Vietnam Journal of Mathematics **29**:2 (2001) 165–177

Vietnam Journal of MATHEMATICS

Dimension and Width of Linearly Compact Modules and the Co-Localization of Artinian Modules*

Le Thanh Nhan

Institute of Mathematics, P. O. Box 631, Bo Ho, Hanoi, Vietnam

Received April 4, 2000

Abstract. Some properties of dimension and width of linearly compact modules with respect to the discrete topology and the co-localization of Artinian modules are given.

1. Introduction

This paper is concerned with linearly compact modules over Noetherian rings. Note that the linear compactness was introduced by Lefschetz [8] for vector spaces of arbitrary dimension and extended to modules by Zelinsky [20] and it plays important role for duality in algebra (see [9]). These modules were studied later by Zöschinger [21] and others. Although the class of linearly compact modules with respect to the discrete topology contains strictly all Artinian modules, we can show that many results of dimension and width of these two classes are similar.

Melkersson and Schenzel [12] defined a so-called co-localization $\mathrm{Hom}_R(R_S; M)$ of an R-module M with respect to a multiplicative set S in R. When M is Artinian, they showed that $\mathrm{Hom}_R(R_S; M)$ is almost never Artinian, but it has many good facts inherited from M. Moreover, it is linearly compact by [2], but it is not usually linearly compact with respect to the discrete topology.

The purpose of this paper is to study the dimension and the width of linearly

* This work was supported in part by the National Basic Research Programm in Natural Science, Vietnam.

compact modules with respect to the discrete topology and the co-localization of Artinian modules.

This paper is divided into five sections. In the next section we study linearly compact modules with respect to discrete topology. In Sec. 3 we are interested in the co-localization of Artinian modules. We will prove that the result of Ooishi [13] of width for Artinian modules is still true for linearly compact modules with respect to the discrete topology (Theorem 2.8) and for the co-localization of Artinian modules (Theorem 3.3). Roberts [14] introduced the notion of Krull dimension for all modules and gave some basic results of Krull dimension for Artinian modules. Kirby [7] changed the terminology of Roberts and referred to Noetherian dimension (N-dim) to avoid confusion with Krull dimension defined for Noetherian modules. In this note we use the terminology of Kirby [7]. Denote by $\dim M$ the Krull dimension of the Noetherian ring $R/\operatorname{Ann} M$. We shall give some relations of dimension of linearly compact modules with respect to the discrete topology (Theorem 2.4). Moreover, we extend the main result of [14] for Artinian modules to the co-localization (Theorem 3.6). In Sec. 4, we consider co-Cohen-Macaulay modules which were studied by Tang and Zakeri [17], Denizler and Sharp [4] etc. In Sec. 5 we give several examples to clarify the results in this paper.

2. Linearly Compact Modules with Respect to the Discrete Topology

First we recall the concept of linearly compact modules by using the terminology of Macdonald [9].

Definition 2.1. (i) *A topological module M over a topological ring R is said to be linearly topologized if M has a nuclear base $\mathcal{M}$ consisting of open submodules which satisfies the condition: given $x \in M$ and $N \in \mathcal{M}$, there exists a nucleus U of R such that $Ux \subseteq N$.*
(ii) *A Hausdorff linearly topologized R-module M is said to be linearly compact if M has the following property: if $\mathcal{F}$ is a family of closed cosets (i.e., the cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection.*

Observe that M is linearly compact with respect to the discrete topology if and only if any finitely solvable system of congruences $x = x_k (\operatorname{mod} M_k)$, where M_k are submodules of M, is solvable. So we can forget the topological structure of these modules. It should be mentioned that the class of linearly compact modules with respect to the discrete topology contains strictly all Artinian modules.

From now on, we always assume that R is a Noetherian ring and M an R-module. Here are some facts which are often used in this paper.

Lemma 2.2. [21] *Suppose that M is linearly compact with respect to the discrete topology. Then we have*

(i) *There exists a Noetherian submodule B of M such that M/B is Artinian.*
(ii) *If $f : M \longrightarrow M$ is a surjective homomorphism then* $\operatorname{Ker} f$ *is Artinian.*

As mentioned in the introduction, the notion of Krull dimension for modules is given by Roberts [14]. Kirby [7] changed the terminology of Roberts and referred to Noetherian dimension (N-dim) to avoid confusion with Krull dimension defined for Noetherian modules. We recall first this concept using the terminology of Kirby [7].

Definition 2.3. *The Noetherian dimension of M, denoted by* N-dim M, *is defined inductively as follows: When $M = 0$, put* N-dim $M = -1$. *Then by induction, for an integer $d \geq 0$, we put* N-dim $M = d$ *if* N-dim $M < d$ *is false and for every ascending sequence $M_0 \subseteq M_1 \subseteq \dots$ of submodules of M, there exists n_0 such that* N-dim$(M_n/M_{n+1}) < d$ *for all $n > n_0$.*

It is clear that N-dim $M = 0$ if and only if M is a non-zero Noetherian module.

The theory of secondary representation, which is in some sense dual to that of primary decomposition of Noetherian modules, is due to Macdonald [10]. A module M is said to be representable if M has a secondary representation. If M is representable then the set of attached prime ideals is denoted by $\operatorname{Att} M$. Any Artinian module is representable (see [10] for more details).

Remark. We denote by $\dim M$ the Krull dimension of the Noetherian ring $R/\operatorname{Ann} M$. For convenience we put $\dim M = -1$ if $M = 0$. On the other hand, Yassemi in [18] also defined the set of coassociated prime ideals ($\operatorname{Coass} M$) of M and then defined in [19] the notion of magnitude ($\operatorname{mag} M$) of M. Note that if M is representable then $\operatorname{Coass} M$ is just the set $\operatorname{Att} M$ (see [18]) and the set of minimal prime ideals of $\operatorname{Ann} M$ is just the set of minimal elements of $\operatorname{Att} M$ (see [10]). In this case, $\operatorname{mag} M$ is therefore nothing else but $\dim M$.

We summarize some known facts of dimension of modules in [1, 3, 7, 19] as follows.

(i) Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$ be an exact sequence of R-modules. Then we have
$\operatorname{mag} M = \max\{\operatorname{mag} M', \operatorname{mag} M"\}$; N-dim $M = \max\{$N-dim M', N-dim $M"\}$.

(ii) If M is Noetherian then N-dim $M = \operatorname{mag} M = 0$.

(iii) Suppose that M is Artinian. The following statements are true:

(a) N-dim $M < \infty$.

(b) Let $J(M)$ be the intersection of all elements in $\operatorname{Supp} M$ (note that $J(M) = \mathfrak{m}$ if $(R, \mathfrak{m})$ is a local ring). Then for n large enough $\ell\big((0 : J(M)^n)_M\big)$ is a polynomial with rational coefficients and

$$\begin{aligned}\text{N-dim}\, M &= \deg\big(\ell((0 : J(M)^n)_M)\big)\\ &= \inf\{t \geq 0 \mid \exists x_1, \dots, x_t \in J(M) : \ell(0_M : (x_1, \dots, x_t)R) < \infty\}.\end{aligned}$$

(c) N-dim $M = 0$ if and only if $\dim M = 0$. In this case, the length of M is finite and the ring $R/\operatorname{Ann} M$ is Artinian.

(d) N-dim $M \leq \dim M$.

Note that there exist by [3, 4.1] Artinian modules M over Noetherian local rings R for which N-dim $M < \dim M$. Moreover, N-dim M is always finite, but $\dim M$ may be infinite when R is non-local (see [3, 4.2]).

Although the class of linearly compact modules with respect to the discrete topology contains strictly all Artinian modules, we have the following general result.

Theorem 2.4. *Let M be linearly compact with respect to the discrete topology. Then we have*

(i) N-dim $M < \infty$,

(ii) N-dim $M \leq \operatorname{mag} M \leq \dim M$.

Proof. (i) There exists an exact sequence $0 \longrightarrow B \longrightarrow M \longrightarrow M/B \longrightarrow 0$ by Lemma 2.2 (i), where B is Noetherian and M/B is Artinian. Since N-dim $B = 0$, we have N-dim $M =$ N-dim(M/B). Since N-dim(M/B) is finite, so is N-dim M. (ii) Because $\operatorname{mag} B = 0$, we get $\operatorname{mag} M = \operatorname{mag}(M/B) = \dim(M/B)$. Now the assertion follows from the above remark. ■

Now we recall the concept of width of modules, which is in some sense dual to the notion of depth of modules (see [13]).

Definition 2.5. (i) *A sequence of elements $x_1, \dots, x_n$ in R is called an M-cosequence if it satisfies the following conditions*

(a) $x_i(0 : (x_1, \dots, x_{i-1})R)_M = (0 : (x_1, \dots, x_{i-1})R)_M$ *for* $1 \leq i \leq n$,

(b) $(0 : (x_1, \dots, x_n)R)_M \neq 0$.

(ii) *For an ideal $\mathfrak{a}$ of R, the width of M in $\mathfrak{a}$, denoted by* $\operatorname{Width}_{\mathfrak{a}} M$, *is the supremum of the lengths of M-cosequences in $\mathfrak{a}$. If $(R, \mathfrak{m})$ is a local ring then we write* Width M *for* $\operatorname{Width}_{\mathfrak{m}} M$.

Ooishi [13] proved that, if M is Artinian and $\mathfrak{a}$ is an ideal of R such that $(0 : \mathfrak{a})_M \neq 0$ then the length of an M-cosequence in $\mathfrak{a}$ is finite, two maximal M-cosequences in $\mathfrak{a}$ have the same length, and

$$\operatorname{Width}_{\mathfrak{a}} M = \inf\{t : \operatorname{Tor}_t^R(R/\mathfrak{a}; M) \neq 0\}.$$

We shall show below that this result is still true for linearly compact modules with respect to the discrete topology. First we need the following lemmas.

Lemma 2.6. *Let $\mathfrak{a}$ be an ideal of R. Then we have*

$$\operatorname{Width}_{\mathfrak{a}} M \leq \text{N-dim}\, M.$$

In particular, if M is linearly compact with respect to the discrete topology then $\operatorname{Width}_{\mathfrak{a}} M < \infty$.

Proof. Let $x_1, \dots, x_t$ be a M-cosequence in $\mathfrak{a}$. We will show that $t \leq$ N-dim M. If $t = 0$ then there is nothing to do. Suppose that $t > 0$. Since $x_1 M = M$, there exists an exact sequence

$$0 \longrightarrow (0 : x_1^n R)_M \longrightarrow (0 : x_1^{n+1} R)_M \xrightarrow{.x_1^n} (0 : x_1 R)_M \longrightarrow 0$$

for all $n > 0$. Hence $(0 : x_1^{n+1}R)_M/(0 : x_1^n R)_M \cong (0 : x_1 R)_M$ for all $n > 0$. From the ascending sequence of submodules of M

$$(0 : x_1 R)_M \subseteq (0 : x_1^2 R)_M \subseteq \ldots \subseteq (0 : x_1^n R)_M \subseteq \ldots$$

we have $\text{N-dim}\,\big((0 : x_1^{n+1}R)_M/(0 : x_1^n R)_M\big) \leq \text{N-dim}\, M - 1$ for n large enough. Therefore $\text{N-dim}(0 : x_1 R)_M \leq \text{N-dim}\, M - 1$. Now the lemma follows by induction on t. ∎

The following lemma can be proved easily by induction.

Lemma 2.7. *Let $\mathfrak{a}$ be an ideal of R and $x_1, \ldots, x_n$ an M-cosequence in $\mathfrak{a}$. Then we have*
(i) $\operatorname{Tor}_i^R(M; R/\mathfrak{a}) = 0$ *for* $i \leq n-1$.
(ii) $\operatorname{Tor}_n^R(M; R/\mathfrak{a}) \cong \big(0 : (x_1, \ldots, x_n)R\big)_M \otimes R/\mathfrak{a}$.

Theorem 2.8. *Let M be a linearly compact R-module with respect to the discrete topology and $\mathfrak{a}$ an ideal of R. Then we have*
(i) *The length of an M-cosequence in $\mathfrak{a}$ is finite.*
(ii) *If $(0 : \mathfrak{a})_M \neq 0$ then two maximal M-cosequences in $\mathfrak{a}$ have the same length and*

$$\operatorname{Width}_{\mathfrak{a}} M = \inf\{n \geq 0 : \operatorname{Tor}_n^R(M; R/\mathfrak{a}) \neq 0\}.$$

Proof. (i) follows from Lemma 2.6.
(ii) Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be maximal M-cosequences in $\mathfrak{a}$. Assume that $n > m$. We have by Lemma 2.7 that

$$\operatorname{Tor}_m^R(M; R/\mathfrak{a}) = 0 \cong \big(0 : (y_1, \ldots, y_m)R\big)_M \otimes R/\mathfrak{a}.$$

Since $n > 0$ and $y_1, \ldots, y_m$ is a maximal M-cosequence in $\mathfrak{a}$, it follows that $m > 0$. Hence $\big(0 : (y_1, \ldots, y_m)R\big)_M$ is a non-zero Artinian module by Lemma 2.2 (ii). Since $\big(0 : (y_1, \ldots, y_m)R\big)_M = \mathfrak{a}\big(0 : (y_1, \ldots, y_m)R\big)_M$, there exists $y_{m+1} \in \mathfrak{a}$ such that

$$\big(0 : (y_1, \ldots, y_m)R\big)_M = y_{m+1}\big(0 : (y_1, \ldots, y_m)R\big)_M.$$

Because $(0 : \mathfrak{a})_M \neq 0$, it follows that $y_1, \ldots, y_{m+1}$ is an M-cosequence in $\mathfrak{a}$. It gives a contradiction. Now the remainder of the claim is derived by Lemma 2.7 ∎

Corollary 2.9. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a} \subseteq \mathfrak{m}$ an ideal of R and M a linearly compact R-module with respect to the discrete topology. Denote by $\operatorname{soc} M$ the largest sum of simple submodules of M. Then the lengths of two maximal M-cosequences in $\mathfrak{a}$ are finite and the same, and*
(i) *If* $\operatorname{soc} M = 0$ *then* $\operatorname{mag} M \leq 1$ *and* $\operatorname{Width} M = 0$.
(ii) *Otherwise,* $\operatorname{Width}_{\mathfrak{a}} M = \inf\{n \geq 0 : \operatorname{Tor}_n^R(M; R/\mathfrak{a}) \neq 0\}$.

Proof. (i) If $\operatorname{soc} M = 0$ then we get by [21, 1.6] and [18, 1.15] that $\operatorname{mag} M \leq 1$ and $(0 : xR)_M = 0$ for all $x \in R$ satisfying $xM = M$. Thus $\operatorname{Width} M = 0$.
(ii). Clearly, $\operatorname{soc} M = (0 : \mathfrak{m})_M$. So $(0 : \mathfrak{m})_M \neq 0$. Now the assertion follows from Theorem 2.8. ∎

Remark. The condition $(0 : \mathfrak{a})_M \neq 0$ in Corollary 2.9 (ii) is necessary (see Example 5.1).

3. Co-Localization of Artinian Modules

In this section we assume in addition that M is an Artinian R-module.

Melkersson and Schenzel in [12] defined so-called co-localization of all modules. In this paper we only study the co-localization of Artinian modules. First we recall this notion.

Definition 3.1. *Let S be a multiplicative set in R. The co-localization of M with respect to S is the module* $\operatorname{Hom}_R(R_S; M)$.
The set $\operatorname{Cos} M = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{Hom}_R(R_\mathfrak{p}; M) \neq 0\}$ *is called the co-support of M.*

For convenience, we write ${}_SM$ for $\operatorname{Hom}_R(R_S; M)$. In particular, for each prime ideal $\mathfrak{p}$ of R, we write ${}_\mathfrak{p}M$ for $\operatorname{Hom}_R(R_\mathfrak{p}; M)$. We recall some facts in [12] which are often used in the rest of this paper.

Lemma 3.2. *The following statements are true:*
(i) $\operatorname{Cos} M$ *is the set of all prime ideal containing* $\operatorname{Ann} M$.
(ii) *For all* $\mathfrak{p} \in \operatorname{Cos} M$, *the co-localization* ${}_\mathfrak{p}M$ *is representable and*

$$\operatorname{Att}_{R_\mathfrak{p}}({}_\mathfrak{p}M) = \{\mathfrak{q}R_\mathfrak{p} : \mathfrak{q} \subseteq \mathfrak{p}, \mathfrak{q} \in \operatorname{Att} M\}.$$

(iii) *Suppose that* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *is an exact sequence of Artinian modules. Then for any multiplicative set S of R, the following sequence is exact*

$$0 \longrightarrow {}_S M' \longrightarrow {}_S M \longrightarrow {}_S M'' \longrightarrow 0.$$

Note that the co-localization of Artinian module is not usually Artinian (see [12]). However, Ooishi's result of the width for Artinian modules is still true for the co-localization.

Theorem 3.3. *Let* $\mathfrak{p} \in \operatorname{Cos} M$ *and* $\mathfrak{a} \subseteq \mathfrak{p}$ *an ideal of R. If* $(0 : \mathfrak{a}R_\mathfrak{p})_{\mathfrak{p}M} \neq 0$ *then the length of every* ${}_\mathfrak{p}M$*-cosequence in* $\mathfrak{a}R_\mathfrak{p}$ *is finite, two maximal* ${}_\mathfrak{p}M$*-cosequences in* $\mathfrak{a}R_\mathfrak{p}$ *have the same length and*

$$\operatorname{Width}_{\mathfrak{a}R_\mathfrak{p}}({}_\mathfrak{p}M) = \inf\{n \geq 0 : \operatorname{Tor}_n^{R_\mathfrak{p}}({}_\mathfrak{p}M; R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}) \neq 0\}.$$

Proof. For any element $x \in \mathfrak{p}$, observe that $(0 : xR_\mathfrak{p})_{{}_\mathfrak{p}M} \cong {}_\mathfrak{p}((0 : xR)_M)$. So $(0 : xR_\mathfrak{p})_{{}_\mathfrak{p}M}$ is representable by Lemma 3.2 (ii). Let $x_1, \dots, x_t$ be an arbitrary

${}_{\mathfrak{p}}M$-cosequence in $\mathfrak{a}R_{\mathfrak{p}}$. Then by induction on t we can show that $t \leq \dim_{R_{\mathfrak{p}}}({}_{\mathfrak{p}}M)$. Therefore

$$\mathrm{Width}_{\mathfrak{a}R_{\mathfrak{p}}}({}_{\mathfrak{p}}M) \leq \dim_{R_{\mathfrak{p}}}({}_{\mathfrak{p}}M) < \infty.$$

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be maximal ${}_{\mathfrak{p}}M$-cosequences in $\mathfrak{a}R_{\mathfrak{p}}$. Assume $n > m$. We have by Lemma 2.7 that

$$\mathrm{Tor}_m^{R_{\mathfrak{p}}}({}_{\mathfrak{p}}M; R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}) = 0 \cong \big(0 : (y_1, \ldots, y_m)R_{\mathfrak{p}}\big)_{{}_{\mathfrak{p}}M} \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}.$$

So

$$\big(0 : (y_1, \ldots, y_m)R_{\mathfrak{p}}\big)_{{}_{\mathfrak{p}}M} = \mathfrak{a}R_{\mathfrak{p}}\big(0 : (y_1, \ldots, y_m)R_{\mathfrak{p}}\big)_{{}_{\mathfrak{p}}M}.$$

Since $\big(0 : (y_1, \ldots, y_m)R_{\mathfrak{p}}\big)_{{}_{\mathfrak{p}}M}$ is representable, there exists $y_{m+1} \in \mathfrak{a}R_{\mathfrak{p}}$ such that

$$\big(0 : (y_1, \ldots, y_m)R_{\mathfrak{p}}\big)_{{}_{\mathfrak{p}}M} = y_{m+1}\big(0 : (y_1, \ldots, y_m)R_{\mathfrak{p}}\big)_{{}_{\mathfrak{p}}M}.$$

Because $\big(0 : \mathfrak{a}R_{\mathfrak{p}}\big)_{{}_{\mathfrak{p}}M} \neq 0$, it follows that $y_1, \ldots, y_{m+1}$ is a ${}_{\mathfrak{p}}M$-cosequence in $\mathfrak{a}R_{\mathfrak{p}}$. It gives a contradiction. Now the remaining statement is derived by Lemma 2.7. ∎

Note that $\mathrm{Ann}(N/\mathfrak{p}N) = \mathfrak{p}$ for any finitely generated R-module N and any prime ideal $\mathfrak{p}$ containing $\mathrm{Ann}\, N$. However, the dual result for Artinian modules is not true. Therefore, we need the following definition (see [3, 4.3]).

Definition 3.4. *Suppose that $(R, \mathfrak{m})$ is a local ring. We say that M satisfies the condition $(*)$ if $\mathrm{Ann}(0 : \mathfrak{p})_M = \mathfrak{p}$ for all prime ideal $\mathfrak{p}$ containing $\mathrm{Ann}\, M$.*

Remark. (i) It is known by [3, 4.4] that there exists an Artinian module M over local ring R for which M does not satisfy the condition $(*)$. However, there are still many Artinian modules which satisfy the condition $(*)$ (see [3, 4.5]).
(ii). Note that the co-localization of Artinian modules, which satisfy the condition $(*)$, may have infinitely many associated primes (see Example 5.2). So they have infinite Goldie dimension. Therefore they are not linearly compact with respect to the discete topology. However, they are linearly compact by [2].

Corollary 3.5. *Suppose that $(R, \mathfrak{m})$ is a local ring and M satisfies the condition $(*)$. Let $\mathfrak{p} \in \mathrm{Cos}\, M$ and $\mathfrak{a}$ be an ideal contained in $\mathfrak{p}$. Then the length of any ${}_{\mathfrak{p}}M$-cosequence in $\mathfrak{a}R_{\mathfrak{p}}$ is finite, two maximal ${}_{\mathfrak{p}}M$-cosequences in $\mathfrak{a}R_{\mathfrak{p}}$ have the same length and*

$$\mathrm{Width}_{\mathfrak{a}R_{\mathfrak{p}}}({}_{\mathfrak{p}}M) = \inf\{n \geq 0 : \mathrm{Tor}_n^{R_{\mathfrak{p}}}({}_{\mathfrak{p}}M; R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}) \neq 0\}.$$

Proof. We have only to prove by Theorem 3.3 that $\big(0 : \mathfrak{a}R_{\mathfrak{p}}\big)_{{}_{\mathfrak{p}}M} \neq 0$. Since M satisfies the condition $(*)$, $\mathrm{Ann}(0 : \mathfrak{a})_M \subseteq \mathrm{Ann}(0 : \mathfrak{p})_M = \mathfrak{p}$. So $\mathfrak{p} \in \mathrm{Cos}(0 : \mathfrak{a})_M$. Hence ${}_{\mathfrak{p}}\big((0 : \mathfrak{a})_M\big) \neq 0$. It leads to $\big(0 : \mathfrak{a}R_{\mathfrak{p}}\big)_{{}_{\mathfrak{p}}M} \neq 0$. ∎

Remark. The condition $(*)$ in Corollary 3.5 is necessary (see Example 5.3 (i)).

Let $(R,\mathfrak{m})$ be a local ring. Then the main result of Roberts [14] is that

$$\text{N-dim}\, M = \inf\{t : \exists x_1,\dots,x_t \in \mathfrak{m} : \ell(0:(x_1,\dots,x_t)R)_M < \infty\}.$$

It is known by [3,6] that N-dim $M = \dim M$ if M satisfies the condition $(*)$. Therefore, the theorem below is an extension of this result to the co-localization.

Theorem 3.6. *Suppose that $(R,\mathfrak{m})$ is a local ring and M satisfies the condition $(*)$. Let $\mathfrak{p} \in \operatorname{Cos} M$. Then we have*

$$\dim_{R_\mathfrak{p}}({}_\mathfrak{p}M) = \inf\{t : \exists x_1,\dots,x_t \in \mathfrak{p}R_\mathfrak{p} : \dim\big(0:(x_1,\dots,x_t)R_\mathfrak{p}\big)_{{}_\mathfrak{p}M} = 0\}.$$

Proof. Let $\mathfrak{q}R_\mathfrak{p}$ be a prime ideal of $R_\mathfrak{p}$ which contains $\operatorname{Ann}_{R_\mathfrak{p}}({}_\mathfrak{p}M)$. Since M satisfies the condition $(*)$ and $\mathfrak{q} \in \operatorname{Cos}_R M$, we have $\operatorname{Ann}_R(0:\mathfrak{q})_M = \mathfrak{q}$. Hence $\mathfrak{q} \in \operatorname{Att}_R(0:\mathfrak{q})_M$. So we get by Lemma 3.2 that $\mathfrak{q}R_\mathfrak{p} \in \operatorname{Att}_{R_\mathfrak{p}}\big(0:\mathfrak{q}R_\mathfrak{p}\big)_{{}_\mathfrak{p}M}$. Hence $\operatorname{Ann}_{R_\mathfrak{p}}\big(0:\mathfrak{q}R_\mathfrak{p}\big)_{{}_\mathfrak{p}M} = \mathfrak{q}R_\mathfrak{p}$. Therefore

$$\operatorname{rad}\big(\operatorname{Ann}_{R_\mathfrak{p}}(0:xR_\mathfrak{p})_{{}_\mathfrak{p}M}\big) = \operatorname{rad}\big(xR_\mathfrak{p} + \operatorname{Ann}({}_\mathfrak{p}M)\big), \qquad (1)$$

for all elements x in $\mathfrak{p}$. Let $\dim_{R_\mathfrak{p}}({}_\mathfrak{p}M) = d$. Now we prove the theorem by induction on d. If $d = 0$, then it is trivial. Suppose $d > 0$. Since $\operatorname{Att}_{R_\mathfrak{p}}({}_\mathfrak{p}M)$ is a finite set, there are only finitely many prime ideals $\mathfrak{p}_1R_\mathfrak{p},\dots,\mathfrak{p}_kR_\mathfrak{p} \in \operatorname{Att}_{R_\mathfrak{p}}({}_\mathfrak{p}M)$ such that $d = \dim R_\mathfrak{p}/\mathfrak{p}_iR_\mathfrak{p}$ for all $i = 1,2,\dots,k$. Since $d > 0$, we can choose an element $x \in \mathfrak{p} \setminus \bigcup_{1\le i\le k} \mathfrak{p}_i$. So $\dim_{R_\mathfrak{p}}\big(0:xR_\mathfrak{p}\big)_{{}_\mathfrak{p}M} = d-1$ by (1). By induction hypothesis we can find $d-1$ elements $x_2,\dots,x_d$ in $\mathfrak{p}$ such that $\dim_{R_\mathfrak{p}}\big(0:(x,x_1,\dots,x_d)R_\mathfrak{p}\big)_{{}_\mathfrak{p}M} = 0$. Conversely, suppose that $x_1,\dots,x_t \in \mathfrak{p}R_\mathfrak{p}$ satisfying $\dim_{R_\mathfrak{p}}\big(0:(x_1,\dots,x_t)R_\mathfrak{p}\big)_{{}_\mathfrak{p}M} = 0$. Then we have by (1) that $d \,\square\, t$ because $R_\mathfrak{p}$ is a local ring. It finishes the proof. ∎

Remark. Theorem 3.6 may be not true when M does not satisfy the condition $(*)$ (see Example 5.3 (ii)).

Note that ${}_\mathfrak{p}M$ is representable. Therefore $\operatorname{mag}_{R_\mathfrak{p}}({}_\mathfrak{p}M)$ and $\dim_{R_\mathfrak{p}}({}_\mathfrak{p}M)$ are the same for any prime ideal $\mathfrak{p}$ in R. However, $\text{N-dim}_{R_\mathfrak{p}}({}_\mathfrak{p}M)$ and $\dim_{R_\mathfrak{p}}({}_\mathfrak{p}M)$ may not be so. Moreover, from $\dim_{R_\mathfrak{p}}({}_\mathfrak{p}M) = 0$ one cannot imply $\ell_{R_\mathfrak{p}}({}_\mathfrak{p}M) < \infty$ (see Example 5.4). Therefore we give below a characterization of representable modules of finite length.

Proposition 3.7. *Let R be a Noetherian ring and S a representable R-module. Then the following statements are equivalent:*

(i) $\ell(S) < \infty$;

(ii) N-dim $S = 0$;

(iii) $\dim S = 0$ *and S has finite Goldie dimension.*

Proof. (i)⇒(ii). Trivial.

(ii)⇒(iii). Suppose N-dim $S = 0$. Then S is Noetherian and hence $\operatorname{mag} S = 0$. Since S is representable, $\dim S = 0$.

(iii) $\Rightarrow$ (i). Suppose $\dim S = 0$. Then $\operatorname{Att} S \subseteq \operatorname{Max} R$. Let $\operatorname{Att} S = \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ and $S = S_1 + \dots + S_r$ be a secondary representation of S with S_i being $\mathfrak{m}_i$-secondary for $1 \leq i \leq r$. For each i, since R is Noetherian, there exists a positive integer n_i such that $\mathfrak{m}_i^{n_i} S_i = 0$. We can easily check that if $\mathfrak{m}_i^t S_i \neq 0$ then $\mathfrak{m}_i^t S_i$ is $\mathfrak{m}_i$-secondary for all positive integers t. Now, consider the exact sequence

$$0 \longrightarrow \mathfrak{m}_i^{t+1} S_i \longrightarrow \mathfrak{m}_i^t S_i \longrightarrow \mathfrak{m}_i^t S_i / \mathfrak{m}_i^{t+1} S_i \longrightarrow 0.$$

Since $\mathfrak{m}_i^t S_i$ and $\mathfrak{m}_i^{t+1} S_i$ are representable of finite Goldie dimension, $\mathfrak{m}_i^t S_i / \mathfrak{m}_i^{t+1} S_i$ has finite Goldie dimension by [11, 3.3]. Therefore it is an $R/\mathfrak{m}_i$-vector space of finite dimension. Now from the sequence $0 = \mathfrak{m}_i^{n_i} S_i \subseteq \mathfrak{m}_i^{n_i - 1} S_i \subseteq \dots \subseteq S_i$ we get $\ell(S_i) < \infty$ for all $i = 1, \dots, r$. Therefore $\ell(S) < \infty$. ∎

4. Co-Cohen-Macaulay Modules

In this section, assume that R is a commutative ring (not necessarily Noetherian) and M an Artinian R-module.

Recall that the inverse polynomial module $M[X_1^{-1}, \dots, X_n^{-1}]$ can be considered as an Artinian module over the polynomial ring $R[X_1, \dots, X_n]$ (see [6]). The following result is related to these modules.

Proposition 4.1. *With the same notations as above, let* $S = R[X_1, \dots, X_n]$ *and* $K = M[X_1^{-1}, \dots, X_n^{-1}]$. *Then we have*

$$\text{N-dim}_S K = \text{N-dim}_R M + n.$$

Proof. By induction, we have only to prove the proposition for the case $n = 1$. Let $X_1 = X$ and $K = M[X^{-1}]$. First, we can easily check that $XK = K$. Next, we claim that $X \in J(K)$, where we denote by $J(K)$ the intersection of all elements in $\operatorname{Supp}_S K$. In fact, we have by [15, 1.4] that $M = M_1 \oplus M_2 \oplus \dots \oplus M_r$, where $\operatorname{Supp} M = \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ and $M_i = \bigcup_{n \geq 0} (0 : \mathfrak{m}_i^n)_M$ for $i = 1, \dots, r$. Let

$$\mathfrak{m}_i^* = \mathfrak{m}_i \oplus RX \oplus \dots \oplus RX^n \oplus \dots, \quad \text{for } i = 1, \dots, r.$$

Clearly, $\mathfrak{m}_i^*$ is a maximal ideal of S and

$$K = M_1[X^{-1}] \oplus M_2[X^{-1}] \oplus \dots \oplus M_r[X^{-1}]. \tag{a}$$

Moreover

$$\bigcup_{n \geq 0} (0 : (\mathfrak{m}_i^*)^n)_K \supseteq M_i[X^{-1}] \neq 0 \tag{b}$$

for $i = 1, \dots, r$. We get by (a), (b) and [15, 1.4] that $\operatorname{Supp} K = \{\mathfrak{m}_1^*, \dots, \mathfrak{m}_r^*\}$. Thus $X \in J(K)$. The claim is proved. Since X is a K-coregular element in $J(K)$, we have

$$\text{N-dim}_S K - 1 = \text{N-dim}_S (0 : X)_K = \text{N-dim}_S M = \text{N-dim}_R M. \quad ∎$$

Now we need the notion of co-Cohen-Macaulay modules (see [4] and [17]).

Definition 4.2. (i) *An Artinian module M over a quasi-local ring $(R, \mathfrak{m})$ is called co-Cohen-Macaulay (CCM, for short) if* $\text{N-dim}\, M = \text{Width}\, M$.
(ii) *An Artinian module M over a commutative ring R is called co-Cohen-Macaulay if $M_{\mathfrak{m}}$ is co-Cohen-Macaulay as $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$.*

Theorem 4.3. *Let M, S, K be as in Proposition* 4.1. *Then M is a CCM R-module if and only if K is a CCM S-module.*

Proof. By induction, we have only to prove for the case $n = 1$. By using the same notations in the proof of Proposition 4.1 we have

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_r; \quad \text{Supp}\, M = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$$

and $M_{\mathfrak{m}_i} \cong M_i$ for $i = 1, \ldots, r$. Therefore

$$K = K_1 \oplus \ldots \oplus K_r; \quad \text{Supp}\, K = \{\mathfrak{m}_1^*, \ldots, \mathfrak{m}_r^*\}$$

and $K_{\mathfrak{m}_i^*} \cong K_i$ for $i = 1, \ldots, r$, where $K_i = M_i[X^{-1}]$. So, for $i = 1, \ldots, r$, we obtain by Proposition 4.1 and [15, 1.7] that

$$\text{N-dim}_{S_{\mathfrak{m}_i^*}} K_i = \text{N-dim}_S K_i = \text{N-dim}_R M_i + 1 = \text{N-dim}_{R_{\mathfrak{m}_i}} M_i + 1. \quad (2)$$

Next, let $a_1, \ldots, a_n$ be a maximal M_i-cosequence in $\mathfrak{m}_i$. Then $a_1, \ldots, a_n, X$ is a maximal K_i-cosequence in $\mathfrak{m}_i^*$. In fact, clearly $a_1, \ldots, a_n, X$ is a K_i-cosequence in $\mathfrak{m}_i^*$. Suppose that $a_1, \ldots, a_n, X, Y$ is a K_i-cosequence in $\mathfrak{m}_i^*$. Let $Y = b_0 + b_1 X + \ldots + b_t X^t$ with $b_1, \ldots, b_t \in R$ and $b_0 \in \mathfrak{m}_i$. Then

$$Y\big(0 : (a_1, \ldots, a_n, X) S_{\mathfrak{m}_i^*}\big)_{K_i} = \big(0 : (a_1, \ldots, a_n, X) S_{\mathfrak{m}_i^*}\big)_{K_i}.$$

It leads to $b_0\big(0 : (a_1, \ldots, a_n) R_{\mathfrak{m}_i}\big)_{M_i} = \big(0 : (a_1, \ldots, a_n) R_{\mathfrak{m}_i}\big)_{M_i}$. Hence $a_1, \ldots, a_n, b_0$ is a M_i-cosequence in $\mathfrak{m}_i$. It gives a contradiction. Therefore, we have by [15, 1.7; 1.9] that

$$\text{Width}_{S_{\mathfrak{m}_i^*}} K_i = \text{Width}_{\mathfrak{m}_i^*} K_i = \text{Width}_{\mathfrak{m}_i} M_i + 1 = \text{Width}_{R_{\mathfrak{m}_i}} M_i + 1. \quad (3)$$

Now, we have by (2) and (3) that K_i is a CCM $S_{\mathfrak{m}_i^*}$-module if and only if M_i is a CCM $R_{\mathfrak{m}_i}$-module for all $i = 1, \ldots, r$. Thus K is a CCM S-module if and only if M is a CCM R-module. ∎

Next, we summarize some properties of dimension and width of co-localization.

Lemma 4.4. *Let $\mathfrak{p} \in \text{Cos}\, M$. Then we have*

(i) $\text{Width}({}_{\mathfrak{p}}M) \le \dim({}_{\mathfrak{p}}M)$,
(ii) $\dim({}_{\mathfrak{p}}M) + \dim R/\mathfrak{p} \le \dim M$,
(iii) $\text{Width}_{\mathfrak{p}}(M) \le \text{Width}({}_{\mathfrak{p}}M)$ *if $(R, \mathfrak{m})$ is Noetherian local and M satisfies the condition* $(*)$,
(iv) *if R is Noetherian local and M is CCM which satisfies the condition $(*)$ then*

(a) $\mathrm{Width}_{\mathfrak{p}}(M) = \dim({}_{\mathfrak{p}}M)$,
(b) $\dim({}_{\mathfrak{p}}M) + \dim R/\mathfrak{p} = \dim M$.

Proof. (i) is obvious.
(ii) is an immediate consequence of Lemma 3.2 (ii).
(iii) is derived by the condition (∗) and the exactness of co-localization.
(iv) follows from (ii) and (iii). ∎

As an application, we get a characterization of CCM modules.

Corollary 4.5. *Suppose that* $(R, \mathfrak{m})$ *is a local ring and* M *satisfies the condition* (∗). *Then* M *is CCM if and only if* $\dim({}_{\mathfrak{p}}M) = \mathrm{Width}({}_{\mathfrak{p}}M)$, *for all* $\mathfrak{p} \in \mathrm{Cos}\, M$.

Proof. It can be immediately derived by Lemma 4.4, (iii), (iv). ∎

Remark. The condition (∗) in Corollary 4.5 is necessary (see Example 5.3 (iii)).

5. Examples

Example 5.1. *There exist modules* M *over a local ring* $(R, \mathfrak{m})$ *such that* M *is linearly compact with respect to the discrete topology and*

$$\mathrm{Width}\, M \neq \inf\{n \geq 0 : \mathrm{Tor}_n^R(M; R/\mathfrak{m}) \neq 0\}.$$

Proof. Let N be a linearly compact module with respect to the discrete topology over a local ring $(R, \mathfrak{m})$ such that $\mathrm{soc}\, N = 0$ and $\mathrm{mag}\, N = 1$. Then there exists by [21, 1.4] a non-zero minimal submodule M of N such that M has no maximal submodule. Clearly, M is a linearly compact module with respect to the discrete topology. It follows by [21, p. 126] that $\mathrm{mag}\, M = 1$, $\mathrm{soc}\, M = 0$ and $\mathrm{Coass}\, M = \mathrm{Ass}\, M$. Hence $\mathfrak{m} \notin \mathrm{Coass}\, M$. Then there exists $x \in \mathfrak{m}$ such that $xM = M$. Hence $(0 : xR)M = 0$ by [21, 1.6]. So the multiplication by x on M is an isomorphism. Therefore $\mathrm{Tor}_i^R(M; R/\mathfrak{m}) = x.\, \mathrm{Tor}_i^R(M; R/\mathfrak{m}) = 0$ as $x \in \mathfrak{m}$, for all $i \geq 0$. Thus, $\inf\{n \geq 0 : \mathrm{Tor}_n^R(M; R/\mathfrak{m}) \neq 0\} = \infty$. ∎

Example 5.2. *A co-localization* ${}_{\mathfrak{p}}M$ *of an Artinian module* M *over a local ring* $(R, \mathfrak{m})$ *such that* M *satisfies the condition* (∗), *but* $\mathrm{Ass}_{R_{\mathfrak{p}}}({}_{\mathfrak{p}}M)$ *is an infinite set:*

Let $(R, \mathfrak{m})$ be a local ring of $\dim R > 2$ and $\mathfrak{p} \neq \mathfrak{m}$ a prime ideal of R of $\mathrm{ht}(\mathfrak{p}) > 1$. Let M be an Artinian R-module which contains the injective hull of $R/\mathfrak{m}$. Then M satisfies the condition (∗). It can be easily derived by [12, 4.1] that $\mathrm{Ass}({}_{\mathfrak{p}}M) = \mathrm{Spec}(R_{\mathfrak{p}})$. Since $\dim_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}) > 1$, it follows that $\mathrm{Spec}(R_{\mathfrak{p}})$ is an infinite set. Therefore $\mathrm{Ass}({}_{\mathfrak{p}}M)$ is an infinite set. It also follows that ${}_{\mathfrak{p}}M$ has infinite Goldie dimension and therefore it is not Artinian.

Example 5.3. *There exist co-Cohen-Macaulay modules* M *over a local ring* $(R, \mathfrak{m})$ *such that* M *does not satisfy the condition* (∗) *and* M *has the following properties:*

(i) *There exists a prime ideal* $\mathfrak{p} \in \mathrm{Cos}\, M$ *such that*

$$\mathrm{Width}_{R_\mathfrak{p}}({}_\mathfrak{p}M) \neq \inf\{t : \mathrm{Tor}_t^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}; {}_\mathfrak{p}M) \neq 0\}.$$

(ii) *There exists a prime ideal* $\mathfrak{p} \in \mathrm{Cos}\, M$ *such that* $\dim_{R_\mathfrak{p}}({}_\mathfrak{p}M) = 1$, *but* $\dim_{R_\mathfrak{p}}\left(0 : xR_\mathfrak{p}\right)_{{}_\mathfrak{p}M} = -1$ *for all* $0 \neq x \in \mathfrak{p}$.

(iii) *There exists a prime ideal* $\mathfrak{p} \in \mathrm{Cos}\, M$ *such that* $\mathrm{Width}_{R_\mathfrak{p}}({}_\mathfrak{p}M) = 0$, *but* $\text{N-dim}_{R_\mathfrak{p}}({}_\mathfrak{p}M) > 0$ *and* $\dim_{R_\mathfrak{p}}({}_\mathfrak{p}M) > 0$.

Proof. Ferrand and Raynaud [5] constructed a local domain $(R, \mathfrak{m})$ of dimension 2 for which the $\mathfrak{m}$-adic completion $\hat{R}$ has an associated prime of dimension 1. Let $M = H^1_\mathfrak{m}(R)$, the local cohomology of R with respect to the maximal ideal $\mathfrak{m}$. We have by [3, 4.1; 4.4] that $\text{N-dim}\, M = 1$, $\dim M = 2$ and M does not satisfy the condition $(*)$. Let $0 \neq x \in \mathfrak{m}$. We can easily check that M is the 0-secondary. Therefore x is M-coregular. So $\mathrm{Width}\, M = 1$. Therefore M is co-Cohen-Macaulay.

(i) Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} \neq 0$ and $\mathfrak{p} \neq \mathfrak{m}$. Let $0 \neq x \in \mathfrak{p}$. Since $(0 : xR)_M$ has finite length, $\mathrm{Cos}(0 : xR)_M = \{\mathfrak{m}\}$. It follows by Lemma 3.2 (i) that

$$\left(0 : xR_\mathfrak{p}\right)_{{}_\mathfrak{p}M} \cong {}_\mathfrak{p}(0 : xR)_M = 0.$$

So $\mathrm{Width}_{R_\mathfrak{p}}({}_\mathfrak{p}M) = 0$. Since x is M-coregular, we get $xM = M$. It follows by the exactness of co-localization that $x({}_\mathfrak{p}M) = {}_\mathfrak{p}M$. Since $\left(0 : xR_\mathfrak{p}\right)_{{}_\mathfrak{p}M} = 0$ and $x \in \mathfrak{p}$, we get

$$\mathrm{Tor}_t^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}; {}_\mathfrak{p}M) \cong x\,\mathrm{Tor}_t^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}; {}_\mathfrak{p}M) = 0$$

for all t. It follows that

$$\inf\{t : \mathrm{Tor}_t^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}; {}_\mathfrak{p}M) \neq 0\} = \infty.$$

(ii) Let $\mathfrak{p}$ be as in (i). Then $\mathrm{ht}(\mathfrak{p}) = 1$. Since $0 \in \mathrm{Att}_R\, M$, we get by Lemma 3.2 (ii) that $0 \in \mathrm{Att}_{R_\mathfrak{p}}({}_\mathfrak{p}M)$. Therefore $\mathrm{Ann}_{R_\mathfrak{p}}({}_\mathfrak{p}M) = 0$. Hence $\dim_{R_\mathfrak{p}}({}_\mathfrak{p}M) = 1$. Let $0 \neq x \in \mathfrak{p}$. Then we have $\left(0 : xR_\mathfrak{p}\right)_{{}_\mathfrak{p}M} = 0$. Therefore $\dim_{R_\mathfrak{p}}\left(0 : xR_\mathfrak{p}\right)_{{}_\mathfrak{p}M} = -1$.

(iii) Let $\mathfrak{p}$ be as in (i). We get $\mathrm{Width}_{R_\mathfrak{p}}({}_\mathfrak{p}M) = 0$ and $\dim_{R_\mathfrak{p}}({}_\mathfrak{p}M) = 1$ by (i) and (ii). Let $0 \neq x \in \mathfrak{p}$. Since ${}_\mathfrak{p}M \neq 0$ and $x({}_\mathfrak{p}M) = {}_\mathfrak{p}M$, it follows that ${}_\mathfrak{p}M$ is not Noetherian by Nakayama lemma. So $\text{N-dim}_{R_\mathfrak{p}}({}_\mathfrak{p}M) > 0$. ∎

Example 5.4. *An Artinian module* M *over a local ring* R *such that there exists a co-localization* ${}_\mathfrak{p}M$ *for which* $\text{N-dim}_{R_\mathfrak{p}}({}_\mathfrak{p}M) > 0, \dim_{R_\mathfrak{p}}({}_\mathfrak{p}M) = 0$ *and* $\ell_{R_\mathfrak{p}}({}_\mathfrak{p}M) = \infty$ *:*

Let $(R, \mathfrak{m})$ be the local domain as in Example 5.3. Let M be the injective hull of $R/\mathfrak{m}$ and let $\mathfrak{p} = (0)$. Then M is Artinian, $R_\mathfrak{p} = K$ is a field and ${}_\mathfrak{p}M$ is a K-vector space. We know by [12, p. 127] that ${}_\mathfrak{p}M$ has infinite length over K. So $\text{N-dim}_K({}_\mathfrak{p}M) > 0$ and $\dim_K({}_\mathfrak{p}M) = 0$.

Acknowledgement. The author is greatly indebted to her advisor - Professor Nguyen Tu Cuong for his suggestion and guidance during the preparation of this paper.

References

1. N. T. Cuong and L. T. Nhan, Dimension, multiplicity and Hilbert function of Artinian modules, *East-West J. Math.* **1** (1999) 179–196.
2. N. T. Cuong and L. T. Nhan, On representable linearly compact modules which are representable, *Proc. Amer. Math. Soc.* (to appear).
3. N. T. Cuong and L. T. Nhan, On the Noetherian dimension of Artinian modules, *Vietnam J. Math.* (to appear).
4. I. H. Denizler and R. Y. Sharp, Co-Cohen-Macaulay modules over commutative rings, *Glasgow Math. J.* **38** (1996) 359–366.
5. D. Ferrand and M. Raynaund, Fibres formelles d'un anneau local Noetherian, *Ann. Sci. Ecole Norm. Sup.* **3** (1970) 295–311.
6. D. Kirby, Artinian modules and Hilbert polynomials, *Quart. J. Math. Oxford* **24** (2) (1973) 47–57.
7. D. Kirby, Dimension and length of Artinian modules, *Quart. J. Math. Oxford* **41** (1990) 419–429.
8. S. Lefschetz, Algebraic Topology, *Colloq. Lect. Amer. Soc.* **27** (1942).
9. I. G. Macdonald, Duality over complete local rings, *Topology* **1** (1962) 213–235.
10. I. G. Macdonald, Secondary representation of modules over a commutative ring, *Symposia Mathematica* **11** (1973) 23–43.
11. L. Melkersson, Cohomological properties of modules with secondary representations, *Math. Scand.* **77** (1995) 197–208.
12. L. Melkersson and P. Schenzel, The co-localization of an Artinian module, *Proc. Edinburgh Math. Soc.* **38** (1995) 121–131.
13. A. Ooishi, Matlis duality and the width of a module, *Hiroshima Math. J.* **6** (1976) 573–587.
14. R. N. Roberts, Krull dimension for Artinian modules over quasi-local commutative rings, *Quart. J. Math. Oxford* **26** (1975) 269–273.
15. R. Y. Sharp, A method for the study of Artinian modules with an application to asymptotic behaviour, in: *Commutative Algebra* (Math. Sciences Research Inst. Publ. No. 15, Spinger-Verlag), 1989, pp. 177–195.
16. D. W. Sharpe and P. Va'mos, *Injective modules*, University Prees Cambridge, 1972.
17. Z. Tang and H. Zakeri, Co-Cohen-Macaulay modules and modules of generalized fractions, *Comm. Algebra* **22** (1994) 2173–2204.
18. S. Yassemi, Coassociated primes, *Comm. Algebra* **23** (1995) 1473–1498.
19. S. Yassemi, Magnitute of modules, *Comm. Algebra* **23** (1995) 3993–4008
20. D. Zelinsky, Linearly compact modules and rings, *Amer. J. Math.* **75** (1953) 79–90.
21. H. Zöschinger, Linear–Compakte Modulen über Noetherschen Ringen, *Arch. Math.* **41** (1983) 121–130.