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Two-Weight Inequality for Fractional Integral Operators and Adams Inequality

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Abstract. For a given weight $u(.)$ another weight $v(.) = (\mathcal{R}u)(.)$ is found such that the fractional integral operator I_α, $0 < \alpha < n$, is bounded from the weighted Lebesgue space $L^p(\mathbb{R}^n, v(x)dx)$ into $L^p(\mathbb{R}^n, u(x)dx)$ whenever $1 < p < n/\alpha$.

1. Introduction

An inequality due to Adams [1] states that for $1 < p < n/\alpha$ and $1 < \tau < n/\alpha p$ there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x) g(x) dx \leq C \int_{\mathbb{R}^n} f^p(y) (M_{\alpha p \tau} g^\tau)^{1/\tau}(y) dy. \tag{1.1}$$

for all locally integrable functions $f(.)$, $g(.) \geq 0$. As usual n is a nonnegative integer. Recall that the fractional operator I_α, $0 < \alpha < n$, is defined by

$$(I_\alpha f)(x) = \int_{\mathbb{R}^n} |x - y|^{\alpha - n} f(y) dy.$$

And M_γ, $0 \leq \gamma < n$, denotes the fractional maximal function

$$(M_\gamma g)(x) = \sup_{t>0} \Big\{ t^{\gamma - n} \int_{B(x,t)} |g(y)| dy \Big\},$$

where $B(x,t) = \{z \in \mathbb{R}^n;\ |x - z| < t\}$.

Inequality (1.1) is a sort of control of the L^p-norm of $(I_\alpha f)(.)$ and can be viewed as a two-weight inequality for the operator I_α. Its introduction is motivated by studies of Schrödinger operators and weighted Sobolev inequalities [3].

Pérez [3] improved (1.1) by showing that

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)w(x)dx \leq C \int_{\mathbb{R}^n} f^p(y)(M_{\alpha p} M^{[p]} w)(y)dy, \quad \text{for all } f(.) \geq 0. \tag{1.2}$$

Here $[p]$ is the integer part of p and $M^k = M \circ \cdots \circ M$ denotes the k-th iteration of the Hardy–Littlewood maximal operator $M = M_0$. The relevance in this Pérez's result appears through the double inequality

$$(M_{\alpha p} w)(.) \leq (M_{\alpha p} M^{[p]} w)(.) \leq c_1 (M_{\alpha p \tau} w^\tau)^{1/\tau}(.)$$

where $c_1 > 0$ is a fixed constant which does not depend on the weight $w(.)$. It is found in [4] that $(M_{\alpha p} M^{[p]} w)(.)$ in (1.2) can be replaced by $(M_{\alpha p} w)(.)$ whenever $w(.)$ satisfies some reverse doubling condition and certain growth assumption on annuli.

Now according to the author's viewpoint, inequalities as (1.1) and (1.2) are not satisfactory if they are considered as weighted inequalities for I_α. Indeed

$$\text{if } w(.) = 1 \text{ then } (M_{\alpha p} w)(.) = \infty, \tag{1.3}$$

however it is well-known (see for instance [5]) that

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)dx \leq C \int_{\mathbb{R}^n} f^p(y)|y|^{\alpha p}dy, \quad \text{for all } f(.) \geq 0. \tag{1.4}$$

Facts described in (1.3) and (1.4) lead to the question of finding another operator $\mathcal{R}$ for which

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)w(x)dx \leq C \int_{\mathbb{R}^n} f^p(y)(\mathcal{R}w)(y)dy, \quad \text{for all } f(.) \geq 0, \tag{1.5}$$

for some constant $C > 0$ which does not depend neither on $f(.)$ nor on $w(.)$ and such that $(\mathcal{R}w)(y) < \infty$ at least for all power weights $w(y) = |y|^\beta$. Consequently, as (two) weighted inequality for I_α, inequality (1.5) is more acceptable than (1.1) or (1.2). To answer the above question is our main purpose of the present paper.

It should be emphasized that our intention is not to improve (1.2). We just aim to bring a good substitute of (1.1) and (1.2) from the two-weight inequality viewpoint, in the sense that for many weights $w(.)$ we have $(\mathcal{R}w)(.) < \infty$ though $(M_{\alpha p} w)(.) = \infty$. For instance as a replacement of (1.2) or (1.1), we will see in Corollary 2.3 that

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)\psi(|x|)dx \leq C \int_{\mathbb{R}^n} f^p(y)|y|^{\alpha p}\psi(|y|)dy, \quad \text{for all } f(.) \geq 0$$

whenever $\psi(.)$ satisfies some suitable growth assumptions.

The question of finding another operator $\mathcal{L}$ such that

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)(\mathcal{L}w)(x)dx \leq C \int_{\mathbb{R}^n} f^p(y)w(y)dy, \quad \text{for all } f(.) \geq 0 \tag{1.6}$$

will be also considered in this paper. It can be noted that some sufficient (or necessary) conditions for the two-weight inequality

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)u(x)dx \le C \int_{\mathbb{R}^n} f^p(y)v(y)dy, \quad \text{for all } f(.) \ge 0 \tag{1.7}$$

were found by many authors (see references given in [5]).

The gain with (1.5) and (1.6) is that, if one weight is given then the other one for which (1.7) holds can be immediately determined without applying any boundedness criterion. Therefore inequalities (1.5) and (1.6) lead to a little step in the knowledge of the (deep problem) two-weight inequality (1.7) for I_α.

2. Results

Throughout this paper it is assumed that

$$0 < \alpha < n, \quad 1 < p < \infty, \quad p' = \frac{p}{p-1}.$$

The restricted fractional maximal function $\widetilde{M}_\gamma$ associated with M_γ, $0 \le \gamma < n$, is defined by

$$(\widetilde{M}_\gamma g)(x) = \sup_{0<t<2^{-1}|x|} \left\{ t^{\gamma-n} \int_{B(x,t)} |g(y)|dy \right\}.$$

This expression is smaller than $(M_\gamma g)(x)$ and for $u(.) = 1$ then $(\widetilde{M}_\gamma u)(x) \approx |x|^\gamma < \infty$ though $(M_\gamma u)(.) = \infty$. Note that $(\widetilde{M}_\gamma u)(.)$ can be easily estimated for many usual weights. It is for instance the case of those $u(.)$ satisfying the growth assumption

$$\sup_{4^{-1}|x| \le |y| \le 4|x|} u(y) \le C_1 |x|^{-n} \int_{|y| \le C_2|x|} u(z)dz, \tag{$\mathcal{H}$}$$

or merely $u(.) \in \mathcal{H}$. Here the constants C_1, $C_2 > 0$ do not depend on $x \ne 0$. Indeed for all τ with $1 \le \tau < n/\gamma$

$$\text{if } u(.) \in \mathcal{H} \text{ then } (\widetilde{M}_{\gamma\tau} u^\tau)^{\frac{1}{\tau}}(x) \le C_3 |x|^{\gamma-n} \int_{|y| \le C_2|x|} u(z)dz \tag{2.0}$$

for some constant $C_3 > 0$ independent of x. The growth assumption $(\mathcal{H})$ is largely used in [4] and [5] and includes all radial and monotone weights.

Our main result of this paper reads as follows.

Theorem 2.1. *Let $p < \frac{n}{\alpha}$ and $1 < \tau < \frac{n}{\alpha p}$. There exists a constant $C > 0$ such that for each weight $u(.)$ satisfying*

$$\int_{\mathbb{R}^n} u(x)dx = \int_{\mathbb{R}^n} |x|^{(\alpha-n)p} u(x)dx = \infty \tag{2.1}$$

we have

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)u(x)dx \le C \int_{\mathbb{R}^n} f^p(y)(\mathcal{R}u)(y)dy, \quad \text{for all } f(.) \ge 0 \tag{2.2}$$

where the operator $\mathcal{R}$ is defined by

$$(\mathcal{R}u)(x) = \max\left\{ (\widetilde{M}_{\alpha p\tau} u^\tau)^{1/\tau}(x) + \left(|x|^{(\alpha p\tau - n)} \int_{4^{-1}|x|<|z|<4|x|} u^\tau(z)dz\right)^{1/\tau}; \right.$$
$$\left(\int_{|x|<|z|} |z|^{(\alpha-n)p} u(z)dz\right)^p \left(|x|^{(\alpha-n)p} u(x)\right)^{1-p};$$
$$\left.\left(|x|^{(\alpha-n)} \int_{|z|<|x|} u(z)dz\right)^p u^{1-p}(x) \right\}. \tag{2.3}$$

As will be seen in the proof of this result, the operator $\mathcal{R}$ must be modified if the assumption (2.1) is not satisfied. The details are not given for shortness reason and also since (2.1) holds for usual and significant cases.

If $u(.) \in \mathcal{H}$ then, by applying (2.0), the first term in the definition of $\mathcal{R}$ can be easily estimated and consequently we get the following.

Corollary 2.2. *Let p and $u(.)$ be as in Theorem 2.1. If moreover $u(.) \in \mathcal{H}$, then for some constant C (which depends on the constant C_1 involved in property $\mathcal{H}$)*

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)u(x)dx \le C \int_{\mathbb{R}^n} f^p(y)v(y)dy, \quad \textit{for all } f(.) \ge 0$$

where

$$v(y) = \left(|y|^{(\alpha p - n)} \int_{4^{-1}|y|<|z|<4|y|} u(z)dz\right)$$
$$+ \left(\int_{|y|<|z|} |z|^{(\alpha-n)p} u(z)dz\right)^p \left(|y|^{(\alpha-n)p} u(y)\right)^{1-p}$$
$$+ \left(|y|^{(\alpha-n)} \int_{|z|<|y|} u(z)dz\right)^p u^{1-p}(y).$$

The inconvenience related to (1.1) and (1.2) as described in (1.3) for power weights and $u(.) = 1$, vanishes with inequality (2.2) since easy computations lead to $(\mathcal{R}u)(y) \approx |y|^{\alpha p} < \infty$.

To give another illustration of the efficiency of Theorem 2.1 in explicit computations, let us introduce the growth condition $\varphi(.) \in \Delta_\sigma$, $\sigma \ge 0$, by which we mean that $\varphi(.)$ is a nonnegative and increasing function on $]0, \infty[$ such that for some constant $c > 0$

$$\varphi(\lambda s) \le c\lambda^\sigma \varphi(s), \quad \text{for all } s > 0 \text{ and } \lambda > 1.$$

For instance if $\varphi(s) = s^a \ln^b(e+s)$, with $a, b \ge 0$, then $\varphi(.) \in \Delta_\sigma$ where $\sigma = a+b$.

Corollary 2.3. *Let $p < n/\alpha$. Assume that $\varphi(.) \in \Delta_\sigma$ such that $0 \le \sigma < p'(n/p - \alpha)$. Then*

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)\psi(|x|)dx \le C \int_{\mathbb{R}^n} f^p(y)|y|^{\alpha p}\psi(|y|)dy, \quad \textit{for all } f(.) \ge 0 \tag{2.4}$$

where $\psi(|x|) = |x|^{-\alpha p}\varphi^{1-p}(|x|)$, and $C > 0$ depends on α, n, p and the constants in assumption Δ_σ.

An answer to the question raised about (1.6) is as follows.

Theorem 2.4. *Let $p' < n/\alpha$ and $1 < \tau < n/\alpha p'$. Then there exists a constant $C > 0$ exists such that for each weight $v(.)$ satisfying*

$$\int_{\mathbb{R}^n} v^{1-p'}(y)dy = \int_{\mathbb{R}^n} |y|^{(\alpha-n)p'} v^{1-p'}(y)dy = \infty, \quad \text{with } \alpha < \frac{n}{p'} \tag{2.5}$$

we have

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)(\mathcal{L}v)(x)dx \le C \int_{\mathbb{R}^n} f^p(y)v(y)dy, \quad \text{for all } f(.) \ge 0, \tag{2.6}$$

where the operator $\mathcal{L}$ is defined by

$$\begin{aligned}(\mathcal{L}v)(x) = \\ \max\Bigg\{&\left(\widetilde{M}_{\alpha p'\tau} v^{(1-p')\tau}\right)^{1/\tau}(x) + \left(|x|^{(\alpha p'\tau - n)} \int_{4^{-1}|x|<|z|<4|x|} v^{(1-p')\tau}(z)dz\right)^{1/\tau};\\ &\left(\int_{|x|<|z|} |z|^{(\alpha-n)p'} v^{1-p'}(z)dz\right)^{p'} \left(|x|^{(\alpha-n)p} v^{1-p'}(x)\right)^{1-p'};\\ &\left(|x|^{(\alpha-n)} \int_{|z|<|x|} v^{1-p'}(z)dz\right)^{p'} \left(v^{1-p'}(x)\right)^{1-p'}\Bigg\}^{1-p}.\end{aligned} \tag{2.7}$$

3. Proofs of Results

To derive our main result we will need the following analogues of (1.5) and (1.6) for the n-dimensional Hardy operators $(Hf)(x) = \int_{|y|<|x|} f(y)dy$ and $(H^*f)(x) = \int_{|x|<|y|} f(y)dy$.

Proposition 3.1. *There is a constant $C > 0$ such that for all weights $w(.)$ with $\int_{\mathbb{R}^n} w(z)dz = \infty$ we have*

$$\int_{\mathbb{R}^n} \left[\int_{|y|<|x|} g(y)dy\right]^p w(x)dx \le C \int_{\mathbb{R}^n} g^p(y) \left(\int_{|y|<|z|} w(z)dz\right)^p w^{1-p}(y)dy \tag{3.1}$$

for all $g(.) \ge 0$.

And for all weights $w(.)$ with $\int_{\mathbb{R}^n} w^{1-p'}(z)dz = \infty$ we have

$$\int_{\mathbb{R}^n} \left[\int_{|y|<|x|} g(y)dy\right]^p \left(\int_{|z|<|x|} w^{1-p'}(z)dz\right)^{-p} w^{1-p'}(x)dx \le C \int_{\mathbb{R}^n} g^p(y)w(y)dy \tag{3.2}$$

for all $g(.) \geq 0$. Again the constant C in (3.2) is independent of the weight $w(.)$ and the function $g(.)$.

Similarly for all weights $w(.)$ with $\int_{\mathbb{R}^n} w(z)dz = \infty$ we have

$$\int_{\mathbb{R}^n} \Big[\int_{|x|<|y|} h(y)dy\Big]^p w(x)dx \leq C \int_{\mathbb{R}^n} h^p(y)\Big(\int_{|z|<|y|} w(z)dz\Big)^p w^{1-p}(y)dy \quad (3.3)$$

for all $h(.) \geq 0$.

Finally for all weights $w(.)$ with $\int_{\mathbb{R}^n} w^{1-p'}(z)dz = \infty$ we have

$$\int_{\mathbb{R}^n} \Big[\int_{|x|<|y|} h(y)dy\Big]^p \Big(\int_{|x|<|z|} w^{1-p'}(z)dz\Big)^{-p} w^{1-p'}(x)dx \leq C \int_{\mathbb{R}^n} h^p(y)w(y)dy \quad (3.4)$$

for all $h(.) \geq 0$.

This result is based on the following two lemmas.

Lemma 3.2. *Suppose that for some $A > 0$*

$$\Big[\int_{R<|x|} u(x)dx\Big]^{\frac{1}{p}} \Big(\int_{|y|<R} v^{1-p'}(y)dy\Big)^{\frac{1}{p'}} \leq A, \quad \text{for all } R > 0. \quad (3.5)$$

Then for some constant $c > 0$ depending only on n and p

$$\int_{\mathbb{R}^n} \Big[\int_{|y|<|x|} g(y)dy\Big]^p u(x)dx \leq (cA)^p \int_{\mathbb{R}^n} g^p(y)v(y)dy, \quad \text{for all } g(.) \geq 0. \quad (3.6)$$

Lemma 3.3. *Let $\varphi(.)$ and $\psi(.)$ be nonnegative and measurable functions defined on $]0, \infty[$. Then for all $R > 0$*

$$\int_0^R \Big[\int_t^\infty \varphi(r)dr\Big]^{-p'} \varphi(t)dt \approx \Big(\int_R^\infty \varphi(t)dt\Big)^{1-p'}, \quad \text{whenever } \int_0^\infty \varphi(t)dt = \infty \quad (3.7)$$

and

$$\int_R^\infty \Big[\int_0^t \psi(r)dr\Big]^{-p} \psi(t)dt \approx \Big(\int_0^R \psi(t)dt\Big)^{1-p}, \quad \text{whenever } \int_0^\infty \psi(t)dt = \infty. \quad (3.8)$$

Here and in the sequel a notation like $a(R) \approx b(R)$ means that for some fixed constants $c_1, c_2 > 0$: $c_1 a(R) \leq b(R) \leq c_2 a(R)$ for all $R > 0$.

Actually (3.5) is a necessary condition for inequality (3.6) to hold. And a proof of Lemma 3.2 can be found in [2]. Elementary computations yield identities (3.7) and (3.8).

We first prove Proposition 3.1 and Theorem 2.4. Next the proofs of Theorem

2.1 and Corollary 2.3 will be performed.

Proof of Proposition 3.1.

We first begin with the proof of inequality (3.1) which is the same as (3.6) with the weights $u(x) = w(x)$ and $v(y) = \left(\int_{|y|<|z|} w(z)dz\right)^p w^{1-p}(y)$. By Lemma 3.2, the task remains to check the test condition (3.5). It is suitable to make use of polar coordinates as follows

$$\int_{|y|<R} v^{1-p'}(y)dy = \int_{|y|<R}\left[\int_{|y|<|z|} w(z)dz\right]^{-p'} w(y)dy$$

$$\approx \int_0^R \left[\int_{t<|z|} w(z)dz\right]^{-p'} \widetilde{w}(t)dt$$

(here $\widetilde{w}(t) = t^{n-1}\int_{S_{n-1}} w(t\sigma)d\sigma$ and $d\sigma$ is the area measure on the unit sphere S_{n-1})

$$\approx \int_0^R \left[\int_t^\infty \widetilde{w}(r)dr\right]^{-p'} \widetilde{w}(t)dt$$

$$\approx \left(\int_R^\infty \widetilde{w}(t)dt\right)^{1-p'}$$

(by (3.7) and since $\int_0^\infty \widetilde{w}(t)dt \approx \int_{\mathbb{R}^n} w(y)dy = \infty$)

$$\approx \left(\int_{R<|x|} w(x)dx\right)^{1-p'} = \left(\int_{R<|x|} u(x)dx\right)^{1-p'}.$$

It means that condition (3.5) appears with the constant $A \approx 1$ (depending only on n and p) and consequently, by Lemma 3.2, the inequality (3.1) is satisfied with a constant C which depends only on n and p but not on the weight $w(.)$.

Next let us prove inequality (3.2) which is the same as (3.6) with the weights $u(x) = \left(\int_{|z|<|x|} w^{1-p'}(z)dz\right)^{-p} w^{1-p'}(x)$ and $v(y) = w(y)$. By Lemma 3.2, the task remains to check the test condition (3.5). Again by making use of polar coordinates then

$$\int_{R<|x|} u(x)dx = \int_{R<|x|}\left[\int_{|z|<|x|} w^{1-p'}(z)dz\right]^{-p} w^{1-p'}(x)dx$$

$$\approx \int_R^\infty \left[\int_0^t \widetilde{w}_1(r)dr\right]^{-p} \widetilde{w}_1(t)dt$$

(here $\widetilde{w}_1(t) = t^{n-1}\int_{S_{n-1}} w^{1-p'}(t\sigma)d\sigma$)

$$\approx \left(\int_0^R \widetilde{w}_1(t)dt\right)^{1-p}$$

(by (3.8) and since $\int_0^\infty \widetilde{w}_1(t)dt \approx \int_{\mathbb{R}^n} w^{1-p'}(y)dy = \infty$)

$$\approx \left(\int_{|y|<R} w^{1-p'}(y)dy\right)^{1-p} = \left(\int_{|y|<R} v^{1-p'}(y)dy\right)^{1-p}.$$

Therefore (3.5) appears with the constant $A \approx 1$ (depending only on n and p) and consequently inequality (3.2) is satisfied with a constant C independent of the weight w and the function $g(.)$.

Now to derive inequality (3.3) we will use the result (3.2).

Since $\int_{\mathbb{R}^n} (w^{1-p'})^{1-p}(z)dz = \int_{\mathbb{R}^n} w(z)dz = \infty$, and applying (3.2) (with the index p' and the weight $w^{1-p'}(.)$) then

$$\int_{\mathbb{R}^n}\left[\int_{|y|<|x|} g(y)dy\right]^{p'}\left[\left(\int_{|z|<|x|} w(z)dz\right)^{p} w^{1-p}(x)\right]^{1-p'} dx$$
$$= \int_{\mathbb{R}^n}\left[\int_{|y|<|x|} g(y)dy\right]^{p'}\left(\int_{|z|<|x|} w^{(1-p)(1-p')}(z)dz\right)^{-p'} w^{(1-p)(1-p')}(x)dx$$
$$\leq C\int_{\mathbb{R}^n} g^{p'}(y)w^{1-p'}(y)dy.$$

By duality and since $g(.)$ is an arbitrary nonnegative, then from this last inequality it follows that

$$\int_{\mathbb{R}^n}\left[\int_{|x|<|y|} h(y)dy\right]^{p} w(x)dx \leq C\int_{\mathbb{R}^n} h^p(y)\left(\int_{|z|<|y|} w(z)dz\right)^{p} w^{1-p}(y)dy$$

for all $h(.) \geq 0$. This is just the inequality (3.3).

Finally to get inequality (3.4) we will proceed as above. Since $\int_{\mathbb{R}^n} w^{1-p'}(z)dz = \infty$, by applying (3.1) (with the index p' and the weight $w^{1-p'}(.)$) then

$$\int_{\mathbb{R}^n}\left[\int_{|y|<|x|} g(y)dy\right]^{p'} w^{1-p'}(x)dx$$
$$\leq C\int_{\mathbb{R}^n} g^{p'}(y)\left(\int_{|y|<|z|} w^{1-p'}(z)dz\right)^{p'} w^{(1-p')(1-p')}(y)dy$$
$$= C\int_{\mathbb{R}^n} g^{p'}(y)\left[\left(\int_{|y|<|z|} w^{1-p'}(z)dz\right)^{-p} w^{1-p'}(y)\right]^{1-p'} dy.$$

This last inequality combined with a duality argument leads to

$$\int_{\mathbb{R}^n}\left[\int_{|x|<|y|} h(y)dy\right]^{p}\left(\int_{|x|<|z|} w^{1-p'}(z)dz\right)^{-p} w^{1-p'}(x)dx \leq C\int_{\mathbb{R}^n} h^p(y)w(y)dy$$

for all $h(.) \geq 0$. This is just inequality (3.4).

Proof of Theorem 2.4.

Since $\int_{\mathbb{R}^n} v^{1-p'}(y)dy = \int_{\mathbb{R}^n} |y|^{(\alpha-n)p'} v^{1-p'}(y)dy = \infty$ then Theorem 2.1 can be applied to derive

$$\int_{\mathbb{R}^n} (I_\alpha h)^{p'}(x) v^{1-p'}(x)dx \le C \int_{\mathbb{R}^n} h^{p'}(y) \Big[(\mathcal{R}v^{1-p'})^{1-p}(y) \Big]^{1-p'} dy$$

for all $h(.) \ge 0$ with

$$(\mathcal{R}v^{1-p'})(x) =$$

$$\max\Big\{ (\widetilde{M}_{\alpha p'\tau} v^{(1-p')\tau})^{\frac{1}{\tau}}(x) + \Big(|x|^{(\alpha p'\tau - n)} \int_{4^{-1}|x|<|z|<4|x|} v^{(1-p')\tau}(z)dz \Big)^{\frac{1}{\tau}};$$

$$\Big(\int_{|x|<|z|} |z|^{(\alpha-n)p'} v^{1-p'}(z)dz \Big)^{p'} \Big(|x|^{(\alpha-n)p'} v^{1-p'}(x) \Big)^{1-p'};$$

$$\Big(|x|^{(\alpha-n)} \int_{|z|<|x|} v^{1-p'}(z)dz \Big)^{p'} \big(v^{1-p'}(x) \big)^{1-p'} \Big\}.$$

By duality argument and since $\int_{\mathbb{R}^n} (I_\alpha f)(x) g(x)dx = \int_{\mathbb{R}^n} f(x)(I_\alpha g)(x)dx$, then the above inequality is equivalent to

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)(\mathcal{L}v)(x)dx \le C \int_{\mathbb{R}^n} f^p(y)v(y)dy, \quad \text{for all } f(.) \ge 0$$

where $(\mathcal{L}v)(.) = \Big[(\mathcal{R}v^{1-p'})(.) \Big]^{1-p}$. And this is just the expected inequality (2.6).

Proof of Theorem 2.2. To derive the inequality (2.2) consider a function $f(.) \ge 0$. Then for some constant $c > 0$ (which depends only on p)

$$\int_{\mathbb{R}^n} (I_\alpha f)^p(x)u(x)dx \le c\Big(\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 \Big),$$

where

$$\mathcal{S}_1 = \sum_k \int_{E_k} (I_\alpha f 1\!\!1_{L_k})^p(x)u(x)dx$$

$$\mathcal{S}_2 = \sum_k \int_{E_k} (I_\alpha f 1\!\!1_{M_k})^p(x)u(x)dx$$

$$\mathcal{S}_3 = \sum_k \int_{E_k} (I_\alpha f 1\!\!1_{R_k})^p(x)u(x)dx$$

$$E_k = \{x; 2^k < |x| \le 2^{k+1}\}, \quad M_k = \{x; 2^{k-1} \le |x| \le 2^{k+2}\},$$

$$L_k = \{x; |x| < 2^{k-1}\}, \quad R_k = \{x; 2^{k+2} < |x|\}.$$

Estimate of $\mathcal{S}_1$

Since the function $(f 1\!\!1_{L_k})(.)$ has its support contained in the ball $B(0, 2^{k-1}) = \{y; |y| < 2^{k-1}\}$ then for each $x \in E_k$

$$\begin{aligned}\big(I_\alpha f 1\!\!I_{L_k}\big)(x) &= \int_{y\in L_k} |x-y|^{\alpha-n} f(y)dy \\ &\leq \int_{|y|<\frac{1}{2}|x|} |x-y|^{\alpha-n} f(y)dy \\ &\leq c_1|x|^{\alpha-n}\int_{|y|<|x|} f(y)dy \ \text{ since } \frac{1}{2}|x|<|x-y|<\frac{3}{2}|x|.\end{aligned}$$

Using this last inequality and (3.1) in Proposition 3.1 then

$$\begin{aligned}\mathcal{S}_1 &\leq c_2\int_{\mathbb{R}^n}\Big[\int_{|y|<|x|} f(y)dy\Big]^p |x|^{(\alpha-n)p}u(x)dx \\ &\leq c_3\int_{\mathbb{R}^n} f^p(y)\Big[\int_{|y|<|z|}|z|^{(\alpha-n)p}u(z)dz\Big]^{p'}\Big(|y|^{(\alpha-n)p}u(y)\Big)^{1-p'}dy. \end{aligned} \tag{3.9}$$

Estimate of $\mathcal{S}_3$

As above, for each $x\in E_k$: x does not belong to the support of the function $(f1\!\!I_{R_k})(.)$. Then

$$\begin{aligned}(I_\alpha f 1\!\!I_{R_k})(x) &\leq \int_{2|x|<|y|} |x-y|^{\alpha-n} f(y)dy \\ &\leq c_4\int_{|x|<|y|} f(y)|y|^{\alpha-n}dy \ \text{ since } \frac{1}{2}|y|<|x-y|<2|y|.\end{aligned}$$

Consequently this last inequality combined with (3.3) in Proposition 3.1 leads to

$$\begin{aligned}\mathcal{S}_3 &\leq c_5\int_{\mathbb{R}^n}\Big[\int_{\{2|x|<|y|\}} f(y)|y|^{\alpha-n}dy\Big]^p u(x)dx \\ &\leq c_6\int_{\mathbb{R}^n} f^p(y)\Big[|y|^{\alpha-n}\int_{|z|<|y|}u(z)dz\Big]^{p'}u^{1-p'}(y)dy. \end{aligned} \tag{3.10}$$

It should be noted that the constants c_3 and c_6 do not depend on the weight $u(.)$.

Estimate of $\mathcal{S}_2$

Observe that for $y\in M_k$

$$(M_{\alpha p\tau}u^\tau 1\!\!I_{E_k})^{\frac{1}{\tau}}(y) \leq$$
$$c_7\Big[(\widetilde{M}_{\alpha p\tau}u^\tau)^{\frac{1}{\tau}}(y) + \Big(|y|^{(\alpha p\tau-n)}\int_{4^{-1}|y|<|z|<4|y|}u^\tau(z)dz\Big)^{\frac{1}{\tau}}\Big]. \tag{3.11}$$

Of course, the constant $c_7>0$ depends just on n. Indeed if $y\in M_k$ (e.g. $2^{k-1}\leq|y|\leq 2^{k+2}$) and $t>2^{-1}|y|$ then

$$\begin{aligned}t^{(\alpha p\tau-n)}\int_{B(y,t)\cap E_k}u^\tau(z)dz &\leq c_8 2^{(\alpha p\tau-n)k}\int_{E_k}u^\tau(z)dz \\ &\leq c_9|y|^{(\alpha p\tau-n)}\int_{4^{-1}|y|<|z|<4|y|}u^\tau(z)dz.\end{aligned}$$

Now the estimate of $\mathcal{S}_2$ can be performed by using (3.11) and the D. Adams inequality (1.1) as follows

$$\mathcal{S}_2 = \sum_k \int_{E_k} (I_\alpha f \mathbb{I}_{M_k})^p(x)u(x)dx$$
$$\leq c_{10} \sum_k \int_{M_k} f^p(y)\Big(M_{\alpha p\tau}u^\tau \mathbb{I}_{E_k}\Big)^{\frac{1}{\tau}}(y)dy \qquad \text{by (1.1)}$$
$$\leq c_{11} \sum_k \int_{M_k} f^p(y)$$
$$\left[(\widetilde{M}_{\alpha p\tau}u^\tau)^{\frac{1}{\tau}}(y) + \left(|y|^{(\alpha p\tau - n)}\int_{4^{-1}|y|<|z|<4|y|} u^\tau(z)dz\right)^{\frac{1}{\tau}}\right]dy \qquad \text{by (3.11)}$$
$$= c_{12}\int_{\mathbb{R}^n} f^p(y)$$
$$\left[(\widetilde{M}_{\alpha p\tau}u^\tau)^{\frac{1}{\tau}}(y) + \left(|y|^{(\alpha p\tau - n)}\int_{4^{-1}|y|<|z|<4|y|} u^\tau(z)dz\right)^{\frac{1}{\tau}}\right]dy. \qquad (3.12)$$

Therefore (3.9), (3.10) and (3.12) lead to the expected inequality (2.2).

Proof of Corollary 2.3

To get the inequality (2.4) we can apply Corollary 2.2 with the weights $u(.) = \psi(|.|) = |.|^{-\alpha p}\varphi^{1-p}(|.|)$ and $v(.) = |.|^{\alpha p}\psi(|.|)\varphi^{1-p}(|.|)$. Indeed

$$\int_{\mathbb{R}^n} |x|^{(\alpha-n)p}u(x)dx \geq \int_{|x|<1} |x|^{-np}\varphi^{1-p}(|x|)dx$$
$$\geq \varphi^{1-p}(1)\int_{|x|<1} |x|^{-np}dx = \infty,$$

and

$$\int_{\mathbb{R}^n} u(x)dx \geq \int_{1<|x|} |x|^{-\alpha p}\varphi^{1-p}(|x|)dx$$
$$\geq c_0\varphi^{1-p}(1)\int_{1<|x|} |x|^{-[\alpha p+(p-1)\sigma]}dx = \infty,$$

because $\varphi(.) \in \Delta_\sigma$ with $n - [\alpha p + (p-1)\sigma] > 0$. Therefore the task remains to prove that $u(.) \in \mathcal{H}$ and a constant $c > 0$ exists such that

$$\left(|y|^{(\alpha p - n)}\int_{4^{-1}|y|<|z|<4|y|} u(z)dz\right) = U_1(y) \leq c\varphi^{1-p}(|y|) = cv(y), \qquad (3.13)$$

$$\left(\int_{|y|<|z|} |z|^{(\alpha-n)p}u(z)dz\right)^p \left(|y|^{(\alpha-n)p}u(y)\right)^{1-p} = U_2(y) \leq c\varphi^{1-p}(|y|) = cv(y), \qquad (3.14)$$

and

$$\left(|y|^{(\alpha-n)}\int_{|z|<|y|} u(z)dz\right)^p u^{1-p}(y) = U_3(y) \leq c\varphi^{1-p}(|y|) = cv(y). \qquad (3.15)$$

The assumption $u(.) \in \mathcal{H}$ is satisfied since from $\varphi(.) \in \Delta_\sigma$ then

$$\begin{aligned}\sup_{4^{-1}|x|\leq|y|\leq 4|x|} u(y) &\approx (2|x|)^{-\alpha p}\varphi^{1-p}(2|x|)\\ &\approx |x|^{-n}\int_{2^{-1}|x|<|z|<2|x|}(2|x|)^{-\alpha p}\varphi^{1-p}(2|x|)dz\\ &\leq |x|^{-n}\int_{|z|<2|x|}u(z)dz.\end{aligned}$$

Inequalities (3.13) and (3.14) follow from $\varphi(.) \in \Delta_\sigma$ since

$$U_1(y) \leq c_1|y|^{\alpha p} \times |y|^{-\alpha p}\varphi^{1-p}(|y|) = c_1\varphi^{1-p}(|y|)$$

and

$$\begin{aligned}U_2(y) &\leq c_1\Big(\varphi^{1-p}(|y|)\int_{|y|<|z|}|z|^{-np}dz\Big)^p\Big(|y|^{-np}\varphi^{1-p}(|y|)\Big)^{1-p}\\ &\leq c_1\Big(|y|^{n(1-p)}\varphi^{1-p}(|y|)\Big)^p\Big(|y|^{-np}\varphi^{1-p}(|y|)\Big)^{1-p}\\ &= c_1\varphi^{1-p}(|y|).\end{aligned}$$

Finally inequality (3.15) appears after using $\varphi(.) \in \Delta_\sigma$ with the restriction $n - [\alpha p + \sigma(p-1)] > 0$ since

$$\begin{aligned}U_3(y) &= \Big(|y|^{\alpha p-n}\int_{|z|<|y|}|z|^{-\alpha p}\varphi^{1-p}(|z|)dz\Big)^p\Big(\varphi^{1-p}(|y|)\Big)^{1-p}\\ &\leq c_2\Big(|y|^{(\alpha p-n)+\sigma(p-1)}\varphi^{1-p}(|y|)\int_{|z|<|y|}|z|^{-[\alpha p+\sigma(p-1)]}dz\Big)^p\Big(\varphi^{1-p}(|y|)\Big)^{1-p}\\ &\leq c_3\Big(\varphi^{1-p}(|y|)\Big)^p\Big(\varphi^{1-p}(|y|)\Big)^{1-p} = c_3\varphi^{1-p}(|y|).\end{aligned}$$

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