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Rings Used in Modal Logic and Their Radicals

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Abstract. We define modal Boolean rings without identity and show that all such can be embedded in modal Boolean rings with identity. We then show that there are radical-theoretic ways of viewing many of the most important classes of modal logics, such as T and $S4$.

1. Introduction

In modal logic, Boolean algebras equipped with additional unary operations □ and ◇ ("possibly" and "necessarily") and satisfying certain identities are the natural algebraic models, just as general Boolean algebras are the natural models in classical propositional logic. Such objects were perhaps first studied in [9], where McKinsey and Tarski defined "closure algebras" in order to "algebraize" point-set topology. These Boolean algebras with closure operation (satisfying the usual Kuratowski sorts of conditions) are also the algebraic models for the so-called "$S4$" version of modal logic, where the closure operation applied to a proposition p is interpreted as "possibly p".

Now it is well-known that the varieties of Boolean algebras and Boolean rings with identity are term equivalent, a fact that easily extends to closure algebras and Boolean rings with identity having closure operation. But Boolean rings having no identity are of interest, and it is an important fact that every Boolean ring without identity can be embedded as a maximal ideal in a Boolean ring with identity. Indeed a similar embedding is possible for general rings without identity (although the embedded ring is not usually a maximal ideal of the bigger ring). A big advantage of varieties of rings without identity is that they are closed under ideals, and one can do radical theory (see Wiegandt [12]) among other things.

So far there has been no study of "modal Boolean rings without identity", in which both possibility and necessity operators are present. One reason for this is that Boolean algebras rather than Boolean rings have traditionally been used in logic. Also, the identity element is critical in the logical interpretation: it models the proposition "true", and only if complements exist (which means the Boolean ring must have an identity) can one define the possibility and necessity operations in terms of one-another.

We shall define "modal rings" not necessarily having an identity, and show that any such object can be embedded as a maximal closed ideal (an ideal respecting the closure and interior operations) in a modal ring with identity in a way that preserves important properties. We then apply radical-theoretic ideas to establish connections between some of the most important sorts of modal rings.

For recent references concerning the algebraic study of modal logic, see those cited in Sections V.4 and V.8 of [5].

2. Modal Boolean Rings

Throughout this section, let R be a Boolean ring, that is, an associative ring satisfying $a^2 = a$. Then R is commutative and is of characteristic 2. R is partially ordered by defining $a \leq_R b$ if and only if $ab = a$; defining $a \vee b = a + b + ab$ and $a \wedge b = ab$, $a \leq_R b$ if and only if $a \vee b = b$ if and only if $a \wedge b = a$, and $(R, \vee, \wedge)$ is a relatively complemented distributive lattice.

R is a *closure Boolean ring* if it possesses a unary operation $[\,]_c$ satisfying $[0]_c = 0$ and for all $a, b \in R$, $[a \vee b]_c = [a]_c \vee [b]_c$, and we call $[\,]_c$ a *closure operation.* Up to term equivalence, these are the same objects as Blok's "generalized interior algebras"; see [1]. See also [2] where the existence of a ring identity is assumed.

R is an *interior ring* if it possesses a unary operation $[\,]_i$ satisfying $[0]_i = 0$ and for all $a, b \in R$, $[ab]_i = [a]_i[b]_i$. We call $[\,]_i$ an *interior operation.*

R is a *modal Boolean ring* if it possesses a closure operation $[\,]_c$ and an interior operation $[\,]_i$ linked by the rule

$$[a]_c[b]_i = [ab + b]_i + [b]_i$$

for all $a, b \in R$. Notation: $(R, [\,]_c, [\,]_i)$, or just R if no confusion can result. We call $[\,]_c$ and $[\,]_i$ the *modal operations* on $(R, [\,]_c, [\,]_i)$.

In the modal Boolean ring R, both modal operations are obviously order-preserving, and for all $a \in R$, $[a]_c[a]_i = [a]_i$, so $[a]_i \leq [a]_c$. Any closure Boolean ring is a modal Boolean ring in at least two ways: one can define $[a]_i = a$ for all a or $[a]_i = 0$ for all a.

If a modal Boolean ring with identity R satisfies $[1]_i = 1$, or equivalently, $[a]_c = [a + 1]_i + 1$ for all $a \in R$, then we say R is a *strong modal Boolean ring.* Any closure Boolean ring with identity is obviously a strong modal Boolean ring in exactly one way.

It is easy to check that normal deontic modal algebras in the sense of [7] are, when viewed as Boolean rings, exactly strong modal Boolean rings in the current sense. The global assumption that $[0]_c = [0]_i = 0$ ensures that the objects of study are *multi-operator groups* in the sense of Kurosh [6].

Let R be a modal Boolean ring. $R' \subseteq R$ is a *modal subring* of R if it is a subring which is closed under the modal operations.

We say the modal Boolean ring R is

- *idempotent* if $[[x]_c]_c = [x]_c$ and $[[x]_i]_i = [x]_i$ for all $x \in R$;
- *classical* if $[x]_c = x$ and $[x]_i = x$ for all $x \in R$;
- of *type T* if $x[x]_c = x$ and $x[x]_i = [x]_i$ for all $x \in R$;
- of *type S4* if it is of type T and idempotent;
- of *type S5* if it has an identity and $1 + [x]_c = [1 + [x]_c]_c$ for all $x \in R$.

The various types of closure Boolean rings are defined by reference to $[\ ]_c$ only, omitting the part of the definition (if any) referring to $[\ ]_i$.

Denote by $\mathcal{I}$, $\mathcal{C}$, $\mathcal{T}$, $\mathcal{S}_4$ and $\mathcal{S}_5$ the classes of idempotent, classical, type T, type $S4$ and type $S5$ modal Boolean rings respectively (and similarly for closure Boolean rings). All of these classes are varieties, and the following inclusions hold:

$$\mathcal{C} \subset \mathcal{S}_4 = \mathcal{I} \cap \mathcal{T}.$$

In the strong modal Boolean ring case, we also have $\mathcal{S}_5 \subseteq \mathcal{S}_4$. Any Boolean ring R is a classical modal Boolean ring in precisely one way, and is an $S4$ modal Boolean ring if one defines $[a]_c = [a]_i = 0$ for all $a \in R$.

For strong modal Boolean rings, each type may be defined in terms of either the closure or the interior operation, but even for non-strong modal Boolean rings having identity, we have the following:

Lemma 2.1. *Suppose the modal Boolean ring R has identity.*

1. *If $a[a]_c = a$ for all $a \in R$, then $R \in \mathcal{T}$.*
2. *If $[a]_i = a$ for all $a \in R$, then $R \in \mathcal{C}$.*

Proof. For all $a \in R$, $[a+1]_c[1]_i = [(a+1)1 + 1]_i + [1]_i = [a]_i + [1]_i$, so

$$a[a]_i + [a]_i = (a+1)[a]_i = (a+1)[a+1]_c[1]_i + (a+1)[1]_i,$$

so R is of type T if $a[a]_c = a$ for all $a \in R$. If $[a]_i = a$ for all $a \in R$, then $[a]_c = a + 1 + 1 = a$ so R is classical. ∎

Example 1. Subsets of topological spaces.

Let X be a topological space. Letting $R = 2^X$, define $[a]_c$ to be the closure of $a \in R$. Then R is a strong $S4$ modal Boolean ring with identity. Moreover all $S4$ modal Boolean rings arise as modal subrings of such modal Boolean rings (see [9]).

Example 2. Sets of natural numbers.

Let $R = 2^N$, N the natural numbers. There are a number of ways of making R into a strong modal Boolean ring. If we define $[X]_c = \{1, 2, \ldots, n\}$ if the largest element of $X \in R$ is n, with $[X]_c = 1$ if no largest element exists, then the sets of the form $[X]_c$ are the closed sets of a topology on the natural numbers, so R is a strong $S4$ modal Boolean ring. A type T example which is not $S4$ may be obtained by setting $[X]_c = \{n|n \in X \text{ and } n+1 \in X\}$ and $[X]_i = \{n|n \in X \text{ or } n+1 \in X\}$. These examples appear in [10].

Example 3. Sets of real numbers.

Each of the above examples contains a modal subring without identity, namely the modal subring of finite subsets of natural numbers. Another $S4$ modal Boolean ring without identity is the Boolean ring of bounded subsets of the real line together with the usual closure and interior operations, an ideal of the Boolean ring of all subsets of the real line which is additionally a modal subring.

Example 4. Subsets under a transformation.

Despite arising quite naturally, modal systems weaker than type T are seldom considered; for instance they are not covered at all in [4]. Let $f : X \to X$ be a transformation defined on the set X. Obviously $f(\emptyset) = \emptyset$, and it is easy to check that $f(S \cup T) = f(S) \cup f(T)$ for all $S, T \in 2^X$. So defining $[S]_c = f(S)$ for all $S \in 2^X$ makes 2^X into a closure Boolean ring, idempotent if f is, and not of type T unless f is the identity transformation, as can be seen by considering the images of singleton sets.

Modal Boolean rings not of type T may also be constructed from type T modal Boolean rings as follows.

Theorem 2.2. *For R a closure Boolean ring and for $a \in R$, with a not an identity for R, define $[x]_c^a = a[x]_c$ for all $x \in R$. Then $(R, [\,]_c^a)$ is a closure ring not of type T. If R is $S4$, then $(R, [\,]_c^a)$ is idempotent.*

Proof. For any $x, y \in R$,

$$[x \vee y]_c^a = a[x \vee y]_c = a([x]_c \vee [y]_c) = (a[x]_c) \vee (a[y]_c) = [x]_c^a \vee [y]_c^a.$$

Further, $[0]_c^a = a[0]_c = 0$ and so $(R, [\,]_c^a)$ is a closure Boolean ring.

Also, $[x]_c^a \leq a$ for all $x \in (R, [\,]_c^a)$, so if a is not an identity, there exists $b \not\leq a$, so that $b \not\leq a[b]_c = [b]_c^a$ and so $(R, [\,]_c^a)$ is not of type T.

If R is $S4$, then for all $x \in R$, $[a[x]_c]_c \leq [[x]_c]_c = [x]_c$ so $a[a[x]_c]_c \leq a[x]_c$; further, $a[x]_c \leq a$ and $a[x]_c \leq [a[x]_c]_c$ so $a[x]_c \leq a[a[x]_c]_c$; hence $a[a[x]_c]_c = a[x]_c$, so $(R, [\,]_c^a)$ is idempotent. ■

The converse process is always possible too. It is routine to show that if R is a modal Boolean ring, then defining $[x]_{c'} = x \vee [x]_c$ and $[x]_{i'} = x[x]_i$, $(R, [\,]_{c'}, [\,]_{i'})$ is a type T modal ring; furthermore, if R is idempotent or classical, so is $(R, [\,]_{c'}, [\,]_{i'})$.

3. Closed Ideals

A *closed ideal* I of a modal Boolean ring R is a ring ideal which is closed with respect to $[\,]_c$: $[i]_c \in I$ for all $i \in I$. Note that such an ideal is a modal subring since $[a]_i[a]_c = [a]_i$, so closed ideals are exactly modal subrings which are ideals.

Proposition 3.1. *Suppose I is a ring ideal of R. The following are equivalent:*
1. *I is a closed ideal.*
2. *The ring congruence induced by I respects the modal operations on R.*

Proof. Suppose I is a closed ideal of R and let $i \in I$, $a \in R$. Then

$$\begin{aligned}
&([a+i]_c + [a]_c)[i]_c \\
&\quad = ([a+i]_c + [i]_c + [a+i]_c[i]_c) + ([a]_c + [i]_c + [a]_c[i]_c) + [a+i]_c + [a]_c \\
&\quad = [a+i]_c \vee [i]_c + [a]_c \vee [i]_c + [a+i]_c + [a]_c \\
&\quad = [(a+i) \vee i]_c + [a \vee i]_c + [a+i]_c + [a]_c \\
&\quad = [a+i+i+ai+i]_c + [a+i+ai]_c + [a+i]_c + [a]_c \\
&\quad = [a+i]_c + [a]_c.
\end{aligned}$$

Hence $[a+i]_c + [a]_c = ([a+i]_c + [a]_c)[i]_c \in I$, so congruence modulo I respects $[\,]_c$ also. Moreover,

$$[a+i]_i + [a]_i = [(a \vee i) + (a \vee i)ai]_i + [a]_i = [ai]_c[a \vee i]_i + [a \vee i]_i + [a]_i.$$

Now $[ai]_c[a \vee i]_i \in I$, so $[a+i]_i + [a]_i \in I$ if and only if $[a \vee i]_i + [a]_i \in I$. But

$$\begin{aligned}
[a]_i + [a \vee i]_i &= [ai + i + (a \vee i)]_i + [a \vee i]_i \\
&= [(ai+i)(a \vee i) + (a \vee i)]_i + [a \vee i]_i \\
&= [ai+i]_c[a \vee i]_i \in I.
\end{aligned}$$

The converse direction is obvious: let $a = 0$ in the assertion that $[a+i]_c - [a]_c \in I$ for all $a \in R$, $i \in I$. ■

Thus closed ideals are exactly the kernels of modal Boolean ring homomorphisms (ring homomorphisms which respect both modal operations), and are the relevant variant of *multi-operator group ideals* in the sense of Kurosh [6] for the variety of modal Boolean rings. The version of this result for strong $S4$ modal Boolean rings was shown in [11].

4. Embedding Modal Boolean Rings in Modal Boolean Rings with Identity

Let R be a Boolean ring; then R is a left unital module over $\mathbb{Z}_2$ in a natural way. Let $R^* = R \times \mathbb{Z}_2$ as an additive group, with multiplication defined as follows:

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$$

for all $a, b \in R$ and $\alpha, \beta \in \mathbb{Z}_2$. Identifying R with its copy $\{(a, 0) \mid a \in R\}$ in R^*, every element of R^* has one of the two forms a or $a+1$ where $a \in R$ and 1 is the identity element of R^*.

The modal operations on the modal Boolean ring R may be extended to R^* as follows: for all $a \in R$, define

$$[a+1]_i = [a]_c + 1, [a+1]_c = [a]_i + 1.$$

Theorem 4.1. *R^* together with the extensions of the modal operations on R just defined is a strong modal Boolean ring having R as a maximal closed ideal. Moreover R^* is idempotent/type T/classical if R is.*

Proof. To show that $[a]_c \vee [b]_c = [a \vee b]_c$ for all $a, b \in R^*$, we consider three cases for a, b. If $a, b \in R$, the result is immediate. If $a \in R$ and $b = c+1$ for some $c \in R$, then

$$\begin{aligned}[a \vee (c+1)]_c &= [c + ac + 1]_c = [c + ac]_i + 1 = [a]_c[c]_i + [c]_i + 1 \\ &= [a]_c \vee ([c]_i + 1) = [a]_c \vee [c+1]_c.\end{aligned}$$

If $a = c+1, b = d+1$ for some $c, d \in R$, then

$$\begin{aligned}[(c+1) \vee (d+1)]_c &= [cd+1]_c = [cd]_i + 1 = [c]_i[d]_i + 1 \\ &= ([c]_i + 1) \vee ([d]_i + 1) = [c+1]_c \vee [d+1]_c.\end{aligned}$$

Thus $[\ ]_c$ is a closure operation on R^*. Furthermore, for all $b \in R^*$, $[b+1]_i = [b]_c + 1$, as can be checked by looking at two cases for b, as in the above arguments. Obviously $[0]_c = [0]_i = 0$ in R^*. So R^* is a strong modal Boolean ring with respect to $[\ ]_c$ and $[\ ]_i$, and the restrictions of $[\ ]_c$ and $[\ ]_i$ to R are the original modal operations on R. Furthermore, R is a maximal ideal of R^* which is a closed ideal, so R is a maximal closed ideal of R^*.

Now suppose R is idempotent. If $a \in R$ then $[[a]_c]_c = [a]_c$ is immediate; if $a = b+1$ for some $b \in R$, then $[[a]_c]_c = [[b+1]_c]_c = [[b]_i + 1]_c = [[b]_i]_i + 1 = [b]_i + 1 = [b+1]_c = [a]_c$. But R^* is strong, so $[[b]_i]_i = [b]_i$ for all $b \in R^*$, and R^* is idempotent.

Suppose R is of type T. Then for all $a \in R$, $a[a]_c = a$, and $(a+1)[a+1]_c = (a+1)([a]_i + 1) = a[a]_i + a + [a]_i + 1 = a + 1$; so $b[b]_c = b$ for all $b \in R^*$, so R^* is of type T.

Suppose R is classical. Then for all $a \in R$, $[a]_i = [a]_c = a$, so $[a+1]_i = [a]_c + 1 = a + 1$ so R^* is classical. ■

Corollary 4.2. *Every $R \in S_4$ may be embedded in $(2^X, C)$ for some topological space X having closure operator C.*

5. Radical and Semisimple Classes

Radical theory began as a branch of the structure theory of abelian groups and rings, but has since been generalized to multi-operator groups and other more general classes of algebras. There is a notion of *ideal* in multi-operator groups which reduces to the usual notion of ideal for rings, normal subgroup for groups and closed ideal for modal rings; essentially they are kernels of multi-operator group homomorphisms. Thus the following generalities in particular apply to modal Boolean rings. Here we link up various natural classes of modal Boolean rings by means of radical theory.

Given a universal class $\mathcal{U}$ of multi-operator groups (that is, a class closed under taking homomorphic images and multi-operator group ideals), a *radical class* $\mathcal{R}$ in $\mathcal{U}$ is a non-empty class for which

1. $A \in \mathcal{R}, A \cong B$ imply $B \in \mathcal{R}$,
2. I is an ideal of $A \in \mathcal{R}$ implies $A/I \in \mathcal{R}$,
3. $\mathcal{R}(A) = \sum\{J : J \text{ is an ideal of } A, J \in \mathcal{R}\} \in \mathcal{R}$ for any $A \in \mathcal{U}$ and
4. $\mathcal{R}(A/\mathcal{R}(A)) = \{0\}$ for any $A \in \mathcal{U}$.

(See [3].) We call $\mathcal{R}(A)$ the *$\mathcal{R}$-radical of R*; it is the largest $\mathcal{R}$-ideal (ideal in $\mathcal{R}$) of R. $A \in \mathcal{U}$ is said to be *$\mathcal{R}$-radical* if $\mathcal{R}(A) = A$ (equivalently, if $A \in \mathcal{R}$) and *$\mathcal{R}$-semisimple* if $\mathcal{R}(A) = \{0\}$.

The first two properties above can be summarized by saying that $\mathcal{R}$ is closed under homomorphic images. An equivalent definition of radical classes which will be useful is as follows: the non-empty class $\mathcal{R}$ of multi-operator groups is a radical class if and only if

1. $\mathcal{R}$ is homomorphically closed;
2. $\mathcal{R}$ is *closed under extensions*, that is, if I is an ideal of A and $I, A/I \in \mathcal{R}$ then $A \in \mathcal{R}$; and
3. $\mathcal{R}$ contains all unions of chains of $\mathcal{R}$-ideals.

The last condition says that for any *chain* $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of a ring A in $\mathcal{R}$, if all the I_j are $\mathcal{R}$-ideals, then so is their union (itself an ideal as is easily checked).

Given a universal class of multi-operator groups, a *semisimple class* $\mathcal{H}$ is a class such that there is a radical class $\mathcal{R}$ for which $A \in \mathcal{H}$ if and only if $\mathcal{R}(A) = \{0\}$, and we say that such $A \in \mathcal{H}$ is *$\mathcal{R}$-semisimple*. For any A, $\mathcal{R}(A)$ is the smallest ideal of A which when factored out, makes the factor ring semisimple. Radical and semisimple classes come in pairs: given a semisimple class $\mathcal{H}$, the radical class giving rise to it consists of all R having no non-trivial homomorphic images in $\mathcal{H}$. (See [3] for the details.)

For many kinds of algebraic objects, it is possible to characterize semisimple classes independently of the concept of a radical class. This is true of modal Boolean rings.

Theorem 5.1. *Let $\mathcal{V}$ be a variety of modal Boolean rings, $\mathcal{H}$ a subclass of $\mathcal{V}$. Then $\mathcal{H}$ is a semisimple class if and only if*

1. *$\mathcal{H}$ is closed under ideals, that is, $A \in \mathcal{H}$ and I is an ideal of A imply $I \in \mathcal{H}$;*
2. *$\mathcal{H}$ is closed under extensions; and*
3. *$\mathcal{H}$ is closed under subdirect products.*

Proof. Note first that closed ideals satisfy the *transitivity property*: if I is a closed ideal of J which is in turn a closed ideal of R then I is a closed ideal of R. The proof is easy: if I is a closed ideal of J which is a closed ideal of R then in particular I is a Boolean ring ideal of R since ideals in Boolean rings satisfy transitivity; it is then immediate that I is a closed ideal of R. The desired result is now immediate from Theorem 3.2 in Chapter 3 of [3]. ■

Let $\mathcal{V}$ be a variety of modal Boolean rings with $F[x, y, z]$ the free modal Boolean ring in $\mathcal{V}$ on the three generators x, y, z. Viewing $F[x, y]$ as embedded in $F[x, y, z]$ in the obvious way, we say $f \in F[x, y]$ is *associating* if there exists

$g \in F[x,y,z]$ for which $f(f(x,y),z) = f(x, g(x,y,z))$. The following is a special case of a result appearing in McConnell and Stokes [8].

Lemma 5.2. *Let $\mathcal{V}$ be a variety of modal Boolean rings with $f \in F[x,y,z]$ associating. Then the class $\mathcal{R}_f$ consisting of those $A \in \mathcal{V}$ for which, for all $a \in A$ there exists $b \in A$ such that $f(a,b) = 0$ is a radical class.*

Here is an example.

Proposition 5.3. *The class $\mathcal{L}$, consisting of modal Boolean rings R such that for every $a \in R$ there exists $b \in R$ for which $a \leq [b]_c$, is a radical class.*

Proof. Note that $\mathcal{L} = \mathcal{R}_f$ where $f(x,y) = x[y]_c + x$, and

$$\begin{aligned} f(f(x,y),z) &= (x[y]_c + x)[z]_c + x[y]_c + x \\ &= x([y]_c[z]_c + [z]_c + [y]_c) + x \\ &= x([y]_c \vee [z]_c) + x \\ &= x[y \vee z]_c + x \\ &= f(x, y \vee z) \end{aligned}$$

so f is associating. Hence $\mathcal{L}$ is a radical class. ∎

We shall consider the associated semisimple class for the special case of the universal class of idempotent modal Boolean rings shortly.

From the radical class definition and from Theorem 5.1, it follows easily that a variety of modal Boolean rings which is closed under extensions is both a semisimple and a radical class, that is, an *SSR class.*

Corollary 5.4. *Let $f(x) \in F[x]$. If $f(x)$ is idempotent, in the sense that $f(f(x)) = f(x)$, then $\mathcal{R}_f$ is an SSR class.*

Proof. For such an f, $\mathcal{R}_f$ is a variety which is also a radical class by Lemma 5.2 and so is closed under extensions, and hence is an SSR class. ∎

Lemma 5.5. *In the variety of modal Boolean rings, both $f(x) = x[x]_c + x$ and $g(x) = x[x]_i + x$ are idempotent.*

Proof.

$$\begin{aligned} f(f(x)) &= (x[x]_c + x)[x[x]_c + x]_c + x[x]_c + x \\ &= x[x]_c[x[x]_c + x]_c + x[x[x]_c + x]_c + x[x]_c + x \\ &= x([x]_c \vee [x[x]_c + x]_c) + x \\ &= x[x \vee (x[x]_c + x)]_c + x \\ &= x[x + x[x]_c + x + x[x]_c + x]_c + x \\ &= x[x]_c + x \\ &= f(x) \end{aligned}$$

and

$$\begin{aligned} g(x)[g(x)]_i &= (x[x]_i + x)[x[x]_i + x]_i = x[x]_i[x[x]_i + x]_i + x[x[x]_i + x]_i \\ &= x[x(x[x]_i + x)]_i + x[x[x]_i + x]_i = x[x[x]_i + x]_i + x[x[x]_i + x]_i \\ &= 0 \end{aligned}$$

so $g(g(x)) = g(x)[g(x)]_i + g(x) = g(x)$. ■

Already we have enough to assert the existence of two SSR varieties of modal Boolean rings, using Lemmas 5.2 and 5.5.

Corollary 5.6. *In the variety $\mathcal{V}$ of closure Boolean rings, $\mathcal{T}$ is an SSR class.*

Proof. In $\mathcal{V}$, $R \in \mathcal{T}$ if and only if $R \in \mathcal{R}_f$, f as in the previous lemma. By Corollary 5.4, this is an SSR class. ■

It follows that every closure Boolean ring R contains a largest type T closed ideal, $\mathcal{T}(R)$, *the $\mathcal{T}$-radical of* R, and $R/\mathcal{T}(R)$ has no non-trivial type T closed ideals. But conversely, every closure Boolean ring contains a largest closed ideal having no type T homomorphic images, *the dual $\mathcal{T}$-radical of R*, which when factored out, gives a type T closure Boolean ring. Denote by $\mathcal{T}'$ the radical class having $\mathcal{T}$ as its semisimple class.

Proposition 5.7. *Let $R = 2^X$, X a set, and let $f : X \to X$ be a function. Defining $[S]_c = f(S)$ for all $S \in R$, in the closure Boolean ring $(R, [\,]_c)$, $\mathcal{T}(R) = 2^M$ where $M = \{x | x \in X, f(x) = x\}$, and $R/\mathcal{T}(R) \cong 2^{X \setminus S}$ with closure operation determined by the partial function obtained from f by the restriction of f to $X \setminus S$.*

Proof. Let I be a closed $\mathcal{T}$-ideal of R (closed ideal of R of type T). Then for all $V \in I$, $V \subseteq f(V)$, so because I is an ideal, $x = f(x)$ for all $x \in V$. Now let $M = \{x | x \in X, f(x) = x\}$; then $f(W) = W$ for all $W \in 2^M$, so $J = 2^M$ is a closed $\mathcal{T}$-ideal of R. Obviously any other such ideal of R is contained in J which is therefore the T-radical of R. The description of $R/\mathcal{T}$ follows easily. ■

We give a nice description of the dual radical (for which $\mathcal{T}$ is the semisimple class) for the case where f is idempotent in a later section.

Proposition 5.8. *Let $(R, [\,]_c) \in \mathcal{T}$ and let $a \in R$ not be an identity for R. Then in $(R, [\,]_c^a)$, $\mathcal{T}(R) = \{ra \mid r \in R\}$, the principal ideal generated by r, and $R/\mathcal{T}(R)$ satisfies $[x]_c = [x]_i = 0$. If R has an identity then $\mathcal{T}'(R) = \langle a+1 \rangle$, the closed ideal generated by $a+1$.*

Proof. Now for all $x \in R$, $[x]_c^a = a[x]_c$, as in Theorem 2.2. A closed $\mathcal{T}$-ideal J of $(R, [\,]_c^a)$ will be such that, for all $x \in J$, $x[x]_c^a = x$, that is, $xa[x]_c = x$, that is, $ax = x$, or $x \leq a$. Let I be the principal ideal generated by a. For $s \in I$, $[s]_c^a = a[s]_c \in I$, so I is closed. Moreover it is immediate that I is the largest closed ideal J of $(R, [\,]_c^a)$ all elements of which are less than or equal to a, that is, it is the largest T-radical closed ideal of $(R, [\,]_c^a)$ and hence is $\mathcal{T}(R)$. The last part is clear.

On the other hand, suppose R has an identity. Then J is a closed ideal of R for which $(R/J, [\,]_c^a)$ is in T if and only if $x[x]_c^a + x \in J$ for all $x \in R$, that is, $xa + x = x(a+1) \in J$ for all $x \in R$, that is, $a+1 \in J$. The smallest such J, and hence the dual T-radical of R, will be the closed ideal generated by $a+1$. ■

6. Idempotent Modal Boolean Rings

Throughout this final section, we concentrate on the universal class $\mathcal{I}$ of idempotent modal Boolean rings. Let $R \in \mathcal{I}$. For $a \in R$, $I_a = \{r[a]_c | r \in R\}$ is a closed principal ideal since for all $r \in R$, $[r[a]_c]_c \leq_R [[a]_c]_c = [a]_c$, and so we call I_a the *principal closed ideal generated by a*.

We begin with the question of simplicity, which in the idempotent case leads to a result analogous to the familiar one stating that a simple commutative ring is a field. We define simple modal Boolean rings in the obvious way, and the usual relationship between maximal closed ideals and simple modal Boolean rings holds. The following result is in effect proved in [11] in the strong $S4$ case; our proof uses a different method.

Proposition 6.1. *R is simple if and only if $[x]_c = 1$ whenever $x \neq 0$.*

Proof. Any such R is obviously simple since for all non-zero $a \in R$, I_a contains $[a]_c = 1$. Conversely, if R is simple and $a \in R$ non-zero, then $I_a = R$ by simplicity, so $[a]_c$ is the maximal element in R and hence is an identity for R. ■

In $\mathcal{I}$, $f(x) = [x]_c$ and $g(x) = [x]_i$ are idempotent, so by Corollary 5.4, the variety $\mathcal{V}_c$ consisting of all R for which $[a]_c = 0$ for all $a \in R$ is an SSR class, and similarly for the variety $\mathcal{V}_i$ consisting of all R satisfying $[a]_i = 0$ for all $a \in R$. Note that $\mathcal{V}_c \subseteq \mathcal{V}_i$.

Viewing $\mathcal{V}_c$ as a semisimple class, the corresponding radical class is all R such that there is no proper closed ideal I of R for which $[a+I]_c = 0 + I$ for all $a \in R$, that is for which $[a]_c \in I$ for all $a \in R$. There is an exactly parallel result for $\mathcal{V}_i$ viewed as a semisimple class.

On the other hand, it is easy to check that the subset $\{a \mid [a]_c = 0\}$ of R is a closed ideal, so $\mathcal{R}$ is $\mathcal{V}_c$-semisimple if and only if, for all $a \in R$, $[a]_c = 0$ implies $a = 0$. For $\mathcal{V}_i$, we have

Proposition 6.2. *R is $\mathcal{V}_i$-semisimple if and only if, for all $a \in R$, $[[a]_c]_i = 0$ implies $a = 0$.*

Proof. Suppose R is $\mathcal{V}_i$-semisimple. Suppose $[[a]_c]_i = 0$ for some $a \in R$. Then $[b]_i \leq [[a]_c]_i = 0$ for all $b \in I_a$, so I_a is trivial because R is $\mathcal{V}_i$-semisimple, and so $[a]_c = 0$. But because $\mathcal{V}_c \subseteq \mathcal{V}_i$, R is $\mathcal{V}_c$-semisimple and so $a = 0$.

Conversely, suppose that for all $a \in R$, $[[a]_c]_i = 0$ implies $a = 0$. If I is a closed ideal of R and $I \in \mathcal{V}_i$, with $a \in I$, then $[a]_c \in I$ and so $[[a]_c]_i = 0$, so $a = 0$ and so I is trivial, so R is $\mathcal{V}_i$-semisimple. ■

Recall the radical class $\mathcal{L}$ of Theorem 4.1, consisting of all R such that for all $a \in R$ there exists $b \in R$ for which $a \leq [b]_c$.

Proposition 6.3. *In $\mathcal{I}$, R is $\mathcal{L}$-semisimple if and only if $R \in \mathcal{V}_c$.*

Proof. Suppose R is $\mathcal{L}$-semisimple. Then there is no non-trivial closed ideal I of R for which, for all $a \in I$, there exists $b \in I$ such that $a \leq [b]_c$. Now for $a \in R$, if the closed ideal I_a is non-trivial, then there exists $r \in I_a$ such that $r \not\leq [b]_c$ for any $b \in I_a$; in particular, $r \not\leq [[a]_c]_c = [a]_c$, a contradiction. So I_a is trivial and $[a]_c = 0$. Hence $R \in \mathcal{V}_c$.

Conversely, if $R \in \mathcal{V}_c$, then for all $a, b \in R$, $a \leq [b]_c$ implies $a = 0$, so in particular R has no non-trivial closed ideals in $\mathcal{R}_0$ and so is $\mathcal{R}_0$-semisimple. ∎

In the idempotent case, the classes of type T and classical modal rings turn out to be SSR classes, as follows from

Lemma 6.4. *Suppose $R \in \mathcal{I}$. $R \in \mathcal{T}$ if and only if $a[a]_c = a$ for all $a \in R$. $R \in \mathcal{C}$ if and only if $a[a]_i = a$ for all $a \in R$.*

Proof. Suppose $a[a]_c = a$ for all $a \in R$. For any $b \in R$, I_b is a modal Boolean ring with identity containing b and is in $\mathcal{T}$ by the first part of Lemma 2.1, so in particular $b[b]_i = [b]_i$, proving the first part.

Now suppose $a[a]_i = a$ for all $a \in R$. Now $[a]_c[b]_i = [ab + b]_i + [b]_i$ for all $a, b \in R$, so letting $b = [a]_i$ gives $[a]_c[a]_i = [a[a]_i + [a]_i]_i + [a]_i$. But $[a]_c[a]_i = [a]_i$ so $[a[a]_i + [a]_i]_i = 0$. So $a[a]_i + [a]_i = (a[a]_i + [a]_i)[a[a]_i + [a]_i]_i = 0$, for all $a \in R$, an identity which therefore holds in I_a also. So by the second part of Lemma 2.1, $I_a \in \mathcal{C}$ and in particular $[a]_i = [a]_c = a$ and the second part is proved. ∎

Corollary 6.5. *In $\mathcal{I}$, $\mathcal{C}$ and $\mathcal{T}$ are SSR classes.*

Proof. Let $f(x) = x[x]_c + x$ and $g(x) = x[x]_i + x$. Then $\mathcal{R}_f$ and $\mathcal{R}_g$ are $\mathcal{T}$ and $\mathcal{C}$ respectively, by Lemma 6.4, and are SSR classes by Corollary 5.4 and Lemma 5.5. ∎

Similar comments to those made after the proof of Corollary 5.6 apply: every idempotent modal Boolean ring contains a largest closed ideal in $\mathcal{T}$ (resp. $\mathcal{C}$) which when factored out, leaves a modal Boolean ring with no closed ideals in $\mathcal{T}$ (resp. $\mathcal{C}$), but also, every idempotent modal Boolean ring has a largest closed ideal having no homomorphic images in $\mathcal{T}$ (resp. $\mathcal{C}$) which when factored out gives a modal Boolean ring in $\mathcal{T}$ (resp. $\mathcal{C}$).

Proposition 6.6. *Let X be a topological space with closure operator C, and let $R = 2^X$. Then in (R, C), $\mathcal{C}(R)$ is all subsets of X all of whose subsets are both open and closed.*

Proof. Let I consist of all subsets of X which are both closed and open and all of whose subsets are. It is clear that I is a closed ideal of R, $I \in \mathcal{C}$, and moreover any closed ideal J of R is contained in I since J is closed under taking subsets of its elements and all of its elements must satisfy $[x]_i = [x]_c = x$. ∎

If X is finite, $\mathcal{C}(2^X)$ consists of the subsets of X all of whose elements define clopen singleton sets. In the semisimple case in which $\mathcal{C}(2^X) = \{\emptyset\}$, there is no $x \in X$ for which $\{x\}$ is clopen.

Proposition 6.7. *Let $R = 2^X$, X a set, and let $f : X \to X$ be an idempotent function. Then defining $[S]_c = f(S)$ for all $S \subseteq X$ and defining $[\,]_i$ so that $(R, [\,]_i, [\,]_c)$ is strong, $T'(R) = 2^M$ where $M = f(X)' \cup f(f(X)')$.*

Proof. Now $f(M) = f(f(X)') \subseteq M$ so $K = 2^M$ is a closed ideal of R. For any $W \in R$, $a \in W \cap f(W)'$ means $a \in W$ but $a \notin f(W)$ so in particular $a \neq f(a)$, so $a \in M$. Thus in R, $W + Wf(W) = W \cap f(W)' \in K$ and so R/K is of type T. But if I is a closed ideal of R for which R/I is of type T, then in particular $f(X)' = X \cap f(X)' = X + Xf(X) \in I$, so $f(f(X)') \in I$ since I is closed, and so $M = f(X)' \cup f(f(X)') \in I$. Thus K is the smallest closed ideal which when factored out, gives a modal Boolean ring in T; that is, it is the dual T-radical of R. ∎

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