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On a Perturbed Boundary Optimal Control System

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Abstract. In this paper, we deal with a class of optimal control problems associated to linear systems obtained by perturbing a Neumann problem for Laplace operator on a regular bounded domain in the n dimensional Euclidean space. The sets of admissible controls are in some wide classes of closed convex subsets of the Hilbert space of all square integrable functions on the boundary, and the cost functionals are quadratic means involving the traces of the states. For these systems, we prove existence of the (perturbed) states and optimal controls, and study their behavior. We establish the systems of optimality conditions and investigate the adjoint states, and prove their strong convergence in some Sobolev spaces.

1. Introduction and Statement of the Problems

In all this paper, Ω will be a connected regular and bounded open subset of the Euclidean space $\mathbf{R}^n$ with a smooth boundary $\partial\Omega =: \Gamma$. We denote by $H^1(\Omega)$ the classical real Sobolev space equipped with its usual inner product and associated norm $\|.\|_{H^1(\Omega)}$. For every $y \in H^1(\Omega)$, we shall denote by $T(y)$ or simply by y the Γ-trace (i.e. the restriction of y to Γ). We know, by the trace theorem, see [4], that the map T is a bounded linear map from the Sobolev space $H^1(\Omega)$ to $L^2(\Gamma)$.

For every $\epsilon > 0$, we want to find

$$\min\{J_\epsilon(v),\, v \in \mathcal{U}_{ad}\}, \tag{Q$_\epsilon$}$$

where $\mathcal{U}_{ad}$ is a closed convex subset of $L^2_0(\Gamma) := \{u \in L^2(\Gamma) : \int_\Gamma u\, d\gamma = 0\}$, and

$$J_\epsilon(v) = \int_\Gamma (y_\epsilon(v) - h)^2\, d\gamma, \tag{1}$$

where h is a fixed (decision) function in $L^2(\Gamma)$, and $y_\epsilon(v)$ is a solution of the following problem:

$$\begin{cases} -\Delta y_\epsilon(v) = 0, \text{ on } \Omega, \\ \dfrac{\partial}{\partial\nu} y_\epsilon(v) + \epsilon y_\epsilon(v) = v, \text{ at } \Gamma = \partial\Omega, \\ y_\epsilon(v) \in V, \end{cases} \tag{P_ϵ)(v}$$

where $\frac{\partial}{\partial\nu} y_\epsilon(v)$ is the normal derivative of $y_\epsilon(v)$, and $V = \{y \in H^1(\Omega) : \int_\Gamma y\, d\gamma = 0\}$. The space V is a Hilbert space when it is endowed with the restriction of the Hilbert structure of $H^1(\Omega)$. We can also consider in V the following inner product and associated norm given by

$$\langle y \mid z \rangle = \int_\Omega \nabla y.\nabla z\, d\omega, \ \ \|y\|_V = \left[\int_\Omega |\nabla y|^2\, d\omega\right]^{\frac{1}{2}} \ \ (y, z \in V). \tag{2}$$

It is well known that the norm $\|.\|_V$ is equivalent to the resriction of the usual norm of $H^1(\Omega)$ to the space V. Therefore, according to the trace theorem, we can find a constant $\lambda > 0$ such that

$$\|y\|_{L^2(\Gamma)} \le \lambda \|y\|_V, \ \forall y \in V. \tag{3}$$

In all this paper, $\mathcal{U}_{ad}$ will be a closed convex subset of $L_0^2(\Gamma)$, verifying one of the following assumptions:

(A_1) *There exists a positive constant* $M > 0$*, such that* $\|u\|_{L^2(\Gamma)} \le M,\ \forall u \in \mathcal{U}_{ad}$.

(A_2) $\mathcal{U}_{ad}$ *is not bounded and for each sequence* $(u_n)_n$ *of elements in* $\mathcal{U}_{ad}$*, satisfying* $\lim_{n\to\infty} \|u_n\|_{L^2(\Gamma)} = +\infty$*, the linear space spanned by the set* $\{u_n : n \in \mathbf{N}\}$ *has finite dimension.*

(A_3) *There exists a finite dimensional subspace* $\mathcal{U}$ *of* $L_0^2(\Gamma)$*, containing* $\mathcal{U}_{ad}$.

The purpose of this work is to prove, under one of the assumptions (A_1), (A_2) or (A_3), existence and uniqueness of the state y_ϵ and the optimal control u_ϵ for the system (P_ϵ)(u_ϵ)&(Q_ϵ). This will be done in the next section. In the third section, we shall study their behavior. More precisely, we shall prove that u_ϵ converges weakly to the unique element $u \in \mathcal{U}_{ad}$ satisfying $J(u) = \min\{J(v) : v \in \mathcal{U}_{ad}\}$ where $J(v) = \int_\Gamma (y(v) - h)^2\, d\gamma$, and $y(v)$ is the unique solution of the following problem:

$$\begin{cases} -\Delta y(v) = 0, \text{ on } \Omega, \\ \dfrac{\partial}{\partial\nu} y(v) = v, \text{ at } \Gamma = \partial\Omega, \\ y(v) \in V, \end{cases} \tag{P_0)(v}$$

The fourth section is devoted to investigate the system of optimality conditions for the optimal controls of the problem (Q_ϵ), and to find the adjoint state p_ϵ associated to the perturbed state y_ϵ. In the section five, we shall establish the strong convergence of the adjoint state p_ϵ to the adjoint state p associated to y

the solution of the system $(P_0)(u)$, where u is the optimal control for the limit problem.

In the papers [1] and [2], the authors considered the problem of finding

$$\min\{I_\epsilon(v),\ v \in \mathcal{U}_{ad}\},$$

where $\mathcal{U}_{ad}$ is any arbitrary finite dimensional subspace of $L_0^2(\Gamma)$, and

$$I_\epsilon(v) = \int_\Gamma (y_\epsilon(v) - z_1)^2\, d\gamma + \int_\Gamma \left(\frac{\partial}{\partial \nu} y_\epsilon(v) - z_2\right)^2 d\gamma,$$

where z_1, z_2 are two fixed (decision) functions in $L^2(\Gamma)$, and $y_\epsilon(v)$ is a solution of the system $(\mathrm{P}_\epsilon)(v)$. In these papers, we have studied only the behavior of the optimal controls and the corresponding states. We have not studied the adjoint states and we have not found the systems of optimality conditions. So, our study was very limited and the cost functional I_ϵ was more easy to deal with than J_ϵ taken here. In this paper, the class of admissible controls is more wide and the methods developed here are more general than those used in the papers [1] and [2]. Moreover, many results stated here remain valid for the optimal control studied in these papers and the methods used here may be applied to the cost functional I_ϵ.

2. Existence and Uniqueness of the State and of the Optimal Control

The space V will be endowed with the inner product (resp. the norm) given by (2). For each $\epsilon > 0$, and for each $v \in L_0^2(\Gamma)$, one can use Lax–Milgram Theorem in the variational formulation for the problem $(P_\epsilon)(v)$, and conclude that there exists a unique element $y_\epsilon \in V$ satisfying the problem $(P_\epsilon)(v)$. To prove the existence of optimal controls, we need the following proposition:

Proposition 2.1. *Let $\mathcal{U}_{ad}$ be a closed convex subset of $L_0^2(\Gamma)$, verifying (A_1) or (A_2) or (A_3). Then for each $\epsilon > 0$, we have*
(2.1.1) *The mapping $S_\epsilon : L_0^2(\Gamma) \to V$, $v \mapsto y_\epsilon(v)$ is injective, linear and continuous having a norm $\|S_\epsilon\| \le \lambda$.*
(2.1.2) *The linear mapping $R_\epsilon : L_0^2(\Gamma) \to L_0^2(\Gamma)$, $v \mapsto y_\epsilon(v)$ is injective and compact having a norm $\|R_\epsilon\| \le \lambda^2$.*
(2.1.3) *The map $J_\epsilon : L_0^2(\Gamma) \to [0, +\infty[$, $v \mapsto J_\epsilon(v)$ is strictly convex and weakly l.s.c. (i.e., lower semicontinuous) on $L_0^2(\Gamma)$.*
(2.1.4) *If $\mathcal{U}_{ad}$ is not bounded, then for every sequence (v_n) in $\mathcal{U}_{ad}$, such that $\|v_n\|_{L^2(\Gamma)} \to +\infty$,, one has $J_\epsilon(v_n) \to +\infty$, when $n \to +\infty$.*

Proof. (2.1.1) It is clear that the mappings S_ϵ and R_ϵ are linear and that S_ϵ is injective. Let v (resp. w) in $L^2(\Gamma)$. Then, by using the variational formulation for the problem $(\mathrm{P}_\epsilon)(v)$ (resp. $(\mathrm{P}_\epsilon)(w)$), we get

$$\int_\Omega \nabla[y_\epsilon(v) - y_\epsilon(w)].\nabla z\, d\omega + \epsilon \int_\Gamma (y_\epsilon(v) - y_\epsilon(w))\, z\, d\gamma =$$

$$\int_\Gamma (v - w) z\, d\gamma, \quad \forall z \in V. \tag{4}$$

We set $z = y_\epsilon(v) - y_\epsilon(w)$ in (4), then we obtain

$$\int_\Gamma (v-w)((y_\epsilon(v) - y_\epsilon(w))\, d\gamma = \|y_\epsilon(v) - y_\epsilon(w)\|_V^2 + \epsilon \int_\Gamma |y_\epsilon(v) - y_\epsilon(w)|^2\, d\gamma.$$

By using the Cauchy–Schwarz inequality and the trace theorem, we deduce from the last inequality that

$$\|y_\epsilon(v) - y_\epsilon(w)\|_V \leq \lambda \|v - w\|_{L^2(\Gamma)}. \tag{5}$$

This proves (2.1.1) and a part of (2.1.2).

(2.1.2) Let $v \in L^2(\Gamma)$ be such that $R_\epsilon(v) = y_\epsilon(v) = 0$ at Γ. Then $y_\epsilon(v) \in H_0^1(\Omega)$ and is a solution of the next problem:

$$\begin{cases} -\Delta y_\epsilon(v) = 0, \text{ on } \Omega, \\ \dfrac{\partial}{\partial \nu} y_\epsilon(v) = v, \text{ at } \Gamma = \partial\Omega, \\ y_\epsilon(v) \in V. \end{cases}$$

Using the variational formulation of this problem, we obtain $\int_\Omega |\nabla y_\epsilon(v)|^2\, d\omega = 0$. This equality gives $y_\epsilon(v) = 0$. Thus R_ϵ is injective. According to (3) and to (2.1.1) we see that its norm is less than λ^2. It remains to show that the mapping R_ϵ is compact. Let B be a bounded subset of $L^2(\Gamma)$. Then $\{y_\epsilon(v) : v \in B\}$ is bounded in $H^1(\Omega)$. Therefore (see [7, Theorem 4, p. 143]) the set of its traces on Γ is conditionally compact in the Hilbert space $L^2(\Gamma)$.

(2.1.3) It is easy to prove that J_ϵ is strictly convex. It remains to show that J_ϵ is weakly l.s.c. on $L_0^2(\Gamma)$. But this is a consequence of (2.1.2) and the fact that the norm in any Hilbert space is weakly l.s.c.

(2.1.4) Let $\mathcal{Z}$ be the linear space spanned in $L^2(\Gamma)$ by the set $\{v_n : n \in \mathbb{N}\}$. Since $\mathcal{Z}$ has finite dimension and R_ϵ is injective, then according to the closed graph theorem, we can find a positive constant $\delta_\epsilon > 0$, such that the following inequality holds true:

$$\delta_\epsilon \|v_{L^2(\Gamma)}\| \leq \|y_\epsilon(v)\|_{L^2(\Gamma)}, \quad \forall v \in \mathcal{Z}. \tag{6}$$

This completes the proof of our proposition. ∎

Now we can prove the existence and uniqueness of optimal controls.

Theorem 2.2. *Let $\mathcal{U}_{ad}$ be a closed and convex subset of $L_0^2(\Gamma)$ verifying* (A_1) *or* (A_2) *or* (A_3). *Then for each $\epsilon > 0$, there exists a unique element $u_\epsilon \in \mathcal{U}_{ad}$, satisfying $J_\epsilon(u_\epsilon) = \min\{J_\epsilon(v); v \in \mathcal{U}_{ad}\}$.*

Proof. The uniqueness of u_ϵ results from the fact that J_ϵ is strictly convex. The existence of u_ϵ is clear when $\mathcal{U}_{ad}$ is bounded. When $\mathcal{U}_{ad}$ is not bounded in $L_0^2(\Gamma)$, then by a classical result of Lions (see [5]) in order to prove the existence of optimal controls, it suffices to verify the two following conditions:

(i) The map: $v \longrightarrow J_\epsilon(v)$ is weakly l.s.c. on the set $\mathcal{U}_{ad}$.
(ii) For every sequence (v_n) in $\mathcal{U}_{ad}$, such that $\|v_n\|_{L^2(\Gamma)} \longrightarrow +\infty$, then $J_\epsilon(v_n) \longrightarrow +\infty$, when $n \longrightarrow +\infty$.

But all these conditions are consequences of Proposition 2.1. ■

3. Convergence of the State y_ϵ and of the Optimal Control u_ϵ

Before stating the main result of this section, we need the following lemma:

Lemma 3.1. (3.1.1) *For every $v \in L_0^2(\Gamma)$, let $y(v)$ be the unique solution of the problem* $(\mathrm{P}_0)(v)$. *Then the linear mapping $R : L_0^2(\Gamma) \longrightarrow L_0^2(\Gamma)$, $v \mapsto y(v)$ is injective and compact having a norm $\|R\| \leq \lambda^2$.*

(3.1.2) *For every $v \in L_0^2(\Gamma)$, the state $y_\epsilon(v)$ converges in the space V to $y(v)$, when $\epsilon \longrightarrow 0$.*

(3.1.3) *For every $\epsilon > 0$, we have*

$$\|y_\epsilon(u_\epsilon) - y(u_\epsilon)\|_V \leq \epsilon\lambda \|y_\epsilon(u_\epsilon)\|_{L^2(\Gamma)}. \tag{7}$$

Proof. (3.1.1) is obtained by the same methods used to prove (2.1.1) and (2.1.2) of Proposition 2.1.

(3.1.2) We use the variational formulations of the problems $(\mathrm{P}_0)(v)$, and $(\mathrm{P}_\epsilon)(v)$. Then we get after some computations that

$$\|y(v) - y_\epsilon(v)\|_V^2 \leq \epsilon \int_\Gamma y_\epsilon(v)[y(v) - y_\epsilon(v)]\, d\gamma.$$

By using Cauchy-Schwarz inequality, Proposition 2.1, and the inequality (3), we obtain

$$\|y(v) - y_\epsilon(v)\|_V \leq \epsilon\lambda^3 \|v\|_{L^2(\Gamma)}, \ \ \forall \epsilon > 0.$$

(3.1.3) In a similar way one uses variational formulations of the problems $(\mathrm{P}_0)(u_\epsilon)$, and $(\mathrm{P}_\epsilon)(u_\epsilon)$, together with the inequality (3) to obtain the needed inequality. This completes the proof of our lemma. ■

Now, we are ready to state and prove the main result of this section dealing with convergence problems. More precisely, we have

Theorem 3.2. *Let $\mathcal{U}_{ad}$ be a closed and convex subset of $L_0^2(\Gamma)$ verifying* (A_1) *or* (A_2) *or* (A_3), *and let $\epsilon \in]0, 1[$. Then*

(3.2.1) *The optimal control u_ϵ converges weakly in $L^2(\Gamma)$, as $\epsilon \longrightarrow 0$, to the unique element $u \in \mathcal{U}_{ad}$, verifying $J(u) = \min\{J(v); v \in \mathcal{U}_{ad}\}$, where $J(v) = \int_\Gamma (y(v) - h)^2\, d\gamma$, and $y(v)$ is the unique solution of the problem $(P_0)(v)$. This convergence turns to be strong when (A_3) holds true.*

(3.2.2) *The state $y_\epsilon(u_\epsilon)$ converges strongly in the space V, as $\epsilon \rightarrow 0$, to $y(u)$ the unique solution of the problem $(P_0)(u)$.*

Proof. (i) If $\mathcal{U}_{ad}$ is bounded then there exists a positive constant $M > 0$ such that $\|u_\epsilon\|_{L^2(\Gamma)} \leq M$ for all $\epsilon \in]0,1[$, and we can extract a subsequence (called again (u_ϵ)) converging weakly to a unique element $u \in \mathcal{U}_{ad}$.

(ii) Suppose that $\mathcal{U}_{ad}$ is not bounded but verifying (A_2) or (A_3). Take and fix an element $w \in \mathcal{U}_{ad}$. Then for every $\epsilon \in]0,1[$, we have $0 \leq J_\epsilon(u_\epsilon) \leq J_\epsilon(w)$. By Proposition 2.1, we have $\|y_\epsilon(w)\|_{L^2(\Gamma)} \leq \lambda^2 \|w\|_{L^2(\Gamma)}$. Thus we can find a positive constant C_1 independent of $\epsilon \in]0,1[$ such that $0 \leq J_\epsilon(u_\epsilon) \leq C_1$. This inequality implies that the set (of traces) $\{y_\epsilon(u_\epsilon) : 0 < \epsilon < 1\}$ is bounded in $L^2(\Gamma)$. By using (7) we deduce that the set (of traces) $\{y(u_\epsilon) : 0 < \epsilon < 1\}$ is bounded in $L^2(\Gamma)$. By (3.1.1) of Lemma 3.1, we conclude that the set $\{u_\epsilon : 0 < \epsilon < 1\}$ must be bounded in $L^2(\Gamma)$. Therefore, we can extract a subsequence (called again (u_ϵ)) converging weakly to a unique element $u \in \mathcal{U}_{ad}$.

(iii) Let us denote u_* the unique element in $\mathcal{U}_{ad}$ verifying $J(u_*) = \min\{J(v) : v \in \mathcal{U}_{ad}\}$, where $J(v) = \int_\Gamma (y(v) - h)^2 \, d\gamma$, and $y(v)$ is the unique solution of the problem $(P_0)(v)$. To simplify the notations, we set $y_\epsilon(u_\epsilon) = y_\epsilon$. We will prove that $J_\epsilon(u_\epsilon)$ converges to $J(u_*)$ when $\epsilon \longrightarrow 0$.

For every $v \in \mathcal{U}_{ad}$, we can write the inequality $J_\epsilon(u_\epsilon) \leq J_\epsilon(v)$. From which we deduce that

$$\limsup_{\epsilon \to 0} J_\epsilon(u_\epsilon) \leq \lim_{\epsilon \to 0} J_\epsilon(v) = J(v). \tag{8}$$

The equality in the right member of (8) is true since $y_\epsilon(v)$ converges strongly in V to $y(v)$ and consequently, the trace $y_\epsilon(v)$ on Γ converges strongly in the Hilbert space $L^2(\Gamma)$ to the trace of $y(v)$.

Since y_ϵ is bounded in V we can find a subsequence (denoted again by y_ϵ) converging weakly to an element $z \in V$. It is no hard to see that we must have $z = y(u)$, where $y(u)$ is the unique solution of the problem $(P_0)(u)$. Now, by using Theorem 4, p. 143 of [7], we can suppose that this subsequence converges also strongly to $y(u)$ in the space $L^2(\Gamma)$. Then according to the lower semicontinuity of the norm in $L^2(\Gamma)$, we can assert that

$$\liminf_{\epsilon \to 0} J_\epsilon(u_\epsilon) \geq J(u). \tag{9}$$

Then we deduce that $u = u_*$ and that $\lim_{\epsilon \to 0} J_\epsilon(u_\epsilon) = J(u) = \int_\Gamma |y(u) - h|^2 \, d\gamma$. We deduce that u_ϵ converges weakly to u_* and that the net of traces (y_ϵ) converges strongly in $L^2(\Gamma)$ to the trace of $y(u)$.

(iv) Now, let us show that the state y_ϵ converges strongly in V to $y(u)$, when $\epsilon \to 0$. To this end, we start by writing the following inequalities:

$$\|y_\epsilon(u_\epsilon) - y(u)\|_V \leq \|y_\epsilon(u_\epsilon) - y(u_\epsilon)\|_V + \|y(u_\epsilon) - y(u)\|_V. \tag{10}$$

The inequality (7) will imply that

$$\|y_\epsilon(u_\epsilon) - y(u_\epsilon)\|_V \leq \epsilon\lambda \|y_\epsilon(u_\epsilon)\|_{L^2(\Gamma)}. \tag{11}$$

By using the variational formulations of the problems $(P_0)(u_\epsilon)$ and $(P_0)(u)$, we obtain

$$\|y(u_\epsilon) - y(u)\|_V \leq \sqrt{2M} \left[\|y(u_\epsilon) - y(u)\|_{L^2(\Gamma)}\right]^{1/2} \tag{12}$$

Now, by using the trace theorem and the inequality (7), we have

$$\begin{aligned}\|y(u_\epsilon)-y(u)\|_{L^2(\Gamma)} &\le \|y(u_\epsilon)-y_\epsilon(u_\epsilon)\|_{L^2(\Gamma)}+\|y_\epsilon(u_\epsilon)-y(u)\|_{L^2(\Gamma)}\\ &\le \epsilon\lambda^2\|y_\epsilon(u_\epsilon)\|_{L^2(\Gamma)}+\|y_\epsilon(u_\epsilon)-y(u)\|_{L^2(\Gamma)}\,. \end{aligned} \tag{13}$$

We conclude that y_ϵ converges strongly to $y(u)$ in the space V when $\epsilon \to 0$. This finishes the proof of our theorem. ∎

4. Adjoint State and System of Optimality Conditions

Let $\epsilon > 0$. For every $v \in L_0^2(\Gamma)$, we consider the following system:

$$\begin{cases} -\Delta p_\epsilon(v) = 0, \text{ on } \Omega,\\ \dfrac{\partial}{\partial\nu}p_\epsilon(v)+\epsilon p_\epsilon(v) = y_\epsilon(v) - h + \dfrac{1}{|\Gamma|}\displaystyle\int_\Gamma h\,d\gamma,\\ p_\epsilon(v)\in V, \end{cases} \tag{$(\mathrm{P}_\epsilon)^*(v)$}$$

where $|\Gamma|$ designates the Lebesgue measure of Γ, and $y_\epsilon(v)$ is the solution of the system $(\mathrm{P}_\epsilon)(v)$. One can use Lax–Milgram Theorem in the variational formulation for the problem $(\mathrm{P}_\epsilon^*)(v)$, and conclude that it has a unique solution $p_\epsilon \in V$. We shall see that $p_\epsilon(v)$ is an adjoint state for $y_\epsilon(v)$. To this end, it is sufficient to prove the following proposition:

Proposition 4.1. *Let $\epsilon \in]0,+\infty[$. Then for each $v, w \in L_0^2(\Gamma)$, we have*

$$\int_\Gamma w p_\epsilon(v)\,d\gamma = \frac{1}{2}J'_\epsilon(v)(w), \tag{14}$$

where $J'_\epsilon(v)$ is the derivative of the cost functional J_ϵ at v, and $J'_\epsilon(v)(w)$ is its value on the vector w.

Proof. It is easy to see that the derivative mapping $J'_\epsilon(v)$ of the cost functional J_ϵ at v is given for every $q \in L_0^2(\Gamma)$ by

$$J'_\epsilon(v)(q) = 2\int_\Gamma y_\epsilon(q)[y_\epsilon(v)-h]\,d\gamma. \tag{15}$$

We have the following equalities:

$$\begin{aligned}\int_\Gamma \frac{\partial}{\partial\nu}y_\epsilon(w)p_\epsilon(v)\,d\gamma &= \int_\Gamma [w-\epsilon y_\epsilon(w)]p_\epsilon(v)\,d\gamma\\ &= \int_\Gamma w p_\epsilon(v)\,d\gamma - \epsilon\int_\Gamma y_\epsilon(w)p_\epsilon(v)\,d\gamma. \end{aligned} \tag{16}$$

Now, by using Green formula, we get

$$\begin{aligned}\int_\Gamma \frac{\partial}{\partial\nu}y_\epsilon(w)p_\epsilon(v)\,d\gamma &= \int_\Gamma y_\epsilon(w)\frac{\partial}{\partial\nu}p_\epsilon(v)\,d\gamma\\ &= -\epsilon\int_\Gamma y_\epsilon(w)p_\epsilon(v)\,d\gamma\\ &\quad+\int_\Gamma y_\epsilon(w)[y_\epsilon(v)-h]\,d\gamma. \end{aligned} \tag{17}$$

By using the relations (16) and (17) we obtain

$$\int_\Gamma w p_\epsilon(v)\, d\gamma = \int_\Gamma y_\epsilon(w)[y_\epsilon(v) - h]\, d\gamma, \tag{18}$$

which is the desired formula. ∎

Let $\mathcal{U}_{ad}$ be a closed and convex subset of $L_0^2(\Gamma)$ verifying (A_1) or (A_2) or (A_3), and let $\epsilon > 0$. Then from Theorem 4.1, we derive the following characterization of the optimal control u_ϵ of the problem (Q_ϵ),

$$\int_\Gamma [v - u_\epsilon] p_\epsilon(u_\epsilon)\, d\gamma \geq 0, \ \forall v \in \mathcal{U}_{ad}.$$

Thus, the system of optimality conditions related to our problem can be written in the following form:

$$\begin{cases} -\Delta p_\epsilon = 0, \ \text{ on } \Omega, \\ -\Delta y_\epsilon = 0, \ \text{ on } \Omega, \\ \dfrac{\partial}{\partial \nu} p_\epsilon + \epsilon p_\epsilon = y_\epsilon - h \ \text{ at } \Gamma, \\ \dfrac{\partial}{\partial \nu} y_\epsilon + \epsilon y_\epsilon = u_\epsilon \ \text{ at } \Gamma, \\ \displaystyle\int_\Gamma [v - u_\epsilon] p_\epsilon(u_\epsilon)\, d\gamma \geq 0, \ \forall v \in \mathcal{U}_{ad}, \\ p_\epsilon \in H^1(\Omega),\ y_\epsilon \in H^1(\Omega),\ \displaystyle\int_\Gamma p_\epsilon\, d\gamma = 0, \text{ and } \int_\Gamma y_\epsilon\, d\gamma = 0. \end{cases}$$

5. Convergence of the Adjoint State p_ϵ

The purpose of this section is to prove the following theorem.

Theorem 5.1. *Let $\mathcal{U}_{ad}$ be a closed and convex subset of $L_0^2(\Gamma)$ verifying (A_1) or (A_2) or (A_3), and let $\epsilon \in]0,1[$. Let u be the optimal control described by (3.2.1) of Theorem 3.2. Then*

(5.1.1) *The adjoint state p_ϵ converges strongly in V, to the unique element $p(u) \in V$, verifying*

$$\begin{cases} -\Delta p(u) = 0, \ \text{ on } \Omega, \\ \dfrac{\partial}{\partial \nu} p(u) = y(u) - h + \dfrac{1}{|\Gamma|}\displaystyle\int_\Gamma h\, d\gamma, \\ p(u) \in H^1(\Omega),\ \displaystyle\int_\Gamma p(u)\, d\gamma = 0. \end{cases} \tag{$(P_0)^*(u)$}$$

Thus, $p(u)$ is the adjoint state corresponding to the state $y(u)$ solution of the problem $(P_0)(u)$.

(5.1.2) *The optimal control u is characterized by the following system of optimality conditions,*

$$\begin{cases} -\Delta p(u) = 0, \ \ \textit{on } \Omega, \\ -\Delta y(u) = 0, \ \ \textit{on } \Omega, \\ \dfrac{\partial}{\partial \nu} p(u) = y - h \ \ \textit{at } \Gamma, \\ \dfrac{\partial}{\partial \nu} y(u) = u \ \ \textit{at } \Gamma, \\ \displaystyle\int_\Gamma [v-u] p(u)\, d\gamma \geq 0, \ \forall v \in \mathcal{U}_{ad}, \\ p(u) \in H^1(\Omega), \ y(u) \in H^1(\Omega), \ \displaystyle\int_\Gamma p(u) d\gamma = 0, \ \textit{and} \ \displaystyle\int_\Gamma y(u)\, d\gamma = 0. \end{cases}$$

Proof. We know (see [7] for example) that we can find a positive constant $\rho > 0$, such that

$$\left[\int_\Omega |\nabla y|^2\, d\omega + \int_\Gamma y^2\, d\gamma\right]^{1/2} \leq \rho \left[\int_\Omega |\nabla y|^2\, d\omega\right]^{1/2} \quad \forall y \in V. \tag{19}$$

Now, by using the variational formulations for the adjoint systems $(\mathrm{P}_\epsilon)^*(u_\epsilon)$ and $(\mathrm{P}_0)^*(u)$, we get after some computations the following inequality

$$\int_\Omega |\nabla(p_\epsilon - p)|^2\, d\omega = -\epsilon \int_\Gamma p_\epsilon [p_\epsilon - p]\, d\gamma + \int_\Gamma [y_\epsilon - y][p_\epsilon - p]\, d\gamma$$

where we have denoted $p_\epsilon := p_\epsilon(u_\epsilon)$, and $p := p(u)$. With the help of the relation (19), we obtain

$$\int_\Omega |\nabla(p_\epsilon - p)|^2\, d\omega \leq \epsilon\rho \|p_\epsilon\|_{L^2(\Gamma)} \|p_\epsilon - p\|_V + \rho^2 \|y_\epsilon - y\|_V \|p_\epsilon - p\|_V. \tag{20}$$

The inequality (20) is equivalent to say that we have

$$\|p_\epsilon - p\|_V \leq \epsilon\rho \|p_\epsilon\|_{L^2(\Gamma)} + \rho^2 \|y_\epsilon - y\|_V .$$

To finish the proof of our theorem, it suffices to see that the net of traces p_ϵ is bounded in $L^2(\Gamma)$. But this fact can be easily proved by using the relation (18). ■

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