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Short Communication

Non-Singular Cubic Surfaces with Star Points*

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1. Introduction

1.1. Non-singular cubic surfaces and star points

A cubic surface in $\mathbb{P}^3$ is given by a non-zero cubic homogeneous polynomial in 4 variables. Fixing an ordering of monomials of degree 3 in the polynomial ring $k[x_0, x_1, x_2, x_3]$, each cubic surface defines a point in $\mathbb{P}^{19}$. The locus $\Delta \subset \mathbb{P}^{19}$ of singular cubic surfaces is a closed subset of codimension 1. A non-singular cubic surface X contains twenty-seven lines. There exist at most 3 lines among these twenty-seven lines through a given point of X. A *star point* (also called *Eckardt point* [1]) on a non-singular cubic surface is the intersection point of three lines on the surface. A non-singular cubic surface does not have more than 18 star points. We denote by H_k the subset of $\mathbb{P}^{19}$ consisting of points corresponding to non-singular cubic surfaces with at least k star points. We study these H_k as subvarieties of $\mathbb{P}^{19} - \Delta$.

1.2. Blowing-up of $\mathbb{P}^2$ at 6 points

One of the main methods used to study non-singular cubic surfaces in the paper is the blowing-up of $\mathbb{P}^2$ at 6 points. The blowing-up of $\mathbb{P}^2$ at 6 points $P_1, \ldots, P_6$ in general position is a variety X which is isomorphic to a non-singular cubic surface [2, 4.7]. The twenty-seven lines of X are the following:

- six exceptional curves $\tilde{P}_i$ corresponding to P_i for $1 \leq i \leq 6$,
- six strict transforms $\tilde{C}_i$ of the conics C_i through $\{P_1, \ldots, P_6\} - \{P_i\}$ for

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$1 \leq i \leq 6$,
- fifteen strict transforms $\tilde{l}_{ij}$ of the lines $l_{ij} = \overline{P_iP_j}$ for $1 \leq i < j \leq 6$.

Star points on a non-singular cubic surface can be recognized by the configuration of these 6 points.

In the paper, we determine all configurations of 6 points in $\mathbb{P}^2$ corresponding to the types of non-singular cubic surfaces with a given number of star points. We consider the irreducibility, the local closedness and the dimension of H_k. Moreover, we determine the inclusion relationship between the irreducible components of these H_k. We work on varieties over an algebraically closed field of characteristic 0.

2. Main Results

2.1. Basic properties

Each non-singular cubic surface X has 45 tritangent planes, i.e. the planes containing 3 lines of X, see [4, pp. 102–103]. In blowing-up of $\mathbb{P}^2$ at 6 points $P_1, \ldots, P_6$ in general position, we see that each tritangent plane is defined uniquely by a triple of lines in the form $(\tilde{P}_i\tilde{C}_j\tilde{l}_{ij})$ or $(\tilde{l}_{ij}\tilde{l}_{mn}\tilde{l}_{kh})$. So, we also use these triples of lines to denote the tritangent planes.

Definition 1. *Let $\mathcal{T}$ denote the set of 45 triples of lines on a given non-singular cubic surface X, which span the tritangent planes. If a triple in $\mathcal{T}$ forms a star point then it is called a star triple.*

Remark 1. Let T_1 and T_2 be 2 triples in $\mathcal{T}$ having no line in common. Each line of T_1 meets exactly one line of T_2. There exists uniquely another triple $T_3 \in \mathcal{T}$ such that each line in T_3 forms one tritangent plane with one line of T_1 and one of T_2. A such set of 3 triples in $\mathcal{T}$ is called a *Steiner set*.

Definition 2. *A Steiner set such that every of the 3 members gives a star point is called a star-Steiner set.*

2.2. A study of H_1, H_2 and H_3

The results in this subsection and in the subsec. 2.1 can be found in [6]. We state them here for the convenience of the readers.

Theorem 1.
(i) *For each $x \in H_1$ there exist 6 points $P_1, \ldots, P_6 \in \mathbb{P}^2$ in general position such that $l_{12} \cap l_{34} \cap l_{56} \neq \emptyset$ and the corresponding cubic surface X_x is isomorphic to the blowing-up of $\mathbb{P}^2$ at these 6 points.*
(ii) *The set H_1 is a closed, irreducible subvariety of $\mathbb{P}^{19} - \Delta$ of dimension 18.*

Definition 3. *Let*

$H_2^{(2)} = \{x \in H_2 \mid$ *the surface X_x has a pair of star triples having one line in common*$\}$,

$$H_2^{(3)} = \{x \in H_2 \mid \textit{the surface } X_x \textit{ has 2 star triples having no line in common}\}.$$

Let

$$K_2^{(2)} = \{\mathcal{P} = (P_1, \ldots, P_6) \in \Phi \mid l_{12} \cap l_{34} \cap l_{56} = \{Q_1\}, l_{12} \cap l_{35} \cap l_{46} = \{Q_2\}\},$$
$$K_2^{(3)} = \{\mathcal{P} = (P_1, \ldots, P_6) \in \Phi \mid l_{12} \cap l_{34} \cap l_{56} = \{Q_1\}, l_{13} \cap l_{45} \cap l_{26} = \{Q_2\}\}.$$

Proposition 1. *For each $x \in H_2^{(i)}$, there exists $\mathcal{P} \in K_2^{(i)}$ such that the surface X_x is isomorphic to the blowing-up of $\mathbb{P}^2$ at $\mathcal{P}$ for $i = 2, 3$.*

Theorem 2. *The set H_2 is closed in $\mathbb{P}^{19} - \Delta$ and has two irreducible components $H_2^{(2)}$ and $H_2^{(3)}$ of dimension 17.*

Corollary 2. *$H_3 = H_2^{(3)}$. Consequently H_3 is a closed, irreducible subvariety of dimension 17 in $\mathbb{P}^{19} - \Delta$.*

2.3. A study of H_4

Recall that H_4 is the set of points corresponding to non-singular cubic surfaces with at least 4 star points. Since $H_4 \subset H_3 = H_2^{(2)}$, this implies that for each $x \in H_4$, the surface X_x has a star-Steiner set. Moreover, if X_x has a star-Steiner $S = \{T_1, T_2, T_3\}$ and another star triple T having 2 lines in common with S, then T has all lines in common with S. This follows from Remark 1. Therefore, the set H_4 consists of elements in one of the 3 following subsets:

$$H_4^{(4)} = \Big\{[X] \in H_4 \mid X \text{ has one star-Steiner } S \text{ and another star triple } T \text{ having 3 lines in common with } S\Big\};$$

$$H_4^{(6)} = \Big\{[X] \in H_4 \mid X \text{ has one star-Steiner } S \text{ and another star triple } T \text{ having 1 line in common with } S\Big\};$$

$$H_4^{(9)} = \Big\{[X] \in H_4 \mid X \text{ has one star-Steiner } S \text{ and another star triple } T \text{ having no line in common with } S\Big\}.$$

Definition 4. *Let*

$$K_4^{(4)} = \{(P_1, \ldots, P_6) \in \Phi \mid l_{13} \cap l_{24} \cap l_{56} = \{Q\};\ l_{12}, l_{13} \textit{ are tangent to } C_1\},$$
$$K_4^{(6)} = \{(P_1, \ldots, P_6) \in \Phi \mid l_{14} \cap l_{23} \cap l_{56} = \{S\};\ l_{12}, l_{13} \textit{ are tangent to } C_1\}.$$

Theorem 3.

(i) *For each $x \in H_4^{(i)}$, there exists $\mathcal{P} \in K_4^{(i)}$ such that the surface X_x is isomorphic to the blowing-up of $\mathbb{P}^2$ at $\mathcal{P}$ for $i = 4, 6$.*

(ii) *The subsets $H_4^{(4)}$ and $H_4^{(6)}$ of H_4 are closed in $\mathbb{P}^{19} - \Delta$, irreducible of dimension 16.*

Corollary 3. *Each $X \in H_4^{(6)}$ has at least 6 star triples, which form 4 star-Steiner sets and each star triple has exactly one line in common with another* *star triple among the* 6 *ones above. Hence* $H_4^{(6)} \subset H_6$.

Definition 5.

$$K_4^{(9)} = \{(P_1,\dots,P_6) \in \Phi \mid l_{12} \cap l_{34} \cap l_{56} = \{S_1\};\ l_{15} \cap l_{24} \cap l_{36} = \{S_2\};\ l_{14} \text{ is tangent to } C_1 \text{ at } P_4\}.$$

Theorem 4.

(i) *For each* $x \in H_4^{(9)}$, *there exists* $\mathcal{P} \in K_4^{(9)}$ *such that the surface* X_x *is isomorphic to the blowing-up of* $\mathbb{P}^2$ *at* $\mathcal{P}$.

(ii) *The subset* $H_4^{(9)} \subset H_4$ *is closed in* $\mathbb{P}^{19} - \Delta$ *and has two irreducible components of dimension* 16.

Corollary 4. *Each cubic surface corresponding to an element of* $H_4^{(9)}$ *contains at least* 9 *star points and at least* 12 *star-Steiner sets.*

Corollary 5. *The subset* $H_2^{(2)}$, *respectively,* $H_2^{(3)}$ *generically consists of point corresponding to cubic surfaces with exactly* 2, *respectively* 3, *star points.*

2.4. A study of H_5 and H_6

Theorem 5. $H_5 = H_6 = H_4^{(6)} \cup H_4^{(9)}$.

2.5. A study of H_7, H_8 and H_9

Recall that H_7, H_8 and H_9 are the subsets corresponding to non-singular cubic surfaces with at least 7, 8 and 9 star points, respectively.

Note that a cubic surface $X \in H_4^{(6)}$ can be assumed to possess a pair $\{S, U\}$, where $S = \{T_1 = (\tilde{C}_1\tilde{P}_2\tilde{l}_{12}),\ T_2 = (\tilde{C}_3\tilde{P}_4\tilde{l}_{34}),\ T_1T_2 = (\tilde{l}_{14}\tilde{l}_{23}\tilde{l}_{56})\}$ and $U = (\tilde{C}_1\tilde{P}_3\tilde{l}_{13})$. We need a lemma.

Lemma 1. *Let* $x \in H_4^{(6)}$ *and let* $T_1,\dots,T_6$ *be the six star triples of* X_x *determined by the given pair* (S, U) *as above. Let* V *be another star triple of* X_x.

(i) *If* V *has all line in common with one of* 4 *star-Steiner sets determined by* $T_1,\dots,T_6$ *then* X_x *has at least* 10 *star points and at least* 10 *star-Steiner sets.*

(ii) *Otherwise, the surface* X_x *has at least* 18 *star points and at least* 42 *star-Steiner sets.*

Definition 6. *Let* $H_{10}^{(10)}$ *and* $H_{10}^{(18)}$ *denote the subsets of* $H_4^{(6)}$ *consisting of all points as in the cases* (i) *and* (ii) *of Lemma* 1, *respectively.*

Corollary 6. $H_7 = H_8 = H_9 = H_{10}^{(10)} \cup H_4^{(9)}$.

Let

$$K_{10}^{(10)} = \{(P_1,\dots,P_6) \in \Phi \mid l_{12} \cap l_{34} \cap l_{56} = \{S_2\};\ l_{14} \cap l_{23} \cap l_{56} = \{S_1\};\ l_{12} \text{ and } l_{13} \text{ are tangent to } C_1\}.$$

Theorem 6.
(i) *For each* $x \in H_{10}^{(10)}$, *there exists* $\mathcal{P} \in K_{10}^{(10)}$ *such that the surface* X_x *is isomorphic to the blowing-up of* $\mathbb{P}^2$ *at* $\mathcal{P}$.
(ii) *The set* $H_{10}^{(10)}$ *is closed in* $\mathbb{P}^{19} - \Delta$, *irreducible of dimension* 15.

Similarly, we can assume that for each $x \in H_{10}^{(18)}$, the surface X_x is isomorphic to the blowing-up of $\mathbb{P}^2$ at one element of the following set

$$K_{10}^{(18)} = \{(P_1, \ldots, P_6) \in \Phi \mid l_{15} \cap l_{24} \cap l_{36} = \{S_1\};\ l_{14} \cap l_{23} \cap l_{56} = \{S_2\};\ l_{12} \text{ and } l_{13} \text{ are tangent to } C_1\}.$$

Theorem 7. *The set* $H_{10}^{(18)}$ *is closed in* $\mathbb{P}^{19} - \Delta$ *and has two irreducible components of dimension* 15.

Proposition 2. $H_{10}^{(10)} \cap H_{10}^{(18)} = \emptyset$.

Corollary 7. *Each cubic surface corresponding to an element of* $H_{10}^{(10)}$ *has exactly* 10 *star points.*

2.6. A study of H_k with $k \geq 10$

Theorem 8. $H_{10} = H_{10}^{(10)} \cup H_{10}^{(18)} = H_4^{(4)} \cap H_4^{(6)}$.

Corollary 8. *A non-singular cubic surface does not have more than* 18 *star points. Consequently* $H_k = \emptyset$ *for* $k > 18$.

Corollary 9. $H_k = H_{10}^{(18)}$ *for* $10 < k \leq 18$.

Proposition 3. $H_{18} = H_4^{(4)} \cap H_4^{(9)} = H_4^{(6)} \cap H_4^{(9)}$.

We give a survey of the results obtained in Figure 1. In the diagram of the figure:
(i) the number n in the left top of the symbol ${}^nH_k^{(m)}$ denotes the dimension of $H_k^{(m)}$;
(ii) the vectors mean the inclusion relations;
(iii) the symbol (m) indicates that generically in the set $H_k^{(m)}$ the corresponding surface has exactly m star points.

Other main results are:
- $H_4 = H_4^{(4)} \cup H_4^{(6)} \cup H_4^{(9)}$ and $\dim H_4 = 16$;
- $H_5 = H_6 = H_4^{(6)} \cup H_4^{(9)}$ and $\dim H_k = 16$, for $k \in \{5, 6\}$;
- $H_7 = H_8 = H_9 = H_4^{(9)} \sqcup H_{10}^{(10)}$ and $\dim H_k = 16$, for $k \in \{7, 8, 9\}$ and the union is disjoint;
- $H_{10} = H_{10}^{(10)} \sqcup H_{10}^{(18)}$ and $\dim H_{10} = 15$ and the union is disjoint.

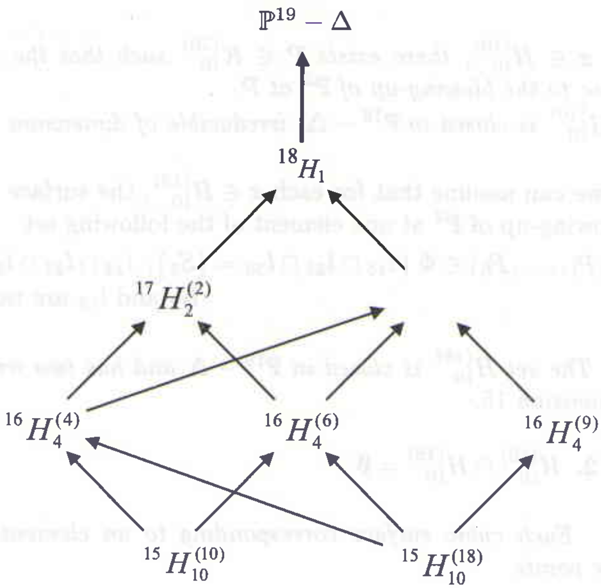

Figure 1. A diagram explaining properties of $H_k^{(m)}$

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