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Iterative Method for Solving a Boundary Value Problem for Triharmonic Equation*

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Abstract. Recently we have developed the boundary operator method and the parametric extrapolation technique for solving a boundary value problem (BVP) for second order elliptic equation with discontinuous coefficients and BVPs for biharmonic and biharmonic type equations. In this paper we use these methods for a BVP for triharmonic equation. Namely, two iterative schemes, which reduce the original problem to a sequence of BVPs for the Poisson equation, are proposed and investigated.

1. Introduction

In earlier papers we have developed the boundary operator method and the parametric extrapolation technique for solving iteratively a BVP for second order elliptic equation with discontinuous coefficients [1], BVPs for biharmonic and biharmonic type equations [2–4]. The idea of the method is to reduce a complicated BVP to a sequence of simpler problems, for which there are available many efficient methods of solving. In this paper we apply this method to the following BVP for triharmonic equation

$$\Delta^3 u = f(x), \quad x \in \Omega, \tag{1}$$

$$u|_\Gamma = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_\Gamma = 0, \quad \Delta u|_\Gamma = 0, \tag{2}$$

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where Δ is the Laplace operator, Ω is a bounded domain in $R^n (n \geq 2), \Gamma$ is the sufficiently smooth boundary of Ω. The solvability and smoothness of the solution of problem (1)–(2) follow from the general theory of elliptic problems (see [6]), namely, if $f \in H^s(\Omega)$ then there exists a unique solution $u \in H^{s+6}(\Omega)$. Here , as usual, $H^s(\Omega)$ is a Sobolev space.

2. Reduction of BVP to Boundary Operator Equation

We set

$$\Delta u = v, \Delta v = w$$

and denote by w_0 the trace of w on Γ , i.e. $w_0 = w|_\Gamma$. Then from (1)–(2) we come to the sequence of problems

$$\begin{aligned} \Delta w &= f, \quad x \in \Omega, \quad w|_\Gamma = w_0, \\ \Delta v &= w, \quad x \in \Omega, \quad v|_\Gamma = 0, \\ \Delta u &= v, \quad x \in \Omega, \quad u|_\Gamma = 0. \end{aligned} \tag{3}$$

The solution u from above problems should satisfy the second condition in (2). Now, we introduce the operator B defined on boundary functions w_0 by the formula

$$Bw_0 = -\left.\frac{\partial u}{\partial \nu}\right|_\Gamma, \tag{4}$$

where u is found from the sequence of problems

$$\begin{aligned} \Delta w &= 0, \quad x \in \Omega, \quad w|_\Gamma = w_0, \\ \Delta v &= w, \quad x \in \Omega, \quad v|_\Gamma = 0, \\ \Delta u &= v, \quad x \in \Omega, \quad u|_\Gamma = 0. \end{aligned} \tag{5}$$

Notice that the operator B primarily defined on smooth functions is extended by continuity on the whole $L_2(\Gamma)$. Its properties will be investigated later.

It is not difficult to verify

Theorem 1. *Suppose that u is the solution of the original problem* (1)–(2). *Then $w_0 = \Delta^2 u|_\Gamma$ satisfies the operator equation*

$$Bw_0 = F, \tag{6}$$

where

$$F = \left.\frac{\partial u_2}{\partial \nu}\right|_\Gamma, \tag{7}$$

u_2 being determined from the problems

$$\begin{aligned} \Delta w_2 &= f, \quad x \in \Omega, \quad w_2|_\Gamma = 0, \\ \Delta v_2 &= w_2, \quad x \in \Omega, \quad v_2|_\Gamma = 0, \\ \Delta u_2 &= v_2, \quad x \in \Omega, \quad u_2|_\Gamma = 0. \end{aligned} \tag{8}$$

Notice that if $f \in H^s(\Omega)$ then $F \in H^{s+9/2}(\Gamma)$. Thus, we have led the original problem to the operator equation (6) for finding w_0. After w_0 is found, by solving the problems (3) we shall find the solution of (1)–(2).

Now, let us study the properties of B. First, for short we denote by H the Hilbert space $L_2(\Gamma)$ with the scalar product $(.,.)$.

Property 1. *B is symmetric and positive in H.*

Proof. For any functions w_0 and $\bar{w}_0$ we have

$$
\begin{aligned}
(Bw_0, \bar{w}_0) &= \int_\Gamma \bar{w}_0 B w_0 d\Gamma = -\int_\Gamma \bar{w}_0 \frac{\partial u}{\partial \nu} d\Gamma = \int_\Gamma (u\frac{\partial \bar{w}}{\partial \nu} - \bar{w}_0 \frac{\partial u}{\partial \nu}) d\Gamma \\
&= \int_\Omega (u\Delta\bar{w} - \bar{w}\Delta u)dx = -\int_\Omega \bar{w}\Delta u dx \\
&= -\int_\Omega \Delta\bar{v} v dx = \int_\Omega \text{grad} v . \text{grad}\bar{v} dx
\end{aligned}
$$

in view of (5) and the same equalities for $\bar{w}_0, \bar{u}, \bar{v}$ and $\bar{w}$. Thus, we obtain

$$(Bw_0, \bar{w}_0) = \int_\Omega \text{grad} v . \text{grad}\bar{v} dx.$$

Hence,

$$(Bw_0, \bar{w}_0) = (B\bar{w}_0, w_0).$$

The symmetry of B is proved.

Furthermore, we have

$$(Bw_0, w_0) = \int_\Omega |\text{grad} v|^2 dx \geq 0.$$

The equality occurs if and only if $w_0 = 0$. So, B is positive and Property 1 is proved.

Property 2. *B is a completely continuous operator in H.*

Proof. From [6] it follows that B is an isomorphism of spaces $H^s(\Gamma)$ and $H^{s+3}(\Gamma)$, $s \geq 0$. Due to the compactness of embedding of $H^{s+3}(\Gamma)$ into $H^s(\Gamma)$ we get that B is completely continuous operator in $H^s(\Gamma)$. In particular, B is a completely continuous in $H = H^0(\Gamma) = L_2(\Gamma)$.

3. Iterative Method for BVP (1)–(2)

Consider the following iterative method for (1)–(2):

i) Give a starting approximation $w_0^{(0)} \in H^{1/2}(\Gamma)$.

ii) Knowing $w_0^{(k)}$, $(k = 0, 1, ...,)$ solve successively three problems

$$\begin{aligned} \Delta w^{(k)} &= f, \quad x \in \Omega, \quad w^{(k)}|_\Gamma = w_0^{(k)}, \\ \Delta v^{(k)} &= w^{(k)}, \quad x \in \Omega, \quad v^{(k)}|_\Gamma = 0, \\ \Delta u^{(k)} &= v^{(k)}, \quad x \in \Omega, \quad u^{(k)}|_\Gamma = 0. \end{aligned} \tag{9}$$

iii) Compute the new approximation of w_0

$$w_0^{(k+1)} = w_0^{(k)} - \tau \frac{\partial u^{(k)}}{\partial \nu}, \quad x \in \Gamma, \tag{10}$$

where τ is a sufficiently small iterative parameter.

Theorem 2. *The above iterative process is a realization of the following iterative scheme*

$$\frac{w_0^{(k+1)} - w_0^{(k)}}{\tau} + Bw_0^{(k)} = F \tag{11}$$

for the operator equation (6).

Proof. Indeed, if we represent $u^{(k)} = u_1^{(k)} + u_2$, $v^{(k)} = v_1^{(k)} + v_2$, $w^{(k)} = w_1^{(k)} + w_2$, where u_2, v_2, w_2 are the solutions of (8) then $u_1^{(k)}$, $v_1^{(k)}$, $w_1^{(k)}$ satisfy the following problems

$$\begin{aligned} \Delta w_1^{(k)} &= 0, \quad x \in \Omega, \quad w_1^{(k)}|_\Gamma = w_0^{(k)}, \\ \Delta v_1^{(k)} &= w_1^{(k)}, \quad x \in \Omega, \quad v_1^{(k)}|_\Gamma = 0, \\ \Delta u_1^{(k)} &= v_1^{(k)}, \quad x \in \Omega, \quad u_1^{(k)}|_\Gamma = 0. \end{aligned} \tag{12}$$

By the definition of B we have

$$Bw_0^{(k)} = -\frac{\partial u_1^{(k)}}{\partial \nu}\bigg|_\Gamma. \tag{13}$$

Taking into account

$$\frac{\partial u^{(k)}}{\partial \nu} = \frac{\partial u_1^{(k)}}{\partial \nu} + \frac{\partial u_2}{\partial \nu},$$

from (10), (13) and (7) we obtain (11) and the theorem is proved.

Since B is shown to be a symmetric, positive, completely continuous operator in H there holds

Theorem 3. *The iterative method* (11) *is convergent if*

$$0 < \tau < \frac{2}{\|B\|}.$$

Proof. See Lemma 2.1 in [4].

4. Accelerated Iterative Method for BVP (1)–(2)

In order to construct a faster convergent iterative process for (1)–(2) we shall use the parametric extrapolation technique, which was developed and used in our works [1- 4]. For this reason we consider the perturbed problem

$$\begin{gathered} \Delta^3 u_\delta = f(x), \quad x \in \Omega, \\ u_\delta|_\Gamma = 0, \quad \Delta u_\delta|_\Gamma = 0, \\ \left(\frac{\partial u_\delta}{\partial \nu} - \delta \Delta^2 u_\delta\right)\Big|_\Gamma = 0. \end{gathered} \tag{14}$$

Analogously as in Sec. 2, it is easy to show that this problem may be reduced to the following operator equation

$$B_\delta w_{\delta 0} = F, \tag{15}$$

where $B_\delta = B + \delta I$, $w_{\delta 0} = \Delta^2 u_\delta|_\Gamma$, I is the identity operator and B and F are defined by (4)-(5) and (7)-(8), respectively. Taking into account the property 1 of B we see that B_δ is a linear, symmetric, positive definite operator in H and $B_\delta \geq \delta I$.

Theorem 4. *Suppose that $f \in H^{s-6}(\Omega)$, $s \geq 6$. Then for the solution of the problem (14) there holds the following asymptotic expansion*

$$u_\delta = u + \sum_{i=1}^{N} \delta^i y_i + \delta^{N+1} z_\delta, \quad x \in \Omega, \quad 0 \leq 3N \leq s - 5/2, \tag{16}$$

where $y_0 = u$ is the solution of (1)–(2), y_i $(i = 1, ..., N)$ are functions independent of δ, $y_i \in H^{s-3i}(\Omega)$, $z_\delta \in H^{s-3N}(\Omega)$ and

$$\|z_\delta\|_{H^2(\Omega)} \leq C_1, \tag{17}$$

C_1 being independent of δ.

Proof. Under the assumption of the theorem, by [6] there exists a unique solution $u \in H^s(\Omega)$ of the problem (14). After substituting (16) into (14) and comparing coefficients of the same powers of δ we see that y_i and z_δ satisfy the following problems

$$\begin{gathered} \Delta^3 y_i = 0, \quad x \in \Omega, \\ y_i|_\Gamma = 0, \quad \Delta y_i|_\Gamma = 0, \\ \frac{\partial y_i}{\partial \nu}\Big|_\Gamma = \Delta^2 y_{i-1}\Big|_\Gamma, i = 1, ..., N, \end{gathered} \tag{18}$$

$$\begin{gathered} \Delta^3 z_\delta = 0, \quad x \in \Omega, \\ z_\delta|_\Gamma = 0, \quad \Delta z_\delta|_\Gamma = 0, \\ \left(\frac{\partial z_\delta}{\partial \nu} - \delta \Delta^2 z_\delta\right)\Big|_\Gamma = \Delta^2 y_N\Big|_\Gamma. \end{gathered} \tag{19}$$

Once again, using [6] it is not difficult to establish successively that (18) has a unique solution $y_i \in H^{s-3i}(\Omega)$ and (19) has a unique solution $z_\delta \in H^{s-3N}(\Omega)$. Clearly, $y_i (i = 1, ..., N)$ do not depend on δ . It remains to estimate z_δ. For this purpose we reduce (19) to a boundary operator equation. We set

$$\Delta z_\delta = v_\delta, \ \ \Delta v_\delta = w_\delta \tag{20}$$

and denote $w_\delta|_\Gamma = w_{\delta 0}$. Then we get

$$\begin{aligned} \Delta w_\delta &= f, \quad x \in \Omega, \quad w_\delta|_\Gamma = w_{\delta 0}, \\ \Delta v_\delta &= w_\delta, \quad x \in \Omega, \quad v_\delta|_\Gamma = 0, \\ \Delta z_\delta &= v_\delta, \quad x \in \Omega, \quad z_\delta|_\Gamma = 0. \end{aligned} \tag{21}$$

By the definition of B we have

$$B w_{\delta 0} = -\frac{\partial z_\delta}{\partial \nu}\bigg|_\Gamma . \tag{22}$$

Now, using the last condition of (19) we obtain

$$B w_{\delta 0} + \delta I w_{\delta 0} = h, \tag{23}$$

where $h = -\Delta^2 y_N|_\Gamma$.
It is not hard to verify that (see [5])

$$(B w_{\delta 0}, w_{\delta 0}) \leq (B w_0, w_0) \tag{24}$$

where w_0 is a solution of the equation $B w_0 = h$. This equation has a solution because the problem (19) with $\delta = 0$ may be reduced to this.

In Sec. 2, when investigating the properties of B we have established that

$$(B w_{\delta 0}, w_{\delta 0}) = \int_\Omega |\text{grad} v_\delta|^2 dx.$$

In view of the Fridrichs inequality [7] we have

$$(B w_{\delta 0}, w_{\delta 0}) \geq C_2 \|v_\delta\|^2_{L_2(\Omega)} \tag{25}$$

On the other hand, since z_δ satisfies the last problem in (21) there holds the estimate

$$\|z_\delta\|_{H^2(\Omega)} \leq \|v_\delta\|^2_{L_2(\Omega)}.$$

Hence, taking into account (25) and (24) we obtain

$$\|z_\delta\|_{H^2(\Omega)} \leq C_1,$$

where $C_1 = \sqrt{\frac{C_3}{C_1}(B w_0, w_0)}, C_2, C_3$ and w_0 being independent of δ . Thus, the theorem is proved.

As usual (see [1–5]), we construct an approximate solution of the original problem (1)–(2) by the formula

$$U^E = \sum_{i=1}^{N+1} \gamma_i u_{\delta/i},$$

where

$$\gamma_i = \frac{(-1)^{N+1-i} i^{N+1}}{i!(N+1-i)!},$$

$u_{\delta/i}$ is the solution of (14) with the parameter δ/i $(i = 1, ..., N+1)$. Then, it is easy to obtain the following estimate

$$\|U^E - u\|_{H^2(\Omega)} \leq C\delta^{N+1},$$

where u is the solution of (1)-(2), C is a constant independent of δ.

For solving (14), which may be reduced to the equation (15) we propose to use the following iterative process under the assumption $f \in L_2(\Omega)$:

i) Give a start approximation $w_{\delta 0}^{(0)} \in H^{1/2}(\Gamma)$.

ii) Knowing $w_{\delta 0}^{(k)}$, $(k = 0, 1, ...)$, solve successively three problems

$$\begin{aligned} \Delta w_\delta^{(k)} &= f, \quad x \in \Omega, \quad w_\delta^{(k)}|_\Gamma = w_{\delta 0}^{(k)}, \\ \Delta v_\delta^{(k)} &= w_\delta^{(k)}, \quad x \in \Omega, \quad v_\delta^{(k)}|_\Gamma = 0, \\ \Delta u_\delta^{(k)} &= v_\delta^{(k)}, \quad x \in \Omega, \quad u_\delta^{(k)}|_\Gamma = 0. \end{aligned} \tag{26}$$

iii) Compute the new approximation of $w_{\delta 0}$

$$w_{\delta 0}^{(k+1)} = w_{\delta 0}^{(k)} - \tau_{\delta,k+1} \frac{\partial u_\delta^{(k)}}{\partial \nu}\Big|_\Gamma, \quad x \in \Gamma, \tag{27}$$

where $\tau_{\delta,k+1}$ is the Chebyshev collection of parameters according to bounds $\gamma_\delta^1 = \delta$, $\gamma_\delta^2 = \delta + \|B\|$ (see [1, 8] for detail). In the case of simple iteration

$$\tau_{\delta,k} \equiv \tau_{\delta,0} = \frac{2}{\gamma_\delta^1 + \gamma_\delta^2}$$

we get

$$\|w_{\delta 0}^{(k)} - w_{\delta 0}\|_H \leq (\rho_\delta)^k \|w_{\delta 0}^{(0)} - w_{\delta 0}\|_H \tag{28}$$

where

$$\rho_\delta = \frac{1-\xi_\delta}{1+\xi_\delta}, \; \xi_\delta = \frac{\gamma_\delta^{(1)}}{\gamma_\delta^{(2)}}$$

and as above $H = L_2(\Gamma)$.

This result follows from the general theory of two-layer iterative schemes [8], applied to the operator equation (15), which is obtained from (14). Using estimates for the solution of elliptic problems [5] and taking into account (28) we get the estimate

$$\|u_\delta^{(k)} - u_\delta\|_{H^{5/2}(\Omega)} \leq C(\rho_\delta)^k \|w_{\delta 0}^{(0)} - w_{\delta 0}\|_H,$$

where C is a constant independent of δ and $w_{\delta 0} = \Delta^2 u_\delta|_\Gamma$ as was mentioned in the beginning of the section.

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