\hat{\ell}_A) = \exp\Big([\Phi(s)\frac{\partial}{\partial s} + i\Psi(s)]|_{(s,t)}\Big).$$

Thus, we obtain the full list of irreducible unitary representations of exponential MD_4-groups.

5. The Case of Groups $G_{4,2,3(\pi/2)}$, $G_{4,2,4}$, $G_{4,3,4(\pi/2)}$, $G_{4,4,1}$

5.1. The local diffeomorphisms

For the group of affine transformations of the complex straight line $G_{4,2,4} = \widetilde{\mathrm{Aff}(\mathbf{C})}$ in [7], we replaced the global diffeomorphism ψ by a local diffeomorphism and obtained

Proposition 5.1. [7] *Fixing the local diffeomorphism* $\psi_k (k \in \mathbf{Z})$

$$\psi_k : \mathbf{C} \times \mathbf{H}_k \longrightarrow \mathbf{C} \times \mathbf{C}_k$$
$$(z, w) \longmapsto (z, e^w),$$

where $\mathbf{H}_k = \{w = q_1 + iq_2 \in \mathbf{C} | -\infty < q_1 < +\infty; 2k\pi < q_2 < 2k\pi + 2\pi\}$, *we have*

1. *For any element* $A \in \mathrm{aff}(\mathbf{C})$, *the corresponding Hamiltonian function* $\widetilde{A}$ *in local coordinates* (z, w) *of the orbit* Ω_F *is of the form*

$$\widetilde{A} \circ \psi_k(z, w) = \frac{1}{2}[\alpha z + \beta e^w + \overline{\alpha z} + \overline{\beta} e^{\overline{w}}]$$

2. *In local coordinates* (z, w) *of the orbit* Ω_F, *the Kirillov form* ω *is of the form*

$$\omega = \frac{1}{2}[dz \wedge dw + d\overline{z} \wedge d\overline{w}].$$

Analogously, for the groups $G_{4,2,3(\varphi)}, G_{4,3,4(\varphi)}$ with $\varphi = \pi/2$, we also replace the global diffeomorphism ψ by local diffeomorphisms $\psi_k (k \in \mathbf{Z})$.

Let us denote $\mathbf{I}_k = (2k\pi, 2\pi + 2k\pi), k \in \mathbf{Z}$.

- For $G = G_{4,2,3(\frac{\pi}{2})}$; $\Omega_F = \{(\alpha, (\beta + i\gamma)e^{is}, t) | s, t \in \mathbf{R}\}$,

$$\psi_k : \mathbf{R} \times \mathbf{I}_k \longmapsto \Omega_F$$
$$(p, q) \longmapsto (\alpha, (\beta + i\gamma)e^{iq}, p).$$

Then the corresponding Hamiltonian function is

$$\tilde{A} \circ \psi_k(p, q) = dp + (b + ic)(\beta + i\gamma)e^{iq} + a\alpha.$$

- For $G = G_{4,3,4(\pi/2)}$; $\Omega_F = \{((\alpha + i\beta)e^{is}, \gamma e^{\lambda s}, t) | s, t \in \mathbf{R}\}$,

$$\psi_k : \mathbf{R} \times \mathbf{I}_k \longmapsto \Omega_F$$
$$(p,q) \longmapsto ((\alpha + i\beta)e^{iq}, \gamma e^{q\lambda}, p).$$

We have $\tilde{A} \circ \psi_k(p,q) = dp + (a+ib)(\alpha+i\beta)e^{iq} + c\gamma e^{q\lambda}$.

At last, for the Lie group $G_{4,4,1} = \mathbf{R} \ltimes_j \mathbf{H}_3$, which is not exponential group, we have

Proposition 5.2. *Each non-trivial orbit Ω_F (in $\mathfrak{g}^*$) of co-adjoint representation of $G_{4,4,1}$ admits local charts $(\mathbf{R} \times \mathbf{I}_k, \psi_k^{-1})$ or $(\mathbf{R}_\pm \times \mathbf{I}_k, \psi_k^{-1})$ such that*

1. *If $\Omega_F = \Omega_F^{(4,1,a)}$ and $A \in \mathfrak{g}_{4,4,1}$, then*

$$\begin{aligned}\tilde{A} \circ \psi_k(p,q) &= dp + \frac{1}{2}\big[a(\alpha+i\beta) + b(\beta - i\alpha)\big]e^{iq} \\ &\quad + \frac{1}{2}\big[a(\alpha - i\beta) + b(\beta + i\alpha)\big]e^{-iq} \\ &= dp + (a\alpha + b\beta)\cos q + (b\alpha - a\beta)\sin q \end{aligned} \tag{37}$$

and the Kirillov form then is $\omega = dp \wedge dq$.

2. *If $\Omega_F = \Omega_F^{(4,1,b)}$ and $A \in \mathfrak{g}_{4,4,1}$, then*

$$\begin{aligned}\tilde{A} \circ \psi_k(p,q) &= \frac{d}{2\gamma}p^2 + \left[\frac{a}{2}(e^{iq} + e^{-iq}) + \frac{b}{2i}(e^{iq} - e^{-iq})\right]p \\ &\quad + c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma} \\ &= \frac{d}{2\gamma}p^2 + (a\cos q + b\sin q)p + c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma}\end{aligned} \tag{38}$$

and we have the Kirillov form to be $\omega = (\gamma/p)dp \wedge dq$.

Proof. 1. We consider the following diffeomorphism

$$\psi_k : \quad \mathbf{R} \times \mathbf{I}_k \longrightarrow \Omega_F$$
$$(p,q) \mapsto (\frac{1}{2}(\alpha+i\beta)e^{iq} + \frac{1}{2}(\alpha - i\beta)e^{-iq}; \frac{1}{2}(\beta - i\alpha)e^{iq} + \frac{1}{2}(\beta+i\alpha)e^{-iq}; 0; p).$$

With each $F' \in \Omega_F$,

$$F' = [\frac{1}{2}(\alpha+i\beta)e^{iq} + \frac{1}{2}(\alpha-i\beta)e^{-iq}]X^* + [\frac{1}{2}(\beta - i\alpha)e^{iq} + \frac{1}{2}(\beta+i\alpha)e^{-iq}]Y^* + pT^*$$

and $A = aX + bY + cZ + dT \in \mathfrak{g}_{4,4,1}$, we have

$$\begin{aligned}\tilde{A}(F') = \langle F', A\rangle &= \frac{a}{2}\big[(\alpha+i\beta)e^{iq} + (\alpha - i\beta)e^{-iq}\big] \\ &\quad + \frac{b}{2}\big[(\beta - i\alpha)e^{iq} + (\beta+i\alpha)e^{-iq}\big] + dp.\end{aligned}$$

It follows that

$$\xi_A(f) = d\frac{\partial f}{\partial q} - \left\{\frac{i}{2}[a(\alpha+i\beta) + b(\beta - i\alpha)]e^{iq} + \frac{i}{2}[a(\alpha - i\beta) + b(\beta+i\alpha)]e^{-iq}\right\}\frac{\partial f}{\partial p}.$$

By analogy,

$$\xi_B(f) = d'\frac{\partial f}{\partial q} - \left\{\frac{i}{2}[a'(\alpha+i\beta)+b'(\beta-i\alpha)]e^{iq}+\frac{i}{2}[a'(\alpha-i\beta)+b'(\beta+i\alpha)]e^{-iq}\right\}\frac{\partial f}{\partial p}.$$

Consider two vector fields

$$\xi_A = d\frac{\partial}{\partial q} - \left\{\frac{i}{2}[a(\alpha+i\beta)+b(\beta-i\alpha)]e^{iq}+\frac{i}{2}[a(\alpha-i\beta)+b(\beta+i\alpha)]e^{-iq}\right\}\frac{\partial}{\partial p}$$

$$\xi_B = d'\frac{\partial}{\partial q} - \left\{\frac{i}{2}[a'(\alpha+i\beta)+b'(\beta-i\alpha)]e^{iq}+\frac{i}{2}[a'(\alpha-i\beta)+b'(\beta+i\alpha)]e^{-iq}\right\}\frac{\partial}{\partial p}.$$

We have

$$\begin{aligned} dp\wedge dq(\xi_A,\xi_B) &= dp(\xi_A)dq(\xi_B) - dp(\xi_B)dq(\xi_A) \\ &= \frac{1}{2}\Big\{[(db'-d'b)(\alpha+i\beta)+(ad'-a'd)(\beta-i\alpha)]e^{iq} \\ &\quad + [(db'-d'b)(\alpha-i\beta)+(ad'-a'd)(\beta+i\alpha)]e^{-iq}\Big\}. \end{aligned} \tag{39}$$

On the other hand, $[A,B] = (db'-d'b)X + (ad'-a'd)Y + (ab'-a'b)Z$ implies (see [12])

$$\begin{aligned} \omega_{F'}(\xi_A,\xi_B) &= \langle F',[A,B]\rangle \\ &= \frac{1}{2}\big[(db'-d'b)(\alpha+i\beta)+(ad'-a'd)(\beta-i\alpha)\big]e^{iq} \\ &\quad + \frac{1}{2}\big[(db'-d'b)(\alpha-i\beta)+(ad'-a'd)(\beta+i\alpha)\big]e^{-iq} \\ &= \frac{1}{2}\Big\{\big[(db'-d'b)(\alpha+i\beta)+(ad'-a'd)(\beta-i\alpha)\big]e^{iq} \\ &\quad + \big[(db'-d'b)(\alpha-i\beta)+(ad'-a'd)(\beta+i\alpha)\big]e^{-iq}\Big\}. \end{aligned} \tag{40}$$

Then, (39) and (40) imply that the Kirillov form is $\omega = dp\wedge dq$.

2. For the case $\gamma \neq 0$ we chose

$$\psi_k : \quad (\mathbf{R}_+\times\mathbf{I}_k) \longrightarrow \Omega_F$$

$$(p,q) \longmapsto (p\cos q, p\sin q, \gamma, \frac{1}{2\gamma}(p^2+2\gamma\delta-\alpha^2-\beta^2))$$

$$(\text{or } \ \psi_k : \quad (\mathbf{R}_-\times\mathbf{I}_k) \longrightarrow \Omega_F).$$

Then, for all $F' \in \Omega_F$,

$$F' = p\cos qX^* + p\sin qY^* + \gamma Z^* + \frac{1}{2\gamma}(p^2+2\gamma\delta-\alpha^2-\beta^2)T^*$$

and $A = aX + bY + cZ + dT \in \mathfrak{g}_{4,4,1}$, we have

$$\begin{aligned} \tilde{A}\circ\psi_k(p,q) &= ap\cos q + bp\sin q + c\gamma + \frac{d}{2\gamma}(p^2+2\gamma\delta-\alpha^2-\beta^2) \\ &= \frac{d}{2\gamma}p^2 + \Big[\frac{a}{2}(e^{iq}+e^{-iq}) + \frac{b}{2i}(e^{iq}-e^{-iq})\Big]p + c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma}. \end{aligned}$$

It follows that

$$\xi_A(f) = \Big[\frac{d}{\gamma}p + \frac{1}{2i}((ai+b)e^{iq} + (ai-b)e^{-iq})\Big]\frac{\partial f}{\partial q} - \frac{p}{2}\Big[(ai+b)e^{iq} - (ai-b)e^{-iq}\Big]\frac{\partial f}{\partial p}.$$

By analogy,

$$\xi_B(f) = \Big[\frac{d'}{\gamma}p + \frac{1}{2i}((a'i+b')e^{iq} + (a'i-b')e^{-iq})\Big]\frac{\partial f}{\partial q} - \frac{p}{2}\Big[(a'i+b')e^{iq} - (a'i-b')e^{-iq}\Big]\frac{\partial f}{\partial p}.$$

From this,

$$dp \wedge dq(\xi_A, \xi_B) = dp(\xi_A)dq(\xi_B) - dp(\xi_B)dq(\xi_A) =$$
$$= \Big\{\frac{p^2}{2\gamma}\Big[\big((da' - d'a)i + (db' - d'b)\big)e^{iq} - \big((da' - d'a)i - (db' - d'b)\big)e^{-iq}\Big] + p(ab' - a'b)\Big\}.$$

Thus,

$$\frac{\gamma}{p}dp \wedge dq(\xi_A, \xi_B) =$$
$$= \frac{p}{2}\Big[\big((a'd - d'a)i + (db' - d'b)\big)e^{iq} - \big((da' - d'a)i - (db' - d'b)\big)e^{-iq}\Big] + \gamma(ab' - a'b). \tag{41}$$

On the other hand,

$$\omega_{F'}(\xi_A, \xi_B) = \langle F', [A, B]\rangle =$$
$$= (db' - d'b)p\cos q + (ad' - a'd)p\sin q + (db' - d'b)\gamma$$
$$= \frac{p}{2}\Big[\big((a'd - d'a)i + (db' - d'b)\big)e^{iq} - \big((da' - d'a)i - (db' - d'b)\big)e^{-iq}\Big] + \gamma(ab' - a'b). \tag{42}$$

From (41) and (42) we see that ω is of the form $(\gamma/p)dp \wedge dq$. ■

5.2. Computation of operators $\hat{\ell}_A^{(k)}$

It is easy to prove that

- If $G = G_{4,2,3(\frac{\pi}{2})}$ then $\hat{\ell}_A^{(k)} = \Big(d\frac{\partial}{\partial_s} + i[(b+ic)(\beta + i\gamma)e^{is} + a\alpha]\Big)\Big|_{(s,t)}$.
- If $G = G_{4,3,4(\frac{\pi}{2})}$ then $\hat{\ell}_A^{(k)} = \Big(d\frac{\partial}{\partial_s} + i[(a+ib)(\alpha + i\beta)e^{is} + c\gamma e^{s\lambda}]\Big)\Big|_{(s,t)}$.

From this we obtain the (local) representations $T(\exp A) = \exp \hat{\ell}_A^{(k)}$.

In [7], we proved the following result for $G_{4,2,4}$.

Let $\mathcal{F}_z(f)$ denote the partial Fourier transform of the function f from the variable $z = p_1 + ip_2$ to the variable $\xi = \xi_1 + i\xi_2$, i.e.

$$\mathcal{F}_z(f)(\xi,w) = \frac{1}{2\pi}\int\int_{R^2} e^{-iRe(\xi\overline{z})} f(z,w)dp_1dp_2$$

and

$$\mathcal{F}_z^{-1}(f)(z,w) = \frac{1}{2\pi}\int\int_{R^2} e^{iRe(\xi\overline{z})} f(\xi,w)d\xi_1 d\xi_2$$

the inverse Fourier transform. Remind that the subindex in the formula of $\mathcal{F}_z^{-1}$ indicates that it is the inverse of the Fourier transformation $\mathcal{F}_z$.

Theorem 5.3. (see [7, Proposition 3.4]) *For each* $A = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \in \text{aff}(\mathbf{C})$ *and for each compactly supported* C^∞*-function* $f \in C_0^\infty(\mathbf{C}\times \mathbf{H}_k)$, *we have*

$$\hat{\ell}_A^{(k)}(f) := \mathcal{F}_z \circ \ell_A^{(k)} \circ \mathcal{F}_z^{-1}(f) =$$

$$= \Big[\alpha\Big(\frac{1}{2}\partial_w - \partial_{\overline{\xi}}\Big)f + \overline{\alpha}\Big(\frac{1}{2}\partial_{\overline{w}} - \partial_\xi\Big)f + \frac{i}{2}\Big(\beta e^{w-\frac{1}{2}\overline{\xi}} + \overline{\beta}e^{\overline{w}-\frac{1}{2}\xi}\Big)f\Big] \quad (43)$$

i.e.

$$\hat{\ell}_A^{(k)} = \alpha\frac{\partial}{\partial u} + \overline{\alpha}\frac{\partial}{\partial \overline{u}} + \frac{i}{2}(\beta e^u + \overline{\beta}e^{\overline{u}});\ \ u = w - \frac{1}{2}\overline{\xi};\ v = w + \frac{1}{2}\overline{\xi}.$$

From this we obtain (directly) all irreducible unitary representations of the group $\widetilde{\text{Aff}(\mathbf{C})}$, the universal covering of $\text{Aff}(\mathbf{C})$. Now we consider the group $G_{4,4,1}$.

Theorem 5.4. *For each* $A \in \mathfrak{g}_{4,4,1}$ *and for each compactly supported* C^∞*-function* $f \in C_0^\infty(\mathbf{R}\times \mathbf{I}_k)$, *the following holds.*

1. *If* $\tilde{A}$ *is defined by* (37) *then*

$$\hat{\ell}_A^{(k)}(f) = \big(d\partial_s f + i[(a\alpha + b\beta)\cos s + (b\alpha - a\beta)\sin s]f\big)|_{(s,t)}.$$

2. *If* $\tilde{A}$ *is defined by* (38) *then*

$$\begin{aligned}\hat{\mathcal{L}}_A^k(f) &= \mathcal{F}_p \circ \ell_A \circ \mathcal{F}_p^{-1}(f)\\ &= i.\Big(\frac{d}{2\gamma}(i\partial_x + \frac{1}{2\gamma}\partial_x\partial_q)^2]f + \Gamma.f\Big)\Big|_{(x,q)}\\ &\quad - \frac{1}{2}\Big(\partial_x[(a-bi)\Delta(f) + (a+bi)\Delta^{-1}(f)]\\ &\quad + \frac{i}{2\gamma}\partial_x[(a-bi)\Theta(f) + (a+bi)\Theta^{-1}(f)]\Big)\Big|_{(x,q)},\end{aligned}$$

where

$$\Gamma = c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma},$$

$$\Delta(f) = \exp\Big[iq + \partial_x\Big(\Big(\frac{x}{2\gamma}\Big).f\Big)\Big] = e^{iq}.\sum_{r=0}\frac{1}{r!}\frac{\partial^r}{\partial_{x^r}}\Big(\Big(\frac{x}{2\gamma}\Big)^r.f\Big),$$

$$\Theta(f) = \exp\Big[iq + \partial_x\Big(\Big(\frac{x}{2\gamma}\Big).\partial_q f\Big)\Big] = e^{iq}.\sum_{r=0}\frac{1}{r!}\frac{\partial^r}{\partial_{x^r}}\Big(\Big(\frac{x}{2\gamma}\Big)^r.\partial_q f\Big).$$

To prove the theorem, we need the following obvious lemma, which is a direct consequence of the definition of $\mathcal{F}_p$ and $\mathcal{F}_p^{-1}$.

Lemma 5.5. *For all $r \geq 1$*

- $\partial^r_{p^r}\big(\mathcal{F}_p^{-1}(f)\big) = i^r \mathcal{F}_p^{-1}(x^r.f)$,
- $\mathcal{F}_p\big(p^r \mathcal{F}_p^{-1}(f)\big) = i^r \partial^r_{x^r}(f)$,
- $\mathcal{F}_p\big(p^r \partial^{r-1}_{p^{r-1}} \mathcal{F}_p^{-1}(f)\big) = i^{2r-1}\partial^r_{x^r}(x^{r-1}.f)$. ∎

Proof of Theorem 5.4.

The proof for the first case is quite similar to that of Theorem 4.4. We prove only the second case.

One can write (38) as

$$\tilde{A} \circ \psi(p,q) = \frac{d}{2\gamma}p^2 + \frac{p}{2}\big[(a-bi)e^{iq} + (a+bi)e^{-iq}\big] + c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma}$$

and remark that

$$\Lambda^{-1} = \begin{pmatrix} 0 & \dfrac{p}{\gamma} \\ -\dfrac{p}{\gamma} & 0 \end{pmatrix}$$

corresponds to the form $\omega = (\gamma/p)dp \wedge dq$. Denoting $\mathcal{F}_p^{-1}(f) = v$, we have

$$P^0 = \Big\{\frac{d}{2\gamma}p^2 + \frac{p}{2}\big[(a-bi)e^{iq} + (a+bi)e^{-iq}\big] + c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma}\Big\}v,$$

$$P^1 = \Big[\frac{dp^2}{\gamma^2} + \frac{p}{2\gamma}((a-bi)e^{iq} + (a+bi)e^{-iq})\Big]\partial_q v$$
$$- \frac{ip^2}{2\gamma}\Big[(a-bi)e^{iq} - (a+bi)e^{-iq}\Big]\partial_p v,$$

$$P^2 = \frac{1}{2\gamma^2}\Big\{i^2\big[(a-bi)e^{iq} + (a+bi)e^{-iq}\big]p^3\partial^2_{p^2}v$$
$$- 2i\big[(a-bi)e^{iq} - (a+bi)e^{-iq}\big]p^2\partial^2_{pq}v\Big\} + \frac{dp^2}{\gamma^3}\partial^2_{q^2}v,$$

$$P^3 = \frac{1}{2\gamma^3}\Big\{(-i)^3\big[(a-bi)e^{iq} - (a+bi)e^{-iq}\big]p^4\partial^3_{p^3}v$$
$$+ 3(-i)^2\big[(a-bi)e^{iq} + (a+bi)e^{-iq}\big]p^3\partial^3_{p^2q}v\Big\},$$

$$P^4 = \frac{1}{2\gamma^4}\Big\{(-i)^4\big[(a-bi)e^{iq} + (a+bi)e^{-iq}\big]p^5\partial^4_{p^4}v$$
$$+ 4(-i)^3\big[(a-bi)e^{iq} - (a+bi)e^{-iq}\big]p^4\partial^4_{p^3q}v\Big\}.$$

By analogy, for all $r \geq 4$ we have

$$P^r = \frac{1}{2\gamma^r}\Big((-i)^r[(a-bi)e^{iq} + (-1)^r(a+bi)e^{-iq}]p^{r+1}\partial^r_{p^r}v\Big)$$
$$+ \frac{r(-i)^{r-1}}{2\gamma^r}\Big([(a-bi)e^{iq} + (-1)^{r-1}(a+bi)e^{-iq}]p^r\partial^r_{p^{r-1}q}v\Big).$$

As $\tilde{A}$ is defined by (38), we obtain

$$
\begin{aligned}
\hat{\mathcal{L}}_A^k(f) &= \mathcal{F}_p(i\tilde{A} \star \mathcal{F}_p^{-1}(f)) i \sum_{r\geq 0}\Big(\frac{1}{2i}\Big)^r \mathcal{F}_p\left(P^r(\tilde{A}, \mathcal{F}_p^{-1}(f)\right) \\
&= i(c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma})\mathcal{F}_p v \\
&\quad + i\Big[\mathcal{F}_p(\frac{d}{2\gamma}p^2 v) + \frac{1}{1!}\frac{1}{2i}\mathcal{F}_p(\frac{d}{\gamma^2}p^2\partial_q v) + \frac{1}{2!}\Big(\frac{1}{2i}\Big)^2\mathcal{F}_p(\frac{d}{\gamma^3}p^2\partial^2_{q^2}v)\Big] \\
&\quad + \sum_{r\geq 0}\frac{1}{r!}\Big(\frac{1}{2i}\Big)^r\frac{1}{2\gamma^r} \\
&\quad \times\Big\{(-i)^r\left[(a-bi)e^{iq} + (-1)^r(a+bi)e^{-iq}\right]\mathcal{F}_p(p^{r+1}\partial^r_{p^r}v) \\
&\qquad + r(-i)^{r-1}[(a-bi)e^{iq} + (-1)^{r-1}(a+bi)e^{-iq}]\mathcal{F}_p(p^r\partial^r_{p^{r-1}q}v)\Big\} \\
&= i(c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma})f + i\frac{d}{2\gamma}(i\partial_x + \frac{1}{2\gamma}\partial_x\partial_q)^2 f \\
&\quad + \frac{i}{2}\sum_{r\geq 0}\frac{1}{r!}\Big(\frac{-1}{2\gamma}\Big)^r(a-bi)e^{iq}\mathcal{F}_p(p^{r+1}\partial^r_{p^r}v) \\
&\quad + \frac{i}{2}\sum_{r\geq 0}\frac{1}{r!}\Big(\frac{1}{2\gamma}\Big)^r(a+bi)e^{-iq}\mathcal{F}_p(p^{r+1}\partial^r_{p^r}v) \\
&\quad + \frac{i}{2}\frac{1}{\gamma}\sum_{r\geq 1}\frac{1}{(r-1)!}\Big(\frac{-1}{2\gamma}\Big)^{r-1}\frac{1}{2i}(a-bi)e^{iq}\mathcal{F}_p(p^r\partial^r_{p^{(r-1)}q}v) \\
&\quad + \frac{i}{2}\frac{1}{\gamma}\sum_{r\geq 1}\frac{1}{(r-1)!}\Big(\frac{1}{2\gamma}\Big)^{r-1}\frac{1}{2i}(a+bi)e^{-iq}\mathcal{F}_p(p^r\partial^r_{p^{(r-1)}q}v) \\
&= i[c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma} + i\frac{d}{2\gamma}(i\partial_x + \frac{1}{2\gamma}\partial_x\partial_q)^2]f \\
&\quad + \frac{i}{2}(a-bi)e^{iq}\sum_{r\geq 0}\frac{1}{r!}\Big(\frac{-1}{2\gamma}\Big)^r i^{2r+1}\partial^{r+1}_{x^{r+1}}(x^r.f) \\
&\quad + \frac{i}{2}(a+bi)e^{-iq}\sum_{r\geq 0}\frac{1}{r!}\Big(\frac{1}{2\gamma}\Big)^r i^{2r+1}\partial^{r+1}_{x^{r+1}}(x^r.f) \\
&\quad + \frac{1}{4\gamma}(a-bi)e^{iq}\sum_{r\geq 1}\frac{1}{(r-1)!}\Big(\frac{-1}{2\gamma}\Big)^{r-1} i^{2r-1}\partial^{r-1}_{p^{r-1}}(x^{r-1}.\partial_q f) \\
&\quad + \frac{1}{4\gamma}(a+bi)e^{-iq}\sum_{r\geq 1}\frac{1}{(r-1)!}\Big(\frac{1}{2\gamma}\Big)^{r-1} i^{2r-1}\partial^{r-1}_{p^{r-1}}(x^{r-1}.\partial_q f) \\
&= i\Big[c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma} + i\frac{d}{2\gamma}\Big(i\partial_x + \frac{1}{2\gamma}\partial_x\partial_q\Big)^2\Big]f \\
&\quad - \frac{1}{2}\partial_x\Big[(a-bi)e^{iq}.\sum_{r\geq 0}\frac{1}{r!}\Big(\frac{1}{2\gamma}\Big)^r\partial^r_{x^r}(x^r.f) \\
&\quad + (a+bi)e^{-iq}.\sum_{r\geq 0}\frac{1}{r!}\Big(\frac{-1}{2\gamma}\Big)^r\partial^r_{x^r}(x^r.f)
\end{aligned}
$$

$$+\frac{i}{2\gamma}(a-bi)e^{iq}.\sum_{r\geq 1}\frac{1}{(r-1)!}\Big(\frac{1}{2\gamma}\Big)^{r-1}\partial^{r-1}_{x^{r-1}}(x^{r-1}.\partial_q f)$$

$$+\frac{i}{2\gamma}(a+bi)e^{-iq}.\sum_{r\geq 1}\frac{1}{(r-1)!}\Big(\frac{-1}{2\gamma}\Big)^{r-1}\partial^{r-1}_{x^{r-1}}(x^{r-1}.\partial_q f)\Big]$$

$$= i\Big[c\gamma + d\delta - d\frac{\alpha^2+\beta^2}{2\gamma} + \frac{d}{2\gamma}\Big(i\partial_x + \frac{1}{2\gamma}\partial_x\partial_q\Big)^2\Big]f$$

$$-\frac{1}{2}\partial_x\Big\{(a-bi)\exp\Big[iq+\partial_x\Big(\Big(\frac{x}{2\gamma}\Big).f\Big)\Big]$$

$$+(a+bi)\exp\Big[-iq+\partial_x\Big(\Big(\frac{-x}{2\gamma}\Big).f\Big)\Big]\Big\}$$

$$-\frac{i}{4\gamma}\partial_x\Big\{(a-bi)\exp\Big[iq+\partial_x\Big(\Big(\frac{x}{2\gamma}\Big).\partial_q f\Big)\Big]$$

$$+(a+bi)\exp\Big[-iq+\partial_x\Big(\Big(\frac{-x}{2\gamma}\Big).\partial_q f\Big)\Big]\Big\}.$$

The theorem is completely proved. ■

From this, we have $[\Omega_F^{(4,1,a)};\hat{\ell}_A^{(k)}]$ are the quantum rotation cylinders; $[\Omega_F^{(4,1,b)};\hat{\mathcal{L}}_A^{(k)}]$ are the quantum rotation paraboloids.

At last, as $\hat{\ell}_A, \hat{\ell}_A^{(k)}, \hat{\mathcal{L}}_A^{(k)}$ are (global or local) representations of the MD$_4$-algebras (for all the cases, Moyal $\star$-product is G-covariant), we have operators $\exp(\hat{\ell}_A)$; $\exp(\hat{\ell}_A^{(k)})$; $\exp(\hat{\mathcal{L}}_A^{(k)})$ are representations of the corresponding connected and simply connected MD$_4$-groups.

We say that they are the representations of MD$_4$-groups arising from the reduction of the procedure of deformation quantization.

Acknowlegement. The author would like to thank Professor Do Ngoc Diep for his guidance and encouragements. He also thanks Dr. Nguyen Viet Dung for valuable suggestions and the referee for his critical comments.

References

1. D. Arnal and J. C. Cortet, $\star$-product and representations of nilpotent Lie groups, *J. Geom. Phys.* **2** (1985) 86–116.
2. D. Arnal and J. C. Cortet, Représentations * des groupes exponentiels, *J. Funct. Anal.* **92** (1990) 103–135.
3. V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, Berlin - New York - Heidelberg, 1984.
4. F. Bayen, M. Flato, C. Fronsdal, and D. Sternheimer, Deformation theory and quantization, I and II, *Annals of Physics* **111** (1978) 61–151.
5. Do Ngoc Diep, Methods of noncommutative geometry for group C*-algebras, *CRC Research Notes in Mathematics Series* 416, Chapman & Hall, 1999.

6. Do Ngoc Diep and Nguyen Viet Hai, Quantum half-planes via deformation quantization, *Beiträge zur Algebra und Geometrie* (to appear).
7. Do Ngoc Diep and Nguyen Viet Hai, Quantum co-adjoint orbits of the group of affine transformations of the complex straight line, *Beiträge zur Algebra und Geometrie* (to appear).
8. B. Fedosov, *Deformation Quantization and Index Theory*, Akademie-Verlag, Berlin, 1993.
9. S. Gutt, Deformation quantization, *ICTP Workshop on Representation Theory of Lie Groups*, SMR 686/14, 1993.
10. I. M. Gelfand and M. A. Naimark, Unitary representations of the group of affine transformations of the straight line, *Dokl. AN SSSR* **55** (1947) 571–574.
11. Nguyen Viet Hai, *Quantum Co-Adjoint Orbits of the Real Diamond Lie Group*, math.QA/0001046, 7 Jan. 2000.
12. A. A. Kirillov, *Elements of the Theory of Representation*, Springer-Verlag, Berlin - New York - Heidelberg, 1976.
13. A. A. Kirillov, Unitary representations of nilpotent Lie groups, *Russian Math. Survey* (1962) 17–52.
14. B. Kostant, On certain unitary representations which arise from a quantization theory, *Lecture Notes in Mathematics* **170** (1970) 237.
15. R. Meise and D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford, 1997.