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A Free Boundary Problem for Heat Equation Arising in Infiltration

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Abstract. This paper deals with an implicit free boundary problem for heat equation. The method of semi-discretization with respect to t is proposed for proving the existence of the solution.

1. Introduction

Consider the following free boundary problem:

Problem (P). Find a pair $\{u(x,t), s(t)\}$ such that the following equation and conditions are satisfied:

1) $s(t) \in C^1[0,T],\ \ s(t) > 0,\ \ \forall t \in [0,T],\ s(0) = b > 0.$
2) $u(x,t) \in C^{2,1}(D_T)$, where $D_T = \{(x,t) : 0 \le x \le s(t),\ 0 \le t \le T\}.$
3)

$$u_{xx} - u_t = 0 \quad \text{in } D_T, \tag{1.1}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le b, \tag{1.2}$$

$$u(0,t) = f_1(t), \quad 0 \le t \le T, \tag{1.3}$$

$$u(s(t),t) = f_2(s(t),t), \quad 0 \le t \le T, \tag{1.4}$$

$$u'_x(s(t),t) = g(s(t),t), \quad 0 \le t \le T, \tag{1.5}$$

$\varphi(x),\ f_1(t),\ f_2(x,t),\ g(x,t)$ being given functions, $b,\ T$ being given positive constants.

This problem is the mathematical model of the earth compaction and seepage with variable porosity taking into account the effect of molecularly bound water [3]. It is observed that the seepage holds if and only if the gradient of pressure u exceeds some threshold value g. Hence the line $x = s(t)$ is the boundary between the region in which the seepage holds and the region in which there is no seepage.

In this problem, the derivative $s'(t)$ does not appear explicitly on the free boundary conditions as in the Stefan-like problems. For this reason the problem (P) is called implicit free boundary problem.

A somewhat unified technique has been developed for solving Stefan-like problem: the problem is transformed into an equivalent integral equation and then attacked by using a fixed point argument (see for instance [4]). The existence of the solution of our problem (P) is established by using the method of semi-discretization with respect to t. This scheme has been used by T. D. Ventsel [9], Nguyen Dinh Tri [7, 8]. Gary G. Sackett [6] for somewhat different problems. Another method for solving Cauchy type free boundary problem for nonlinear parabolic equations can be found in [2].

The plan of this paper is the following. In Sec. 2 the uniqueness of the solution is obtained. The existence of the solution and the asymptotic behaviour of the solution when $t \to +\infty$ are established in Sec. 3.

2. Uniqueness of the Solution

Theorem 1. *Assume that the functions* $f_1'(t)$, $f_{2x}'(x,t)$, $f_{2t}'(x,t)$, $g(x,t)$, $g_x'(x,t)$, $\varphi''(x)$ *are continuous,* $f_1'(t) \le 0$, $f_{2x}'(x,t) \le 0$, $f_{2t}'(x,t) \le 0$, $g(x,t) > 0$, $g_x'(x,t) \ge 0$, $\varphi''(x) \le 0$. *Then the problem* (P) *cannot have more than one solution satisfying* $ds/dt \ge 0$.

Proof. The function $q := u_t$ satisfies the equation

$$q_t - q_{xx} = 0 \quad \text{in } D_T \tag{2.1}$$

and the conditions

$$q(x,0) = \varphi''(x) \le 0, \quad 0 \le x \le b, \tag{2.2}$$

$$q(0,t) = f_1'(t) \le 0, \quad 0 \le t \le T, \tag{2.3}$$

$$q(s(t),t) = [f_{2x}'(s(t),t) - g_x'(s(t),t)]s'(t) + f_{2t}'(s(t),t) \le 0, \quad 0 \le t \le T. \tag{2.4}$$

By the maximum principle, we get $q(x,t) = u_t(x,t) = u_{xx}(x,t) \le 0$ in D_T. Since $u_x(s(t),t) = g(s(t),t) > 0$, we have $u_x(x,t) > 0$ in D_T.

Suppose that the problem (P) has two solutions: $(s_1(t), u_1(x,t))$ and $(s_2(t), u_2(t))$. Then $v(x,t) = u_1(x,t) - u_2(x,t)$ is the solution of the equation

$$v_{xx} - v_t = 0, \quad 0 \le x \le s(t) = \min_{0 \le t \le T} \{s_1(t), s_2(t)\}, \quad 0 \le t \le T, \tag{2.5}$$

satisfying the conditions

$$v(x,0) = 0, \tag{2.6}$$

$$v_x(0,t) = 0. \tag{2.7}$$

Hence v can reach positive maximum or negative minimum only on $x = s(t)$. If $s(t) = s_1(t)$, we have

$$\begin{aligned} v(s(t),t) &= u_1(s_1(t),t) - u_2(s_1(t),t) = f_2(s_1(t),t) - u_2(s_1(t),t) \\ &\geq f_2(s_2(t),t) - u_2(s_1(t),t) = u_2(s_2(t),t) - u_2(s_1(t),t) \\ &= [s_2(t) - s_1(t)]u'_{2x}(\xi(t),t) \geq 0,\ \xi(t) \in (s_1(t), s_2(t)), \end{aligned}$$

$$\begin{aligned} v'_x(s(t),t) &= u'_{1x}(s_1(t),t) - u'_{2x}(s_1(t),t) = g(s_1(t),t) - u'_{2x}(s_1(t),t) \\ &\leq g(s_2(t),t) - u'_{2x}(s_1(t),t) = u'_{2x}(s_2(t),t) - u'_{2x}(s_1(t),t) \\ &= [s_2(t) - s_1(t)]u''_{2xx}(\eta(t),t) \leq 0,\ \eta(t) \in (s_1(t), s_2(t)). \end{aligned}$$

It is a contradiction. We get the same conclusion if $s(t) = s_2(t)$. ■

Corollary. *We have the following estimation*

$$s(t) \leq X, \quad \forall t \in [0,T] \tag{2.8}$$

where X is the solution of the equation

$$xg(x,0) - f_2(x,0) + f_1(T) = 0 \tag{2.9}$$

Proof. Since $u_{xx} \leq 0$, $u_x > 0$ in D_T, we get

$$s(t) \leq \frac{f_2(s(t),t) - f_1(t)}{g(s(t),t)} \leq \frac{f_2(s(t),0) - f_1(T)}{g(s(t),0)}$$

i.e.

$$s(t)g(s(t),0) - f_2(s(t),0) + f_1(T) \leq 0. \tag{2.10}$$

It is easy to check that the equation (2.9) has a unique solution X and from (2.10) we obtain the estimation (2.8). ■

3. Existence of the Solution

Theorem 2. *Under the assumptions*

1) $f_1(t) \in C^2[0,T]$, $\varphi(x) \in C^4[0,b]$, $f_2(x,t) \in C^{2,2}(Q_T)$, $g(x,t) \in C^{2,2}(Q_T)$, $Q_T = \{(x,t) : 0 \leq x \leq X,\ 0 \leq t \leq T\}$;
2) $f'_1(t) < 0$, $\forall t \in [0,T]$; $\varphi''(x) < f'_{2t}(b,0)$, $\forall x \in [0,b]$; $f'_{2x}(x,t) < 0$, $f'_{2t}(x,t) < 0$, $f''_{2tt}(x,t) > 0$, $f'_{2t}(x,t) - f'_1(t) > 0$, $g(x,t) > 0$, $g'_x(x,t) > 0$, $g'_t(x,t) > 0$, $\forall (x,t) \in Q_T$;
3) $\varphi(0) = f_1(0)$, $\varphi(b) = f_2(b,0)$, $\varphi'(b) = g(b,0)$, $\varphi''(0) = f'_1(0)$,

there exists a solution of problem (P) satisfying $s'(t) \geq 0$.

Proof. The semi-discretization method is applied for producing approximations: we approximate the derivative with respect to t by a finite difference and leave x as continuous variable. This scheme leads to a recursive family of free boundary problems for ordinary differential equations of second order.

Let $t_n = n\Delta t$, with $\Delta t > 0$,

$$u_n(x) = u(x,t_n), \quad s_n = s(t_n), \quad s_0 = b.$$

The problem (P) is approximated by the following problems (P_n).

Problem (P_n). Find a pair $\{u_n(x), s_n\}$ satisfying

$$\frac{u_n(x) - u_{n-1}(x)}{\Delta t} = u_n''(x), \quad 0 \le x \le s_n, \tag{3.1}$$

$$u_0(x) = \varphi(x), \quad 0 \le x \le s_0, \tag{3.2}$$

$$u_n(0) = f_1(t_n), \tag{3.3}$$

$$u_n(s_n) = f_2(s_n, t_n), \tag{3.4}$$

$$u_n'(s_n) = g(s_n, t_n). \tag{3.5}$$

We have to prove:
1) the existence and uniqueness of the solution of problems (P_n);
2) the uniform boundedness of some quantities, to be used for establishing the convergence of the approximation scheme;
3) the convergence of the scheme.

3.1. Existence and Uniqueness of the Solution of Problem (P_n)

Proposition 1. *For Δt sufficiently small, the problem (P_n) has a unique solution $\{u_n(x), s_n\}$ such that $s_n > s_{n-1}$, $\forall n$.*

Proof. We put

$$u_n(x) = f_2(s_n, t_n) + g(s_n, t_n)(x - s_n), \quad \text{if } x > s_n.$$

The general solution of (3.1) is

$$u_n(x) = A_n \mathrm{sh}\frac{x}{\sqrt{\Delta t}} + B_n \mathrm{ch}\frac{x}{\sqrt{\Delta t}} - \frac{1}{\sqrt{\Delta t}}\int_0^x \mathrm{sh}\frac{x-\xi}{\sqrt{\Delta t}} u_{n-1}(\xi)d\xi,$$

A_n, B_n being arbitrary constants. From (3.3)–(3.5) we get

$$f_2(s_n, t_n)\mathrm{ch}\frac{s_n}{\sqrt{\Delta t}} - \sqrt{\Delta t}\, g(s_n, t_n)\mathrm{sh}\frac{s_n}{\sqrt{\Delta t}} = f_1(t_n) + \frac{1}{\sqrt{\Delta t}}\int_0^{s_n} u_{n-1}(\xi)\mathrm{sh}\frac{\xi}{\sqrt{\Delta t}}d\xi.$$

Consequently, we have to prove that there exists a unique zero $s_n > s_{n-1}$ for the function

$$\Phi_n(s) = f_2(s, t_n)\mathrm{ch}\frac{s}{\sqrt{\Delta t}} - \sqrt{\Delta t}\, g(s, t_n)\mathrm{sh}\frac{s}{\sqrt{\Delta t}} - f_1(t_n) - \frac{1}{\sqrt{\Delta t}}\int_0^{s} u_{n-1}(\xi)\mathrm{sh}\frac{\xi}{\sqrt{\Delta t}}d\xi. \tag{3.6}$$

For this, it is sufficient to show inductively that for Δt small enough

$$\Phi_n(s_{n-1}) > 0, \quad \Phi_n'(s_{n-1}) < 0, \quad \Phi_n''(s) < 0 \quad \text{for } s > s_{n-1}.$$

We will use the notation

$$\Delta h(.,t_n) = h(.,t_n) - h(.,t_{n-1}),$$
$$\Delta^2 h(.,t_n) = \Delta(\Delta h(.,t_n)).$$

We have

$$\Phi_n'(s_{n-1}) = \frac{\Delta f_2(s_{n-1},t_n)}{\sqrt{\Delta t}}\operatorname{sh}\frac{s_{n-1}}{\sqrt{\Delta t}} + f_{2x}'(s_{n-1},t_n)\operatorname{ch}\frac{s_{n-1}}{\sqrt{\Delta t}}$$
$$- \sqrt{\Delta t}\, g_x'(s_{n-1},t_n)\operatorname{sh}\frac{s_{n-1}}{\sqrt{\Delta t}} - g(s_{n-1},t_n)\operatorname{ch}\frac{s_{n-1}}{\sqrt{\Delta t}},$$

and for $s > s_{n-1}$

$$\Phi_n''(s) = f_{2xx}''(s,t_n)\operatorname{ch}\frac{s}{\sqrt{\Delta t}} + \frac{2}{\sqrt{\Delta t}} f_{2x}'(s,t_n)\operatorname{sh}\frac{s}{\sqrt{\Delta t}}$$
$$+ \frac{1}{\Delta t}\left[f_2(s,t_n) - f_2(s_{n-1},t_{n-1}) - g(s_{n-1},t_{n-1})(s-s_{n-1})\right]\operatorname{ch}\frac{s}{\sqrt{\Delta t}}$$
$$- \sqrt{\Delta t} g_{xx}''(s,t_n)\operatorname{sh}\frac{s}{\sqrt{\Delta t}} - 2g_x'(s,t_n)\operatorname{ch}\frac{s}{\sqrt{\Delta t}}$$
$$- \frac{1}{\sqrt{\Delta t}}\left[g(s,t_n) + g(s_{n-1},t_{n-1})\right]\operatorname{sh}\frac{s}{\sqrt{\Delta t}}\cdot$$

Because $f_{2x}' < 0$, $f_{2t}' < 0$, $g > 0$, $g_x' > 0$, we get $\Phi_n'(s_{n-1}) < 0$ and with Δt sufficiently small $\Phi_n''(s) < 0$ for $s > s_{n-1}$.

Now we will prove by induction that $\Phi_n(s_{n-1}) > 0$. For $n = 1$ we have

$$\Phi_1(b) = f_2(b,t_1)\operatorname{ch}\frac{b}{\sqrt{\Delta t}} - \sqrt{\Delta t}\, g(b,t_1)\operatorname{sh}\frac{b}{\sqrt{\Delta t}}$$
$$- f_1(t_1) - \frac{1}{\sqrt{\Delta t}}\int_0^b \varphi(\xi)\operatorname{sh}\frac{\xi}{\sqrt{\Delta t}}d\xi.$$

By integrating the last term by parts and using the compatibility conditions, we get

$$\Phi_1(b) = \left[f_2(b,t_1) - f_2(b,0)\right]\operatorname{ch}\frac{b}{\sqrt{\Delta t}} - \sqrt{\Delta t}\left[g(b,t_1) - g(b,0)\right]\operatorname{sh}\frac{b}{\sqrt{\Delta t}}$$
$$+ \left[f_1(0) - f_1(t_1)\right] - \sqrt{\Delta t}\int_0^b \varphi''(\xi)\operatorname{sh}\frac{\xi}{\sqrt{\Delta t}}d\xi$$
$$= \Delta\left[f_2(b,t_1) - f_1(t_1)\right] - \sqrt{\Delta t}\Delta g(b,t_1)\left(1 - e^{-\frac{b}{\sqrt{\Delta t}}}\right)$$
$$- \sqrt{\Delta t}\int_0^b \left[\varphi''(\xi) - \frac{\Delta f_2(b,t_1)}{\Delta t} + \sqrt{\Delta t}\frac{\Delta g(b,t_1)}{\Delta t}\right]\operatorname{sh}\frac{\xi}{\sqrt{\Delta t}}\, d\xi.$$

Since $\varphi''(x) - f_{2t}'(b,t) < 0$, for Δt sufficiently small, the kernel of the last term is negative and this term is predominant. Hence $\Phi_1(b) > 0$.

Moving on to the induction step, using the induction hypothesis that $\Phi_{n-1}(s_{n-1}) = 0$, we get

$$\Phi_n(s_{n-1}) = \Phi_n(s_{n-1}) - \Phi_{n-1}(s_{n-1})$$
$$= \Delta f_2(s_{n-1}, t_n)\text{ch}\frac{s_{n-1}}{\sqrt{\Delta t}} - \sqrt{\Delta t}\Delta g(s_{n-1}, t_n)\text{sh}\frac{s_{n-1}}{\sqrt{\Delta t}} - \Delta f_1(t_n)$$
$$-\frac{1}{\sqrt{\Delta t}}\int_0^{s_{n-1}} \Delta u_{n-1}(\xi)\text{sh}\frac{\xi}{\sqrt{\Delta t}}d\xi$$
$$= \Delta[f_2(s_{n-1}, t_n) - f_1(t_n)] - \sqrt{\Delta t}\Delta g(s_{n-1}, t_n)(1 - e^{-\frac{s_{n-1}}{\sqrt{\Delta t}}})$$
$$-\sqrt{\Delta t}\int_0^{s_{n-1}} \Big[\frac{\Delta u_{n-1}(\xi)}{\Delta t} - \frac{\Delta f_2(s_{n-1}, t_n)}{\Delta t} + \sqrt{\Delta t}\frac{\Delta g(s_{n-1}, t_n)}{\Delta t}\Big]\text{sh}\frac{\xi}{\sqrt{\Delta t}}d\xi.$$

We will prove that for Δt sufficiently small and for $0 \le x \le s_{n-1}$,

$$\frac{\Delta u_{n-1}(x)}{\Delta t} - \frac{\Delta f_2(s_{n-1}, t_n)}{\Delta t} < 0, \tag{3.7}$$

the kernel of the last term is negative and this term is predominant. Hence $\Phi_n(s_{n-1}) > 0$. We put

$$q_{n-1}(x) = \frac{\Delta u_{n-1}(x)}{\Delta t},$$
$$Q_{n-1}(x) = q_{n-1}(x) - \frac{\Delta f_2(s_{n-1}, t_n)}{\Delta t}.$$

The function $Q_{n-1}(x)$ satisfies the following relations

$$Q''_n(x) - \frac{Q_{n-1}(x)}{\Delta t} = \frac{\Delta f_2(s_{n-1}, t_n)}{\Delta t^2}, \quad s_{n-2} \le x \le s_{n-1} \tag{3.8}$$

$$Q_{n-1}(s_{n-1}) = \frac{f_2(s_{n-1}, t_{n-1}) - f_2(s_{n-2}, t_{n-2}) - g(s_{n-2}, t_{n-2})(s_{n-1} - s_{n-2})}{\Delta t}$$
$$- \frac{\Delta f_2(s_{n-1}, t_n)}{\Delta t}$$
$$= -\Big[\Delta t\frac{\Delta^2 f_2(s_{n-1}, t_n)}{\Delta t^2} + g(s_{n-2}, t_{n-2})\frac{\Delta s_{n-1}}{\Delta t}$$
$$- f'_{2x}(\xi_1, t_{n-2})\frac{\Delta s_{n-1}}{\Delta t}\Big] < 0,$$

since $f''_{2tt} > 0$, $g > 0$, $f'_{2x} < 0$; $\xi_1 \in (s_{n-2}, s_{n-1})$,

$$Q'_{n-1}(s_{n-1}) = \frac{g(s_{n-1}, t_{n-1}) - g(s_{n-2}, t_{n-2})}{\Delta t}$$
$$= \frac{\Delta g(s_{n-1}, t_{n-1})}{\Delta t} + g'_x(\xi_2, t_{n-2})\frac{\Delta s_{n-1}}{\Delta t} > 0,$$

since $g'_t > 0$, $g'_x > 0$, $\xi_2 \in (s_{n-2}, s_{n-1})$. Because the right-hand side of (3.8) is a negative constant and because $Q_{n-1}(s_{n-1}) < 0$, $Q'_{n-1}(s_{n-1}) > 0$, it follows that

$$Q_{n-1}(x) < 0 \quad \text{for} \quad x \in (s_{n-2}, s_{n-1}), \tag{3.9}$$

and, a fortiori, $Q_{n-1}(s_{n-1}) < 0$. On the interval $[0, s_{n-2}]$ we have

$$Q''_{n-1}(x) - \frac{Q_{n-1}(x) - Q_{n-2}(x)}{\Delta t} = \frac{\Delta^2 f_2(s_{n-1}, t_n)}{\Delta t^2}, \tag{3.10}$$

$$\begin{aligned} Q_{n-1}(0) &= \frac{\Delta f_1(n-1)}{\Delta t} - \frac{\Delta f_2(s_{n-1}, t_n)}{\Delta t} \\ &= -\frac{\Delta[f_2(s_{n-1}, t_n) - f_1(t_n)] + \Delta^2 f_1(t_n)}{\Delta t} < 0, \end{aligned}$$

for Δt small enough, since $f'_{2t}(x,t) - f'_1(t) > 0$,

$$Q_0(x) = \varphi''(x) - \frac{\Delta f_2(b, t_1)}{\Delta t} < 0$$

for t_1 sufficiently small, since $\varphi''(x) - f'_{2t}(b, 0) < 0$. Because the right hand side of (3.10) is positive, $Q_0(x) < 0$, $Q_{n-1}(0) < 0$, $Q_{n-1}(s_{n-2}) < 0$, we obtain

$$Q_{n-1}(x) < 0 \quad \text{for} \quad x \in [0, s_{n-2}]. \tag{3.11}$$

The estimation (3.6) is deduced from (3.8) and (3.10). The proof is complete. ■

3.2. Some a Priori Estimations

Proposition 2. *We have*

$$s_n \le X, \quad \forall n, \tag{3.12}$$

where X is the solution of the equation (2.9), and for every n, for $x \in [0, s_n]$

$$|u''_n(x)| = \left|\frac{u_n(x) - u_{n-1}(x)}{\Delta t}\right| \le M_1, \tag{3.13}$$

$$|u'_n(x)| \le M_2, \tag{3.14}$$

$$|u_n(x)| \le M_3, \tag{3.15}$$

$$\left|\frac{\Delta s_n}{\Delta t}\right| \le M_4; \tag{3.16}$$

M_i are the positive constants which do not depend on x, n, Δt.

Proof. From (3.7) it follows that

$$u''_n(x) = q_n(x) \le \frac{\Delta f_2(x, t_{n+1})}{\Delta t} \le 0, \quad \forall n, \ \forall x \in [0, s_n],$$

and for Δt small enough, since $f'_{2t}(x,t) < 0$, and

$$|u''_n(x)| = |q_n(x)| \le \max_{(x,t)\in Q_T} |f'_{2t}(x,t)| = M_1.$$

Since $u'_n(x)$ decreases on $[0, s_n]$, $u'_n(s_n) = g(s_n, t_n) > 0$, we have $u'_n(x) > 0$ on $[0, s_n]$. Because $u''_n(x) < 0$, $u'_n(x) > 0$ on $[0, s_n]$, we get (3.12) as in the corollary of Proposition 1. The estimations (3.14), (3.15) are deduced easily from (3.13). Because

$$q_n(s_n) = \frac{f_2(s_n,t_n) - f_2(s_{n-1},t_{n-1})}{\Delta t} - g(s_{n-1},t_{n-1})\frac{\Delta s_n}{\Delta t},$$

and both terms of the right-hand side are negative, we obtain

$$g(s_{n-1},t_{n-1})\frac{\Delta s_n}{\Delta t} \le |q_n(s_n)| \le M_1.$$

Consequently

$$\left|\frac{\Delta s_n}{\Delta t}\right| \le \frac{M_1}{\min\limits_{(x,t)\in Q_T} g(x,t)} = M_4.$$

Proposition 3. *We have for every n, for $x \in [0, s_{n-1}]$*

$$|q_n'(x)| = |u_n'''(x)| \le M_5. \tag{3.17}$$

Proof. The function $z_n(x) := q_n'(x)$ satisfies the following relations

$$z_n''(x) = \frac{z_n(x) - z_{n-1}(x)}{\Delta t}, \quad 0 \le x \le s_{n-1},$$

$$z_0(x) = \varphi'''(x), \quad 0 \le x \le b,$$

$$\begin{aligned} z_n(s_{n-1}) &= \frac{g(s_n,t_n) - g(s_{n-1},t_{n-1})}{\Delta t} - \frac{\Delta s_n}{\Delta t} u_n''(\xi_3) \\ &= g_t'(s_n,\tau) + g_x'(\xi_4,t_{n-1})\frac{\Delta s_n}{\Delta t} - u_n''(\xi_3)\frac{\Delta s_n}{\Delta t}, \end{aligned}$$

$\xi_3 \in (s_{n-1}, s_n)$, $\xi_4 \in (s_{n-1}, s_n)$, $\tau \in (t_{n-1}, t_n)$. Hence

$$|z_0(x)| \le \max_{0\le x\le b} |\varphi'''(x)| = M_6, \quad \forall x \in [0,b]$$

$$|z_{n-1}(s_{n-1})| \le \max_{(x,t)\in Q_T} |g_t'(x,t)| + \max_{(x,t)\in Q_T} |g_x'(x,t)| \cdot M_4 + M_1 M_4 = M_7.$$

We need a bound for $z_n(0)$ and to this end we introduce two Bernstein auxiliary functions

$$h_n^{\pm}(x) = q_n(x) - \frac{\Delta f_1(t_n)}{\Delta t} \pm M_8 e^{-M_9 x}, \quad 0 \le x \le \frac{1}{M_9},$$

where the positive constants M_8, M_9 will be chosen later with $\frac{1}{M_8} \le b$ (see [1]). We get

$$\begin{aligned} h_n^{+\prime\prime}(x) - \frac{h_n^+(x) - h_{n-1}^+(x)}{\Delta t} &= q_n''(x) - \frac{q_n(x) - q_{n-1}(x)}{\Delta t} \\ &\quad + M_8 M_9^2 e^{-M_9 x} + \frac{\Delta^2 f_1(t_n)}{\Delta t^2} \\ &= M_8 M_9^2 e^{-M_9 x} + \frac{\Delta^2 f_1(t_n)}{\Delta t^2}, \end{aligned}$$

the right-hand side is positive for $M_8 M_9$ large enough, independent of the sign of $f_1''(t)$. Thus $h_n^+(x)$ takes its maximal value for $n = 0$, $x = 0$, or $x = 1/M_9$. We have

$$h_n^+(0) = M_8,$$

$$h_n^+\left(\frac{1}{M_9}\right) = q_n\left(\frac{1}{M_9}\right) - q_n(0) + \frac{M_8}{e} \leq 2M_1 + \frac{M_8}{e} < M_8,$$

if M_8 is chosen larger than $4M_1$,

$$h_0^{+\prime}(x) = \varphi'''(x) - M_8 M_9 e^{-M_9 x} \leq \max_{0 \leq x \leq b} |\varphi'''(x)| - \frac{M_8 M_9}{2} < 0$$

for M_9 large enough. Consequently, $h_n^+(x)$ takes its maximal value on $x = 0$, hence $h_n^{+\prime}(0) \leq 0$, or

$$q_n'(0) - M_8 M_9 \leq 0.$$

The similar argument applied to $h_n^-(x)$ yields $q_n'(0) + M_8 M_9 \geq 0$. Consequently,

$$|z_n(0)| \leq M_8 M_9,$$

$$|z_n(x)| = |q_n'(x)| \leq \max(M_6, M_7, M_8 M_9) = M_5. \quad ∎$$

Proposition 4. *We have for every n, for $x \in [0, s_{n-2}]$*

$$\left|\frac{q_n(x) - q_{n-1}(x)}{\Delta t}\right| \leq M_{10}, \tag{3.18}$$

$$\left|\frac{s_n - 2s_{n-1} + s_{n-2}}{\Delta t^2}\right| \leq M_{11}. \tag{3.19}$$

Proof. The function $r_n(x) := [q_n(x) - q_{n-1}(x)]/\Delta t$ satisfies the following relations

$$r_n''(x) = \frac{r_n(x) - r_{n-1}(x)}{\Delta t}, \quad 0 \leq x \leq s_{n-2},$$

$$r_0(x) = \varphi^{(4)}(x), \quad 0 \leq x \leq b,$$

$$r_n(0) = \frac{\Delta^2 f_1(t_n)}{\Delta t^2},$$

$$r_n(s_{n-2}) = \frac{u_n(s_{n-2}) - 2u_{n-1}(s_{n-2}) + u_{n-2}(s_{n-2})}{\Delta t^2}$$

$$= \alpha \frac{\Delta^2 s_n}{\Delta t^2} + A, \tag{3.20}$$

where

$$\alpha = f_{2x}'(\xi_5, t_{n-1}) - g(s_{n-1}, t_{n-1}) < 0, \quad \xi_5 \in (s_{n-2}, s_{n-1}),$$

A is uniformly bounded with respect to x, n, Δt,

$$r_n'(s_{n-2}) = \beta \frac{\Delta^2 s_n}{\Delta t^2} + B, \tag{3.21}$$

where

$$\beta = [g_x'(\xi_6, t_{n-1}) - u_{n-1}''(s_{n-1})]$$

$$- [f_{2x}'(\xi_7, t_{n-1}) - g(\xi_{n-1}, t_{n-1})]\left(\frac{\Delta s_n}{\Delta t} + \frac{\Delta s_{n-1}}{\Delta t}\right) > 0,$$

$\xi_6 \in (s_{n-2}, s_{n-1})$, $\xi_7 \in (s_{n-2}, s_{n-1})$. From (3.20), (3.21) we get

$$\beta r_n(s_{n-2}) - \alpha r_n'(s_{n-2}) = \beta A - \alpha B. \tag{3.22}$$

If $|r_n(x)|$ takes its maximal value at $x = 0$, then

$$|r_n(x)| \leq \max_{0 \leq t \leq T} |f_1''(t)| = M_{12}.$$

If $|r_n(x)|$ takes its maximal value on $n = 1$, then

$$|r_n(x)| \leq \max_{0 \leq x \leq b} |\varphi'''(x)| = M_{13}.$$

If $|r_n(x)|$ takes its maximal value at $x = s_{n-2}$, then both $r_n(s_{n-2})$ and $r_n'(s_{n-2})$ are positive, or negative. Thus the two terms of the right-hand side of (3.22) have the same sign. Consequently,

$$|r_n(x)| \leq \frac{1}{\min\limits_{(x,t)\in Q_T} |g_x'(x,t)|} \max_{(x,t)\in Q_T} |\beta A - \alpha B| = M_{14}.$$

In any case, we obtain

$$|r_n(x)| \leq \max(M_{12}, M_{13}, M_{14}) = M_{10}.$$

From (3.17) we get

$$\left|\frac{\Delta^2 s_n}{\Delta t^2}\right| \leq \frac{1}{\min\limits_{(x,t)\in Q_T} g(x,t)} (M_{14} + \max|A|) = M_{11}. \qquad \blacksquare$$

3.3. The Convergence of the Scheme

$u_n(x)$ and s_n are defined only on the line $t = t_n$. By linear interpolation we put for $(n-1)\Delta t \leq t \leq n\Delta t$

$$s^{\Delta t}(t) = \frac{t - (n-1)\Delta t}{\Delta t} s_n + \frac{n\Delta t - t}{\Delta t} s_{n-1},$$

$$u^{\Delta t}(x,t) = \frac{t - (n-1)\Delta t}{\Delta t} u_n(x) + \frac{n\Delta t - t}{\Delta t} u_{n-1}(x).$$

From the estimations (3.12), (3.16), (3.19) it follows that the families

$$\left\{s^{\Delta t}(t)\right\}, \quad \left\{\frac{\Delta s^{\Delta t}(t)}{\Delta t}\right\}$$

are uniformly bounded and equicontinuous. Hence, by Arzela's theorem, there exists a sequence $\Delta t_i \to 0$ for which

$$s^{\Delta t_i}(t) \to s(t),$$

$$\frac{\Delta s^{\Delta t_i}(t)}{\Delta t_i} \to s'(t)$$

uniformly on $[0, T]$. The families

$$\left\{u^{\Delta t}(x,t)\right\}, \quad \left\{\frac{\partial u^{\Delta t}(x,t)}{\partial t}\right\}, \quad \left\{\frac{\partial^2 u^{\Delta t}(x,t)}{\partial x^2}\right\}$$

are equicontinuous and uniformly bounded on D_T because of the estimations (3.13), (3.14), (3.15), (3.17), (3.18). Consequently, there exists a subsequence of $\{\Delta t_i\}$ (which we still denote by Δt_i) such that

$$u^{\Delta t_i}(x,t) \to u(x,t),$$
$$\frac{\partial u^{\Delta t_i}(x,t)}{\partial t} \to \frac{\partial u}{\partial t},$$
$$\frac{\partial^2 u^{\Delta t_i}(x,t)}{\partial x^2} \to \frac{\partial^2 u}{\partial x^2}$$

uniformly on D_T. It is not difficult to check that $u(x,t)$, $s(t)$ are the solution of problem (P). ∎

Theorem 3. *Assume that the assumptions* 1), 2), 3) *of Theorem* 2 *are satisfied for* $t \in \mathbb{R}_+$, $x \in [0, \overline{X}]$, *where* $\overline{X}$ *is the solution of the equation*

$$xg(x,0) - f_2(x,0) + \ell = 0, \tag{3.23}$$

$\ell = \lim_{t\to+\infty} f_1(t)$, *and that*

$$\lim_{t\to+\infty} f_2(x,t) = F_2(x), \quad \lim_{t\to+\infty} g(x,t) = G(x)$$

uniformly on $x \in [0, \overline{X}]$. *Then there exists*

$$\lim_{t\to+\infty} s(t) = S,$$

where S *is the solution of the equation*

$$xG(x) - F_2(x) + \ell = 0 \tag{3.24}$$

and

$$\lim_{t\to+\infty} u(x,t) = G(S)x + \ell.$$

Proof. Since the function $s(t)$ is increasing and bounded above

$$s(t) \le \overline{X}, \quad \forall t \in \mathbb{R}_+,$$

there exists $\lim_{t\to+\infty} s(t)$. Denote by S this limit and by $Y(x)$ the solution of the differential equation

$$Y''(x) = 0 \tag{3.25}$$

satisfying the conditions

$$Y(0) = \ell, \quad Y(S) = F_2(S), \quad Y'(S) = G(S). \tag{3.26}$$

We get

$$Y(x) = G(S)x + \ell,$$

where S is the solution of the equation (3.24).

The function $U(x,t) = u(x,t) - Y(x)$ satisfies the following relations

$$U_{xx} - U_t \quad \text{in } D_\infty, \tag{3.27}$$

$$U\big|_{x=0} = f_1(t) - \ell, \; U_x\big|_{x=s(t)} = g(s(t)) - Y'(s(t)). \tag{3.28}$$

Hence, when $t \to +\infty$,

$$U\big|_{x=0} \to 0, \quad U_x\big|_{x=s(t)} \to 0. \tag{3.29}$$

Using the method of A. Friedman in [5], we can prove that

$$\lim_{t\to+\infty} U(x,t) = 0,$$

i.e.

$$\lim_{t\to+\infty} u(x,t) = Y(x) = G(S)x + \ell$$

uniformly on $x \in [0, \overline{S}]$ ■

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