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Chu Spaces, Fuzzy Sets and Game Invariances

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Abstract. By constructing a covariant functor, called the "fuzzy functor", from the set category into the category of fuzzy spaces, we show that, a Chu space is a fuzzy space if and only if it is fully complete.

1. Introduction

This work is motivated by recent attempts to model information flow in distributed systems [2] as well as the work of Pratt in computer science, in which a general algebraic scheme, known as Chu spaces, is systematically used [5]. In this paper, we are interested in using Chu category framework in uncertainty modeling. We specify an important Chu category: vague (fuzzy) evaluations having the unit interval as the set of truth values. We address fundamental questions in these modeling frameworks towards applications.

In Sec. 2 we consider Chu spaces in general settings. We define some numerical data and prove that these data are invariant in the category of Chu spaces in the sense that any two isomorphic Chu spaces have the same data.

In Sec. 3 we introduce a new class of Chu spaces, called "fuzzy spaces". We construct a covariant functor, called the "fuzzy functor", from the set category into the category of "fuzzy spaces". The "fuzzy functor" characterizes fuzzy spaces as fully complete Chu spaces.

The results in Sec. 3 are extended further in Sec. 4, where each map in the set category is associated with a Chu space, called a "*-fuzzy space". This establishes a covariant functor, called the "*-fuzzy functor", from the "dual" set category into the category of *-fuzzy spaces.

Finally, the theory of Chu spaces applies to game theory in Sec. 5. We define some statistical data as norm, mean, standard deviation of a game space. These data are proved to be game invariances.

2. Chu Spaces in General Settings

A *Chu space* is a triple $\widetilde{C} = (X, A, f)$ consisting of two sets:
1. X, called the *set of events, or players* of $\widetilde{C}$; and
2. A, called the *set of states, or situations* of $\widetilde{C}$.

The two sets X and A are joined by a map $f : X \times A \to K$, where K is an arbitrary set of values. In this paper, we take the set K to be the unit interval $[0,1]$. Then the map $f : X \times A \to [0,1]$ is called the *probability function* of $\widetilde{C}$.

Example 1. Let X be a metric space. Then $\widetilde{C} = (X, X, f)$ is a Chu space, where $f : X \times X \to [0,1]$ is defined by

$$f(x,y) = \min\{d(x,y), 1\} \text{ for } x, y \in X.$$

The notation (x, a) means the event x is at the state a. The value $f(x,a)$ is called the *probability of the event x given that it is in the state a.*

Let $\widetilde{C} = (X, A, f)$ be a Chu space. For $x \in X$ and $a \in A$ we define the *supports* of x and a respectively by

$$\text{supp}(x) = \{a \in A : f(x,a) > 0\} \text{ and } \text{supp}(a) = \{x \in X : f(x,a) > 0\}.$$

For an event $x \in X$ we define the following statistical data:
1. The number $\|x\|^* = \sup\{f(x,a) : a \in A\}$ is called the *upper value* of x.
2. The number $\|x\|_* = \inf\{f(x,a) : a \in A\}$ is called the *lower value* of x.
3. The number $\|x\| = (\|x\|^* + \|x\|_*)/2$ is called the *value* of x.
4. The number $d(x) = \|x\|^* - \|x\|_*$ is called the *deviation* of x.

We can also define the following statistical data for the whole space $\widetilde{C}$:
1. $M^*(X, \widetilde{C}) = \sup\{\|x\|^* : x \in X\}$. The number $M^*(X, \widetilde{C})$ is called the *upper event value* of $\widetilde{C}$.
2. $M_*(X, \widetilde{C}) = \inf\{\| x\|^* : x \in X\}$. The number $M_*(X, \widetilde{C})$ is called the *minimax event value* of $\widetilde{C}$.
3. $m^*(X, \widetilde{C}) = \sup\{\|x\|_* : x \in X\}$.
4. $m_*(X, \widetilde{C}) = \inf\{\|x\|_* : x \in X\}$.

Dually, we can define the values $\|a\|^*$, $\|a\|_*$, $\|a\|$, $d(a)$ for a state $a \in A$, and the numbers $M^*(A, \widetilde{C})$, $M_*(A, \widetilde{C})$, $m^*(A, \widetilde{C})$, $m_*(A, \widetilde{C})$ in the same way. For instance:

$$\|a\|^* = \sup\{f(x,a) : x \in X\}.$$

Roughly speaking, for an event $x \in X$ the upper value $\|x\|^*$ measures the "skill" of x in the best situation and the lower value $\|x\|_*$ measures the "skill" of x in the worst situation. An event $x \in X$ is called a *strong event* if $\|x\| = 1$, or equivalently $f(x,a) = 1$ for every $a \in A$, and x is called a *null event* if $\|x\| = 0$, or equivalently $\text{supp}(x) = \emptyset$.

Dually, for a state $a \in A$ the upper value $\|a\|^*$ describes the quality of the position a if a best player is staying there, and the lower value $\|a\|_*$ describes the quality of the position a if a worst player is staying there. A state $a \in A$ is called a *winning state* if $\|a\| = 1$, or equivalently $supp(a) = X$ and a is called a *dead state* if $\| a\| = 0$, or equivalently $\text{supp}(a) = \emptyset$.

We can define the distances $\|x - y\|$ between two events x, y and $\|a - b\|$ between two states a and b. For instance

$$\|x - y\| = \sup\{|f(x,a) - f(y,a)| : a \in A\}.$$

A Chu space $\widetilde{C}$ is *separated*, see [2], if $\|a - b\| = 0$ implies $a = b$ and $\widetilde{C}$ is *extensional* if $\|x - y\| = 0$ implies $x = y$. If $\widetilde{C}$ is both separated and extensional then we say that $\widetilde{C}$ is *biextensional.*

Clearly, the Chu distance defines pseudometrics on X and A. Hence

Proposition 1. *If $\widetilde{C}$ is separated (resp. extensional) then A (resp. X) is a metric space with the Chu distance. Therefore if $\widetilde{C}$ is biextensional then both A and X are metric spaces.*

We say that a Chu space $\widetilde{C} = (X, A, f)$ is *complete* if for any function $\varphi : X \to [0,1]$ there exists a state $a \in A$ such that $\varphi(x) = f(x,a)$ for every $x \in X$. We say $\widetilde{C}$ *fully complete* if $\widetilde{C}$ is complete and separated.

If (X, A, f) and (Y, B, g) are Chu spaces, then a *Chu morphism* $\Phi : (X, A, f) \to (Y, B, g)$ is a pair of maps $\Phi = (\varphi, \psi)$, where $\varphi : X \to Y$ and $\psi : B \to A$ such that the diagram below commutes:

$$\begin{array}{ccc} X \times B & \xrightarrow{(\varphi, 1_B)} & Y \times B \\ {\scriptstyle (1_X, \psi)} \downarrow & & \downarrow {\scriptstyle g} \\ X \times A & \xrightarrow[f]{} & [0,1] \end{array} \tag{1}$$

where $1_X : X \to X$ denotes the identity map. That is

$$f(x, \psi(b)) = g(\varphi(x), b) \text{ for } x \in X \text{ and } b \in B. \tag{2}$$

If $\Phi = (\varphi, \psi) : \widetilde{C} = (X, A, f) \to \widetilde{D} = (Y, B, g)$ is a Chu morphism, then the Chu space $(X, B, f \times_\Phi g)$, where

$$(f \times_\Phi g)(x, b) = f(x, \psi(b)) = g(\varphi(x), b) \text{ for } (x, b) \in X \times B$$

is called the *cross product of $\widetilde{C}$ and $\widetilde{D}$ over* Φ, denoted by $\widetilde{C} \times_\Phi \widetilde{D}$, see [3].

We say that the diagram (1) upper-commutes if instead of (2) we have

$$f(x, \psi(b)) \leq g(\varphi(x), b) \text{ for } x \in X \text{ and } b \in B. \tag{3}$$

If (3) holds, then we say that $\Phi : (X, A, f) \to (Y, B, g)$ is a Chu upper-morphism.

The *composition* of two morphisms $\Phi_1 = (\varphi_1, \psi_1)$ and $\Phi_2 = (\varphi_2, \psi_2)$ is given by $\Phi_1\Phi_2 = (\varphi_1\varphi_2, \psi_2\psi_1)$. Clearly $1_{\widetilde{C}} = (1_X, 1_A)$ is the identity map of $\widetilde{C} = (X, A, f)$.

An easy proof of the following proposition will be omitted.

Propositon 2. *If Φ_1 and Φ_2 are Chu morphisms (resp. Chu upper-morphisms), then $\Phi_1\Phi_2$ is a Chu morphism (resp. a Chu upper-morphism).*

By Proposition 2 we can define:

1. The *Chu category*, denoted by $\mathcal{C}$, of Chu spaces with Chu morphisms.
2. The *Chu upper-category*, denoted by $\mathcal{C}^*$, of Chu spaces with Chu upper-morphisms.

For Chu spaces $\widetilde{C} = (X, A, f)$ and $\widetilde{D} = (Y, B, g)$ let $\mathcal{M}(\widetilde{C}, \widetilde{D})$ (resp. $\mathcal{M}^*(\widetilde{C}, \widetilde{D})$) denote the set of all Chu morphisms (resp. Chu upper-morphisms) from $\widetilde{C}$ into $\widetilde{D}$. Observe that $\mathcal{M}(\widetilde{C}, \widetilde{D}) = \emptyset$ in many situations. In fact, let X and A be two sets and $t \in [0,1]$. Then $\widetilde{C}_t = (X, A, f_t)$ is a Chu space, where f_t is the constant function defined by:

$$f_t(x,a) = t \text{ for any } (x,a) \in X \times A.$$

It is easy to see that

Proposition 3.

1. *If $t \neq s$, then both $\mathcal{M}(\widetilde{C}_t, \widetilde{C}_s) = \emptyset$ and $\mathcal{M}(\widetilde{C}_s, \widetilde{C}_t) = \emptyset$.*
2. *If $t > s$, then $\mathcal{M}^*(\widetilde{C}_t, \widetilde{C}_s) = \emptyset$ but $\mathcal{M}^*(\widetilde{C}_s, \widetilde{C}_t) \neq \emptyset$.*

More generally we have the following necessary condition for the existance of Chu morphisms between Chu spaces.

Proposition 4. *Let $\widetilde{C} = (X, A, f)$ and $\widetilde{D} = (Y, B, g)$ be Chu spaces. If $\mathcal{M}(\widetilde{C}, \widetilde{D}) \neq \emptyset$ then $M^*(X, \widetilde{C}) \geq M_*(Y, \widetilde{D})$.*

Proof. We prove that $\mathcal{M}(\widetilde{C}, \widetilde{D}) \neq \emptyset$ implies $M^*(X, \widetilde{C}) \geq M_*(Y, \widetilde{D})$. In fact, if it is not the case, then

$$\|x\|^* < \|y\|^* \text{ for any } x \in X \text{ and } y \in Y. \tag{4}$$

On the other hand, since $\mathcal{M}(\widetilde{C}, \widetilde{D}) \neq \emptyset$ there exists a morphism $\Phi = (\varphi, \psi)$, where $\varphi : X \to Y$ and $\psi : B \to A$ such that

$$f(x, \psi(b)) = g(\varphi(x), b) \text{ for } x \in X \text{ and } b \in B.$$

It follows that

$$\begin{aligned}\|x\|^* &= \sup\{f(x,a) : a \in A\} \\ &\geq \sup\{f(x, \psi(b)) : b \in B\} \\ &= \sup\{g(\varphi(x), b) : b \in B\} \\ &= \|\varphi(x)\|^*,\end{aligned}$$

which contradicts (4). Consequently $M^*(X, \widetilde{C}) \geq M_*(Y, \widetilde{D})$ and the proposition is proved. ■

If $\mathcal{M}(\widetilde{C}, \widetilde{D}) \neq \emptyset$, then we say that $\widetilde{C}$ is *dominated* by $\widetilde{D}$ and denote $\widetilde{C} \preceq \widetilde{D}$. We say that $\widetilde{C}$ and $\widetilde{D}$ are *equivalent* , denoted by $\widetilde{C} \approx \widetilde{D}$, if $\widetilde{C} \preceq \widetilde{D}$ and $\widetilde{D} \preceq \widetilde{C}$, and $\widetilde{C}$ and $\widetilde{D}$ are *connected* if either $\widetilde{C} \preceq \widetilde{D}$ or $\widetilde{D} \preceq \widetilde{C}$. A class of Chu spaces $\mathcal{G}$ is called a *connected system* if any two members of $\mathcal{G}$ are connected. If $\widetilde{C} \approx \widetilde{D}$ for $\widetilde{C}, \widetilde{D} \in \mathcal{G}$, then we say that $\mathcal{G}$ is an *equivalent system.* A connected system is called a *closed system* if $\mathcal{G}$ is closed under cross products. That is, $\widetilde{C} \times_{\Phi} \widetilde{D} \in \mathcal{G}$ for any $\widetilde{C}, \widetilde{D} \in \mathcal{G}$ and $\Phi \in \mathcal{M}(\widetilde{C}, \widetilde{D})$. A *complete system* is a closed equivalent system.

We say that $\widetilde{C}$ and $\widetilde{D}$ are *isomorphic*, denoted by $\widetilde{C} \cong \widetilde{D}$, if $\widetilde{C}$ and $\widetilde{D}$ are isomorphic objects in the category $\mathcal{C}$ of Chu spaces. It is easy to see that a Chu morphism $\Phi = (\varphi, \psi) : (X, A, f) \to (Y, B, g)$ is an isomophism if and only if both $\varphi : X \to Y$ and $\psi : B \to A$ are one-to-one and onto. If φ is one-to-one, ψ is onto, then we say that $\widetilde{C} = (X, A, f)$ is a *subspace* of $\widetilde{D} = (Y, B, g)$, denote $\widetilde{C} \subseteq \widetilde{D}$.

It is easy to see that

Proposition 5. *The space $\widetilde{C} = (X, A, f)$ is a subspace of $\widetilde{D} = (Y, B, g)$ if and only if $\Phi = (\varphi, \psi) : \widetilde{C} = (X, A, f) \to \widetilde{D} = (Y, B, g)$ is a monomorphism, that is, if $\Phi_1 = (\varphi_1, \psi_1)$ and $\Phi_2 = (\varphi_2, \psi_2)$ are two morphisms with the same target $\widetilde{C} = (X, A, f)$, then the equality $\Phi\Phi_1 = \Phi\Phi_2$ implies $\Phi_1 = \Phi_2$.*

The Chu space $\widetilde{C}^{\perp} = (X, A, 1 - f)$ is called the *complement* of $\widetilde{C} = (X, A, f)$.

Proposition 6. *For any Chu space $\widetilde{C} = (X, A, f)$ we have*
1. *$m_*(X, \widetilde{C}) \leq m^*(X, \widetilde{C})$ and $M_*(X, \widetilde{C}) \leq M^*(X, \widetilde{C})$.*
2. *$m_*(X, \widetilde{C}^{\perp}) = 1 - M^*(X, \widetilde{C})$ and $m_*(X, \widetilde{C}) = 1 - M_*(X, \widetilde{C}^{\perp})$.*

Proof.
1. An easy proof is omitted.
2. We have

$$\begin{aligned}
m_*(X, \widetilde{C}^{\perp}) &= \inf\{\|x\|_* : x \in X\} \\
&= \inf\{\inf\{(1 - f(x, a)) : a \in A\} : x \in X\} \\
&= \inf\{1 - \sup\{f(x, a)) : a \in A\} : x \in X\} \\
&= 1 + \inf\{-\sup\{f(x, a) : a \in A\} : x \in X\} \\
&= 1 - \sup\{\sup\{f(x, a) : a \in A\} : x \in X\} \\
&= 1 - \sup\{\|x\|^* : x \in X\} \\
&= 1 - M^*(X, \widetilde{C}).
\end{aligned}$$

$$\begin{aligned}
m^*(X, \widetilde{C}^{\perp}) &= \sup\{\|x\|_* : x \in \widetilde{C}^{\perp}\} \\
&= \sup\{\inf\{1 - f(x, a)\} : a \in A\} : x \in \widetilde{C}^{\perp}\} \\
&= 1 + \sup\{-\sup\{f(x, a) : a \in A\} : x \in \widetilde{C}^{\perp}\} \\
&= 1 - \inf\{\sup f(x, a) : a \in A\} : x \in \widetilde{C}^{\perp}\} \\
&= 1 - M_*(X, \widetilde{C}^{\perp}).
\end{aligned}$$

Of course Proposition 6 still holds if the set X of events is replaced by the set A of states.

Observe that if $m^*(X,\widetilde{C}) > M_*(X,\widetilde{C})$, then $\|x\|_* > \|y\|^*$ for some $x, y \in X$. This means that in the worst situation the player x can do better than the player y even when y is in the best situation. Clearly, in this situation the qualification of the set X is "very non-uniform". We say that Chu space $\widetilde{C}$ is *event uniform* (resp. *state uniform*) if $m^*(X,\widetilde{C}) \leq M_*(X,\widetilde{C})$ (resp. $m^*(A,\widetilde{C}) \leq M_*(A,\widetilde{C})$), and $\widetilde{C}$ is *uniform* if it is both event and state uniform. From Proposition 6 we get

Proposition 7. *For any uniform Chu space* $\widetilde{C} = (X, A, f)$:

1. $m_*(X,\widetilde{C}) \leq m^*(X,\widetilde{C}) \leq M_*(X,\widetilde{C}) \leq M^*(X,\widetilde{C})$.
2. $m_*(A,\widetilde{C}) \leq m^*(A,\widetilde{C}) \leq M_*(A,\widetilde{C}) \leq M^*(A,\widetilde{C})$.

The following theorem shows that any two isomorphic Chu spaces have the same data

Theorem 1. *Let* $\widetilde{C} = (X, A, f)$ *and* $\widetilde{D} = (Y, B, g)$ *be Chu spaces. If* $\widetilde{C} \subseteq \widetilde{D}$ *then*

1. $M^*(X,\widetilde{C}) \leq M^*(Y,\widetilde{D})$.
2. $M_*(X,\widetilde{C}) \geq M_*(Y,\widetilde{D})$.
3. $m^*(X,\widetilde{C}) \leq m^*(Y,\widetilde{D})$.
4. $m_*(X,\widetilde{C}) \geq m_*(Y,\widetilde{D})$.

Therefore, if $\widetilde{C}$ *and* $\widetilde{D}$ *are isomorphic, then* $M^*(X,\widetilde{C}) = M^*(Y,\widetilde{D})$, $M_*(X,\widetilde{C}) = M_*(Y,\widetilde{D})$, $m^*(X,\widetilde{C}) = m^*(Y,\widetilde{D})$ *and* $m_*(X,\widetilde{C}) = m_*(Y,\widetilde{D})$.

Proof. 1. We have

$$\begin{aligned} M^*(X,\widetilde{C}) &= \sup\{\, \sup\{f(x,a) : a \in A\} : x \in X\} \\ &= \sup\{\sup\{f(x,\psi(b)) : b \in B\} : x \in X\} \\ &= \sup\{\sup\{g(\varphi(x),b) : b \in B\} : x \in X\} \\ &\leq \sup\{\sup\{g(y,b) : b \in B\} : y \in Y\} \\ &= \sup\{\|y\|^* : y \in Y\} \\ &= M^*(Y,\widetilde{D}). \end{aligned}$$

2. We have

$$\begin{aligned} M_*(X,\widetilde{C}) &= \inf\{\sup\{f(x,a) : a \in A\} : x \in X\} \\ &= \inf\{\sup\{f(x,\psi(b)) : b \in B\} : x \in X\} \\ &= \inf\{\sup\{g(\varphi(x),b) : b \in B\} : x \in X\} \\ &= \inf\{\|\varphi(x)\|^* : x \in X\} \\ &\geq \inf\{\|y\|^* : y \in Y\} \\ &= M_*(Y,\widetilde{D}). \end{aligned}$$

3. We have

$$\begin{aligned} m^*(X,\widetilde{C}) &= \sup\{\inf\{f(x,a) : a \in A\} : x \in X\} \\ &= \sup\{\inf\{f(x,\psi(b)) : b \in B\} : x \in X\} \\ &= \sup\{\inf\{g(\varphi(x),b) : b \in B\} : x \in X\} \\ &= \sup\{\|\varphi(x)\|_* : x \in X\} \\ &\leq \sup\{\|y\|_* : y \in Y\} \\ &= m^*(Y,\widetilde{D}). \end{aligned}$$

4. We have

$$\begin{aligned} m_*(X,\widetilde{C}) &= \inf\{\inf\{f(x,a) : a \in A\} : x \in X\} \\ &= \inf\{\inf\{f(x,\psi(b)) : b \in B\} : x \in X\} \\ &= \inf\{\inf\{g(\varphi(x),b) : b \in B\} : x \in X\} \\ &\geq \inf\{\inf\{g(y,b) : b \in B\} : y \in Y\} \\ &= m_*(Y,\widetilde{D}). \end{aligned}$$

The following subcategories of $\mathcal{C}$ and $\mathcal{C}^*$ will be also considered:

1. $\mathcal{C}_S$ (resp. $\mathcal{C}_S^*$) denotes the category of separated Chu spaces with Chu morphisms (resp. with Chu upper-morphisms).
2. $\mathcal{C}_E$ (resp. $\mathcal{C}_E^*$) denotes the category of extensional Chu spaces with Chu morphisms (resp. with Chu upper-morphisms).
3. $\mathcal{C}_B$ (resp. $\mathcal{C}_B^*$) denotes the category of biextensional Chu spaces with Chu morphisms (resp. with Chu upper-morphisms).
4. $\mathcal{C}_C$ (resp. $\mathcal{C}_C^*$) denotes the category of complete Chu spaces with Chu morphisms (resp. with Chu upper-morphisms).
5. $\mathcal{C}_F$ (resp. $\mathcal{C}_F^*$) denotes the category of full complete Chu spaces with Chu morphisms (resp. with Chu upper-morphisms).

Observe that $\mathcal{C}_S$, $\mathcal{C}_E$, $\mathcal{C}_B$, $\mathcal{C}_C$ and $\mathcal{C}_F$ are full subcategories of $\mathcal{C}$, and $\mathcal{C}_S^*$, $\mathcal{C}_E^*$, $\mathcal{C}_B^*$, $\mathcal{C}_C^*$ and $\mathcal{C}_F^*$ are full subcategories of $\mathcal{C}^*$.

3. Fuzzy Spaces and the Fuzzy Functor

In this section, we introduce a special class of Chu spaces called *fuzzy spaces*. The category of fuzzy spaces is an equivalent system. That is, any two fuzzy spaces are equivalent.

By a *fuzzy subset* of a set X we mean any function $f : X \to [0,1]$, see [4]. Observe that if A is a subset of X, then the characteristic function $\mathcal{X}_A$ of A is a fuzzy subset of X. So by identifying A with $\mathcal{X}_A$ we can say that any subset of X is a fuzzy subet of X. A fuzzy subset of X is also simply called a *fuzzy set*.

Let $\mathcal{S}$ denote the category of sets. For a given set X, let $X^* = [0,1]^X$ denote the collection of all fuzzy sets of X.

For any map $\alpha : X \to Y$ we define the *conjugate* $\alpha^* : Y^* \to X^*$ of α by the formula

$$\alpha^*(a)(x) = a(\alpha(x)) \text{ for every } x \in X \text{ and } a \in Y^*. \tag{5}$$

It is easy to see that

$$(\beta\alpha)^* = \alpha^*\beta^* \text{ for every } \alpha : X \to Y, \beta : Y \to Z.$$

For any set $A \subset X^*$ we define $f_A : X \times A \to [0,1]$ by

$$f_A(x,a) = a(x) \text{ for } (x,a) \in X \times A. \tag{6}$$

Clearly $\widetilde{C} = (X, A, f_A)$ is a Chu space. This space is called a *pre-fuzzy space* on X. In the case $A = X^*$ the Chu space $F(X) = (X, X^*, f_{X^*})$ is uniquely determined by X, and called *fuzzy space associated with* X, or shortly a *fuzzy space*.

We have the following proposition

Proposition 8. *Any pre-fuzzy space is separated, but not necessarily extensional. However, any fuzzy space* $F(X) = (X, X^*, F_{X^*})$ *is fully complete and biextensional.*

Proof. Firstly, we show that pre-fuzzy space is separated. Assume that $\|a-b\| = \sup\{|f(x,a) - f(x,b)| : x \in X\} = 0$. Thus

$$f(x,a) = f(x,b) \text{ for every } x \in X.$$

Hence

$$a(x) = b(x) \text{ for every } x \in X.$$

That is

$$a = b.$$

Therefore, pre-fuzzy space is separated.

In the next we prove that the pre-fuzzy space is not necessary extensional. In fact, let $A = \{a : a(x) = 1 \text{ for all } x \in X\}$. Then for $x \neq y$ we still obtain

$$\|x - y\| = \sup\{|f(x,a) - f(y,a)| : a \in A\} = 0.$$

We show that $F(X) = (X, X^*, f_{X^*})$ is biextensional. Since $F(X)$ is separated, it is enough to show that $F(X)$ is extensional. Assume that

$$\|x - y\| = \sup\{|f(x,a) - f(y,a)| : a \in X^*\} = 0.$$

It follows that

$$f(x,a) = f(y,a) \text{ for all } a \in X^*.$$

Hence

$$a(x) = a(y) \text{ for all } a \in X^*.$$

Let $a = \mathcal{X}_{\{x\}}$ we have $a(x) = 1$. It implies that $a(y) = \mathcal{X}_{\{x\}}(y) = 1$, that is $x = y$.

Finally, we show that $F(X) = (X, X^*, f_{X^*})$ is complete. Let $\varphi : X \to [0,1]$ be a map of a set X into the interval [0,1]. Then $\varphi \in X^*$. Thus with $a = \varphi \in X^*$, we have

$$f(x,a) = \varphi(x) \text{ for } x \in X.$$

The assertion is proved. ■

In particular, any fuzzy space $F(X) = (X, X^*, f_{X^*})$ is biextensional. Therefore by Proposition 1 the Chu distance on X defines a metric. It is easy to see that it is a *discrete metric.*

The category of pre-fuzzy spaces with Chu morphisms is called the *pre-fuzzy category*, denoted by $\mathcal{F}_P$. The *fuzzy category*, denoted by $\mathcal{F}$, is the subcategory of $\mathcal{F}_P$ consisting of fuzzy spaces.

Observe that a Chu morphism $\Phi : (X, A, f_A) \to (Y, B, g_B)$ in the pre-fuzzy category is a pair of maps $\Phi = (\varphi, \psi)$, where $\varphi : X \to Y$ and $\psi : B \to A$ satisfy the condition

$$\psi(b)(x) = b(\varphi(x)) \text{ for } (x, b) \in X \times B.$$

As we have seen, in general Chu spaces are not connected. Fortunately it is not the case in the fuzzy category. In fact, we get

Theorem 2. *The fuzzy category $\mathcal{F}$ is an equivalent system.*

Proof. We need to show that $\mathcal{M}(F(X), F(Y)) \neq \emptyset$ for any fuzzy spaces $F(X) = (X, X^*, f_{X^*})$ and $F(Y) = (Y, Y^*, f_{Y^*})$.

Let $\alpha : X \to Y$ be any map (in the set category). Define $\alpha^* : Y^* \to X^*$ by

$$\alpha^*(y^*)(x) = y^*(\alpha(x)) \text{ for } x \in X \text{ and } y^* \in Y^*.$$

We have

$$\begin{aligned} \alpha^*(y^*)(x) &= f_{X^*}(x, \alpha^*(y^*)) \\ &= y^*(\alpha(x)) \\ &= f_{Y^*}(\alpha(x), y^*). \end{aligned}$$

Therefore the diagram below commutes

$$\begin{array}{ccc} X \times Y^* & \xrightarrow{(\alpha, 1_{Y^*})} & Y \times Y^* \\ {\scriptstyle (1_X, \alpha^*)}\downarrow & & \downarrow {\scriptstyle f_{Y^*}} \\ X \times X^* & \xrightarrow[f_{X^*}]{} & [0,1] \end{array}$$

Thus, $\Phi = (\alpha, \alpha^*) \in \mathcal{M}(F(X), F(Y))$, and the theorem is proved. ∎

We shall show that $F(X) = (X, X^*, f_{X^*})$ is a covariant functor from the set category $\mathcal{S}$ into the fuzzy category $\mathcal{F}$ and F will be called the *fuzzy functor.*

In fact, let $\alpha : X \to Y$ be a map. Define $F(\alpha) : F(X) \to F(Y)$ by

$$F(\alpha) = (\alpha, \alpha^*), \quad \text{where } \alpha^* : Y^* \to X^* \text{ is the conjugate of } \alpha, \text{ see (5).}$$

Observe that

$$F(\beta\alpha) = (\beta\alpha, (\beta\alpha)^*) = (\beta\alpha, \alpha^*\beta^*) = F(\beta)F(\alpha)$$

for any $\alpha : X \to Y$ and $\beta : Y \to Z$. Therefore F preserves the composition.

Now we shall prove the following theorem stating that fuzzy spaces are, in fact, fully complete Chu spaces.

Theorem 3 (Charaterization of fuzzy spaces). *A Chu space $\widetilde{C} = (X, A, f)$ is a fuzzy space if and only if $\widetilde{C}$ is fully complete.*

Proof. By Proposition 8 any fuzzy space $F(X) = (X, X^*, f_{X^*})$ associated with a set X is fully complete.

Conversely, let $\widetilde{C} = (X, A, f)$ be a fully complete Chu space. Then $F(X) = (X, X^*, f_{X^*})$ is a fuzzy space. We shall show that $\widetilde{C}$ and $F(X)$ are isomorphic. To see this we first define $T : A \to X^*$ by

$$T(a)(x) = f(x, a) \text{ for every } x \in X. \tag{7}$$

We claim that T is one-to-one. In fact, assume that $a, b \in A$ and $a \neq b$. Since $\widetilde{C}$ is separated there exists $x \in X$ such that $f(x, a) \neq f(x, b)$. Hence $T(a) \neq T(b)$.

To see that T is onto, let $\varphi \in X^*$. Then $\varphi : X \to [0, 1]$. Since $\widetilde{C}$ is state complete there exists a state $a \in A$ such that

$$\varphi(x) = f(x, a) \text{ for every } x \in X.$$

It follows that $T(a) = \varphi$.

Now we define

$$\Phi = (1_X, T^{-1}) : \widetilde{C} = (X, A, f) \to F(X) = (X, X^*, f_{X^*}).$$

$$\Psi = (1_X, T) : F(X) = (X, X^*, f_{X^*}) \to \widetilde{C} = (X, A, f).$$

From (6) and (7) we get

$$f(x, T^{-1}(a)) = TT^{-1}(a)(x) = a(x) = f_{X^*}(x, a)$$

for every $x \in X$ and $a \in X^*$, and by (7)

$$f_{X^*}(x, T(a)) = T(a)(x) = f(x, a)$$

for every $x \in X$ and $a \in A$. Therefore $\Phi = (1_X, T^{-1})$ and $\Psi = (1_X, T)$ are Chu morphisms.

It is easy to see that $\Psi\Phi = 1_{\widetilde{C}}$ and $\Phi\Psi = 1_{F(X)}$. Consequently $\widetilde{C}$ and $F(X)$ are isomorphic, and the theorem is proved. ∎

Corollary 1. *The two categories $\mathcal{F}$ and C_F are isomorphic.*

Proof. The functor F defined in the proof of Theorem 3 is an isomorphism between the fuzzy category $\mathcal{F}$ and the Category C_F of fully complete Chu spaces.

From Theorem 2 and Corollary 1 we get

Corollary 2. *The category C_F of all fully complete Chu spaces is an equivalent system.*

Let $F(X) = (X, X^*, f_{X^*})$ be a fuzzy space. Observe that if $a \in X^*$ is a winning state in the fuzzy category, then $\|a\| = 1$ which implies that $a = \mathcal{X}_X$, and if $a \in X^*$ is a dead state in the fuzzy category, then $\|a\| = 0$ which implies that $a = \mathcal{X}_\emptyset$. Consequently the whole set X will be called the *winning set* and the empty set $\emptyset$ will be called the *dead set*.

Clearly there are neither strong events nor null events in this category.

Remark 1. Since any subset of a set X is a fuzzy set we can consider the family $A = 2^X \subset X^*$ consisting of all subsets of X. The resulting pre-fuzzy space $D(X) = (X, 2^X, f_{2^X})$ will be called the *crisp space associated with* X, and the category $\mathcal{D}$ of all crisp spaces is called the *crisp category*.

We shall show that

Proposition 9. *Every crisp space is biextensional.*

Proof. By Proposition 8 a crisp space is separated, we shall claim that it is extensional.

Assume that $0 = \|x-y\| = \sup\{|f(x,a)-f(y,a)| : a \in A\}$, then $a(x)-a(y) = 0$ for every $a \in A$. From this it follows that $x = y$, since if it is not the case, setting $a = \mathcal{X}_{\{x\}} \in A$, we get $a(x) = 1$, but $a(y) = 0$.

The crisp category $\mathcal{D}$ is a subcategory of $\mathcal{F}$. Observe that

Proposition 10. *The map D defined in Remark 1 is a covariant functor from the set category $\mathcal{S}$ into the crisp category $\mathcal{D}$.*

Proof. In fact, let $\alpha : X \to Y$ be a map. Then the morphism

$$D(\alpha) : D(X) = (X, 2^X, f_{2^X}) \to D(Y) = (Y, 2^Y, f_{2^Y})$$

is defined by

$$D(\alpha) = (\alpha, \alpha^{-1}), \text{ where } \alpha^{-1}(b) \in 2^X \text{ for every } b \in 2^Y.$$

We shall show that the following diagram commutes

$$\begin{array}{ccc} X \times 2^Y & \xrightarrow{(\alpha, 1_{2^Y})} & Y \times 2^Y \\ {\scriptstyle (1_X, \alpha^{-1})}\downarrow & & \downarrow{\scriptstyle f_{2^Y}} \\ X \times 2^X & \xrightarrow[f_{2^X}]{} & [0,1] \end{array}$$

To do this, we have to show that

$$f_{2^X}(x, \alpha^{-1}(b)) = f_{2^Y}(\alpha x, b) \text{ for every } b \in 2^Y.$$

That is, we need to claim that

$$\alpha^{-1}(b)(x) = b(\alpha x) \text{ for every } b \in 2^Y.$$

Since $\alpha^{-1}(b)$ and b are two characteristic functions of the sets $\alpha^{-1}(b)$ and b in the spaces 2^X and 2^Y, respectively, they admit only two values 0 or 1. If $\alpha^{-1}(b)(x) = 1$, then $x \in \alpha^{-1}(b)$ which implies $\alpha x \in b$, hence $b(\alpha x) = 1$. If $\alpha^{-1}(b)(x) = 0$, then $x \notin \alpha^{-1}(b)$ which implies $\alpha x \notin b$, hence $b(\alpha x) = 0$.

Thus, in both cases we have

$$\alpha^{-1}(b)(x) = b(\alpha x) \text{ for } x \in X.$$

Therefore the proposition is proved. ■

4. *-Fuzzy Spaces and the *-Fuzzy Functor

As we have seen, the fuzzy category $\mathcal{F}$ is an equivalent system. Unfortunately $\mathcal{F}$ is not closed under the cross product, therefore $\mathcal{F}$ is not a complete system. In this section, we expand the fuzzy category $\mathcal{F}$ to a complete system.

Let $\mathcal{S}$ denote the set category. We define the category $\mathcal{S}^*$ as follows:

1. Objects of $\mathcal{S}^*$ are morphisms in $\mathcal{S}$.
2. If $\alpha : X \to Y$ and $\alpha' : X' \to Y'$ are two objects of $\mathcal{S}^*$, then a *morphism* $\varphi : \alpha \to \alpha'$ from α to α' in $\mathcal{S}^*$ is a map (in the set category) $\varphi : Y \to X'$.

Let $\alpha : X \to Y, \alpha' : X' \to Y'$ and $\alpha'' : X'' \to Y''$ be objects in $\mathcal{S}^*$ and $\varphi : \alpha \to \alpha'$, $\varphi' : \alpha' \to \alpha''$ be morphisms of $\mathcal{S}^*$ (i.e., $\varphi : Y \to X'$ and $\varphi' : Y' \to X''$). Then *composition* of φ and φ', denoted by $\varphi' * \varphi$, is given by

$$\varphi' * \varphi = \varphi'\alpha'\varphi : \alpha \to \alpha''.$$

It is easy to check that with the above definition $\mathcal{S}^*$ is a category.

For $\alpha : X \to Y$ we define $F^*(\alpha) = (X, Y^*, f_\alpha)$, where Y^* denote the collection of all fuzzy sets of Y, and $f_\alpha : X \times Y^* \to [0,1]$ is given by

$$f_\alpha(x, a) = a(\alpha(x)) \text{ for every } (x, a) \in X \times Y^*.$$

The Chu space $F^*(\alpha) = (X, Y^*, f_\alpha)$ is called the **-fuzzy space associated with the map* $\alpha : X \to Y$. The category of all *-fuzzy spaces associated with maps in the set category $\mathcal{S}$ is called the **-fuzzy category* and denoted by $\mathcal{F}^*$.

The *-fuzzy category $\mathcal{F}^*$ contains the fuzzy category $\mathcal{F}$ as a subcategory. In fact we have

Theorem 4. *Any fuzzy space is a *-fuzzy space.*

Proof. If $F(X) = (X, X^*, F_{X^*})$ is a fuzzy space, then clearly $F(X) = F^*(1_X)$ is a *-fuzzy space.

Theorem 5. *$\mathcal{F}^*$ is a complete system.*

Proof. Assume that $\Phi = (\varphi, \psi) : F^*(\alpha) = F^*(X, Y^*, f_\alpha) \to F^*(\alpha') = (X', Y'^*, f_{\alpha'})$ is a Chu morphism, where $F^*(\alpha)$ and $F^*(\alpha')$ are *-fuzzy spaces associated with the map $\alpha : X \to Y$ and $\alpha' : X' \to Y'$ respectively.

Putting $\beta = \alpha'\varphi : X \to Y'$ we get the cross product $\widetilde{C} = (X, Y'^*, f_\alpha \times_\Phi f_{\alpha'})$ which is a *-fuzzy space associated with the map β. In fact, for every $(x, b) \in X \times Y'^*$ we have

$$\begin{aligned}(f_\alpha \times_\Phi f_{\alpha'})(x, b) &= f_{\alpha'}(\varphi(x), b) = b(\alpha'\varphi(x)) \\ &= f_{\alpha'\varphi}(x, b) = f_\beta(x, b).\end{aligned}$$

Thus, the category $\mathcal{F}^*$ is closed under the cross product. Therefore the theorem is proved. ∎

Theorem 6. $F^* : \mathcal{S}^* \to \mathcal{F}^*$ *is a covariant functor.*

Proof. For a morphism $\varphi : \alpha \to \alpha'$ in $\mathcal{S}^*$ we define

$$F^*(\varphi) = (\varphi\alpha, \varphi^* {\alpha'}^*),$$

where φ^* and ${\alpha'}^*$ are conjugate of φ and α' respectively, see (5)

We claim that $F^*(\varphi) : F^*(\alpha) = (X, Y^*, f_\alpha) \to F^*(\alpha') = (X', {Y'}^*, f_{\alpha'})$ is a Chu morphism. That is, the following diagram commutes:

$$\begin{array}{ccc} X \times {Y'}^* & \xrightarrow{(\varphi\alpha, 1_{Y'^*})} & X' \times {Y'}^* \\ {\scriptstyle (1_X, \varphi^* {\alpha'}^*)} \downarrow & & \downarrow {\scriptstyle f_{\alpha'}} \\ X \times Y^* & \xrightarrow[f_\alpha]{} & [0,1] \end{array} \tag{8}$$

In fact, for every $x \in X$ and $a \in {Y'}^*$,

$$\begin{aligned} f_\alpha(x, \varphi^* {\alpha'}^*(a)) &= \varphi^* {\alpha'}^*(a)(\alpha x)) \\ &= (\alpha'\varphi)^*(a)(\alpha x) \\ &= a\alpha'\varphi(\alpha x) \\ &= f_{\alpha'}(\varphi\alpha(x), a). \end{aligned}$$

Consequently the diagram (8) commutes. Hence $F^*(\varphi) = (\varphi\alpha, \varphi^* {\alpha'}^*)$ is a Chu morphism.

Now we shall show that F^* preserves the composition. In fact, let $\alpha : X \to Y$, $\alpha' : X' \to Y'$, $\alpha'' : X'' \to Y''$ be objects in the category $\mathcal{S}^*$, and let $\varphi : \alpha \to \alpha'$, $\varphi' : \alpha' \to \alpha''$ be morphisms in $\mathcal{S}^*$ (i. e. , $\varphi : Y \to X'$ and $\varphi' : Y' \to X''$ are maps in the set category). Then by definition we have $\varphi' * \varphi = \varphi'\alpha'\varphi$. Therefore

$$\begin{aligned} F^*(\varphi' * \varphi) &= (\varphi'\alpha'\varphi\alpha, (\varphi'\alpha'\varphi)^* {\alpha''}^*) \\ &= (\varphi'\alpha'\varphi\alpha, \varphi^* {\alpha'}^* {\varphi'}^* {\alpha''}^*) \\ &= F^*(\varphi')F^*(\varphi). \end{aligned}$$

Consequently F^* preserves the composition, and hence $F^* : \mathcal{S}^* \to \mathcal{F}^*$ is a covariant functor.

The functor $F^* : \mathcal{S}^* \to \mathcal{F}^*$ is called the **-fuzzy functor.*

5. Game Spaces and Game Invariance Theorem

Given a set A, by a *game space over* A we mean a Chu space $\widetilde{G} = (X, A, f)$, where:

1. X is a *finite set,* called the *team game.* If $x \in X$, then x is called a *player* of the game $\widetilde{G}$.

2. A is any set, called the *field game*. If $a \in A$, then a is called a *position* in the field game A.
3. $f(x, a)$ is called the *winning probability* of the player x while he is in the position a in the field game.

Example 2. A soccer team X in a soccer field A is a game space $\widetilde{S} = (X, A, f)$, where

1. The team X is the player team. Hence X is a finite set consisting of eleven elements.
2. The field game A is the soccer field. Hence A is an infinite set: Every point in the soccer field is an element of A.
3. (x, a) means the player x is having the ball at the position a in the soccer field, and $f(x, a)$ means the probability that the player x kicks a goal from the position a in the soccer field A.

Example 3. An armed force $\widetilde{S} = (X, A, f)$ is a game space, where:

1. X is the set of soldiers of the force $\widetilde{S}$.
2. The field game A is the arsenal of the force $\widetilde{S}$.
3. (x, a) means the soldier x is having the weapon a at hand, and $f(x, a)$ is the ability of the soldier x to kill an enermy when he has the weapon a at hand.

Example 4. A university $\widetilde{U} = (X, A, f)$ is a game space, where:

1. X is the set of students of the university $\widetilde{U}$.
2. The field game A is the set of courses being taken at the university $\widetilde{U}$.
3. (x, a) means the student x is taking the course a, and $f(x, a)$ is his grade in course a. For instance $f(x, a) = 1$ if the student x gets an "A" in course a. In this example the function $f : X \times A \to [0, 1]$ takes only five values: $A = 1$, $B = \frac{3}{4}$, $C = \frac{1}{2}$, $D = \frac{1}{4}$ and $F = 0$.

Observe that if $\widetilde{S} = (X, A, f)$ is a game space, then the value $\|x\|^*$ describes the "skill" of x in the best situation, and the lower value $\|x\|_*$ describes the "skill" of x in the worst situation.

Dually $\|a\|^*$ describes the "qualification" of the position a in hands of the best players and $\|a\|_*$ describes the "qualification" of the position a in hands of the worst players.

For instance, if we take the "soccer example" $\|x\| = 1$ means that the player x can kick a goal from any point a in the soccer field. (This player is really too good!) On the other hand if $x \in X$ is a null event, then $\|x\| = 0$. This player x is, perhaps, the goal keeper!

Dually, if $a \in A$ is a strong state in the soccer space S, then $\|a\| = 1$. This means that a is, in fact, a "winning position". From this point any player can kick a goal! Clearly there are many "winning positions" in the soccer field. If $a \in A$ is a weak state, then $\|a\| = 0$. This position is clearly a most difficult position in the soccer field. From this point no player can kick a goal.

On the other hand in Example 3 the value $\|x\|^*$ describes the "fighting ability" of the soldier x when he has best weapon at hand, and the lower value

$\|x\|_*$ describes the "fighting ability" of the soldier x when he has no weapon at hand.

Since the team X of a game space $\widetilde{G} = (X, A, f)$ is finite we can define the following statistical data for a game space:

1. The number

$$\|\widetilde{G}\| = \sqrt{\sum_{x \in X} \|x\|^2}$$

is called the *norm* of $\widetilde{G}$.

2. The number

$$D(\widetilde{G}) = \sqrt{\sum_{x \in X} (\|x\|^* - \|x\|_*)^2} = \sqrt{\sum_{x \in X} [d(x)]^2}$$

is called the *standard deviation* of $\widetilde{G}$.

3. The number

$$M(\widetilde{G}) = \frac{1}{|X|} \sum_{x \in X} \|x\|,$$

where $|X|$ denotes the cardinality of X, is called the *mean* of $\widetilde{G}$.

Now given a set A, we define the *game category over the field* A, denoted by $\mathcal{G}_A$, as follows:

1. The objects of $\mathcal{G}_A$ are game spaces over A.
2. If $\widetilde{S} = (X, A, f)$ and $\widetilde{T} = (Y, A, g)$ are two game spaces over A, then a morphism $\Phi = (\varphi, 1_A) : \widetilde{S} \to \widetilde{T}$, where $\varphi : X \to Y$ is a map satisfying the condition:

$$f(x, a) \leq g(\varphi(x), a) \text{ for } x \in X \text{ and } a \in A.$$

Consequently morphisms in the game category are Chu upper-morphisms.

Observe that a morphism $\Phi : \widetilde{S} = (X, A, f) \to \widetilde{T} = (Y, A, g)$ in $\mathcal{G}_A$ is determined by a map $\alpha : X \to Y$ such that

$$f(x, a) \leq g(\alpha(x)) \text{ for } x \in X.$$

The existence of a morphism $\Phi : \widetilde{S} = (X, A, f) \to \widetilde{T} = (Y, A, g)$ in the game category over the field A implies that for any player x of the team X there exists a player $\varphi(x)$ of the team Y such that at any position a in the game field A the player $\varphi(x)$ has better chance to win than the player x at the same position a. It follows that the team Y has some advantages over the team X in the field A. It is straightforward to check that

Proposition 11. *If $\widetilde{S} \subseteq \widetilde{G}$, then $\|\widetilde{S}\| \leq \|\widetilde{G}\|$.*

The following theorem shows that the statistical data as norm, mean and standard deviation are game invariances.

Theorem 7. *(The game invariance theorem.) The number $\|\widetilde{G}\|$, $M(\widetilde{G})$ and $D(\widetilde{G})$ are invariances in the game category over the field A. That is, if $\widetilde{S}$ and $\widetilde{G}$ are isomorphic, then $\|\widetilde{S}\| = \|\widetilde{G}\|$, $M(\widetilde{S}) = M(\widetilde{G})$ and $D(\widetilde{S}) = D(\widetilde{G})$.*

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