Vietnam Journal of Mathematics **30**:1 (2002) 43–54

Vietnam Journal of MATHEMATICS

p-Adic Poisson–Jensen Formula in Several Variables

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Received September 26, 2000
Revised March 9, 2001

Abstract. In this paper we give a p-adic version of the Poisson–Jensen Formula for p-adic holomorphic functions of several variables in the general case of critical points of the forms $(t_1, \dots, t_m)$.

1. Introduction

Nevanlinna theory extends Picard's theorem for meromorphic functions. There are two main theorems which occupy a central place in Nevanlinna theory. The First Main Theorem is just the reformulation of the Possoin–Jensen Formula for meromorphic functions. The Second Main Theorem generalizes the classical Picard's theorem. Recently, Nevanlinna theory was extended to the p-adic case. In [7], Ha Huy Khoai related non-Archimedean function theory in several variables to the combinatorial geometry of higher dimensional analogues of the valuation polygon. Rather than take this approach, Cherry and Ye ([4]) consider a meromorphic function in several variables and restrict it to a generic line through the origin, and prove that the counting function for this one variable function does not depend on the choice of line though the origin. They use this observation to define counting functions as in Ha Huy Khoai's one variable theory, and then a several variable Poisson–Jensen Formula follows. However, their method gives the formula only for the case of critical points of the forms $(t, \dots, t)$.

In this paper by using the ideas in [7] and some arguments in [4] we give a p-adic version of the Poisson–Jensen Formula for p-adic holomorphic functions of several variables in the case of critical points of the forms $(t_1, \dots, t_m)$.

2. Height of p-Adic Holomorphic Functions of Several Variables

Let p be a prime number, $\mathbf{Q}_p$ the field of p-adic numbers and $\mathbf{C}_p$ the p-adic completion of an algebraic closure of $\mathbf{Q}_p$. The absolute value in $\mathbf{Q}_p$ is normalized so that $|p| = p^{-1}$. We further use the notation $v(z)$ for the additive valuation on $\mathbf{C}_p$ which extends ord_p.

We use the notations

$b_{(m)} = (b_1, ..., b_m)$, $b_{(m,i)}(b) = (b_1, ..., b_{i-1}, b, b_{i+1}, ..., b_m)$,

$\widehat{(b_i)} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_m)$,

$D_r = \{z \in \mathbf{C}_p : |z| \le r, r > 0\}$, $D_{<r>} = \{z \in \mathbf{C}_p : |z| = r, r > 0\}$,

$D = \{z \in \mathbf{C}_p : |z| \le 1\}$,

$D_{r_{(m)}} = D_{r_1} \times \cdots \times D_{r_m}$, where $r_{(m)} = (r_1, \ldots, r_m)$ for $r_i \in \mathbf{R}_+$,

$D_{<r_{(m)}>} = D_{<r_1>} \times \cdots \times D_{<r_m>}$,

$D^m = D \times \cdots \times D$ the unit polydisc in $\mathbf{C}_p^m$, $|f|_{r_{(m)}} = |f|_{(r_1, \ldots, r_m)}$;

$\gamma_i \in \mathbf{N}, \gamma = (\gamma_1, ..., \gamma_m)$,

$|\gamma| = \gamma_1 + \cdots + \gamma_m$, $z^\gamma = z_1^{\gamma_1} ... z_m^{\gamma_m}$, $r^\gamma = r_1^{\gamma_1} ... r_m^{\gamma_m}$.

$\log = \log_p$, $t_i = -\log r_i$, $i = 1, ..., m$.

Let f be a non-zero holomorphic function in $D_{r_{(m)}}$ represented by a convergent series

$$f = \sum_{|\gamma| \ge 0} a_\gamma z^\gamma, |z_i| \le r_i \text{ for } i = 1, \ldots, m.$$

Notice that the set of $(r_1, ..., r_m) \in \mathbf{R}_+^m$ such that there exist $x_1, ..., x_m \in \mathbf{C}_p$ with $|x_i| = r_i, i = 1, ..., m$, is dense in $\mathbf{R}_+^m$. Therefore, without loss of generality one can assume that $D_{<r_{(m)}>} \neq \emptyset$.

We define

$$|f|_{r_{(m)}} = \max_{0 \le |\gamma| < \infty} |a_\gamma| r^\gamma.$$

Set $\gamma t = \gamma_1 t_1 + \cdots + \gamma_m t_m$.
Then we have

$$\lim_{|\gamma| \to \infty} (v(a_\gamma) + \gamma t) = +\infty.$$

Hence, there exists a $\vec{\gamma} \in \mathbf{N}^m$ such that $v(a_\gamma) + \gamma t$ is minimal.

Definition 2.1. *The height of the function $f(z_{(m)})$ is defined by*

$$H_f(t_{(m)}) = \min_{0 \le |\gamma| < \infty} (v(a_\gamma) + \gamma t).$$

We also use the notation

$$H_f^+(t_{(m)}) = -H_f(t_{(m)}).$$

Write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}(\widehat{z_i}) z_i^k, \ \ i = 1, 2, \ldots, m.$$

Set

$$I_f(t_{(m)}) = \left\{(\gamma_1, \ldots, \gamma_m) \in \mathbf{N}^m : v(a_\gamma) + \gamma t = H_f(t_{(m)})\right\},$$

$$n^+_{i,f}(t_{(m)}) = \min\left\{\gamma_i : \exists\, (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(t_{(m)})\right\},$$

$$n^-_{i,f}(t_{(m)}) = \max\left\{\gamma_i : \exists\, (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(t_{(m)})\right\},$$

$$n_{i,f}(0,0) = \min\left\{k : f_{i,k}(\widehat{z}_i) \neq 0\right\},$$

$$\nu_f(t_{(m)}) = \sum_{i=1}^{m} \left((n^-_{i,f}(t_{(m)}) - n^+_{i,f}(t_{(m)})\right).$$

Call $t_{(m)}$ a *critical point* if $\nu_f t_{(m)} \neq 0$.

Theorem 2.2. *Let $f(z)$ be a holomorphic function on D_r. Assume that f is not identically zero. Then there exists a polynomial*

$$g(z) = b_0 + b_1 z + \cdots + b_v z^v,\ \deg g = n^-_f(t), t = -\log_p r,$$

and a holomorphic function $h = 1 + \sum_{n=1}^{\infty} c_n z^n$ on D_r such that

1) *$f(z) = g(z)h(z)$,*
2) *$f(z)$ just has $n^-_f(t)$ zeros in D_r,*
3) *$n^-_f(t) - n^+_f(t)$ is equal to the number of zeros of f at $v(z) = t$,*
4) *h has no zeros in D_r.*

For the proof, see Weierstrass Preparation Theorem [5].

Let $f = \sum_{|\gamma|=0}^{\infty} a_\gamma z^\gamma$ be a non-zero entire function on $\mathbf{C}_p^m$. Choose $y = y_{(m)}$ such that

$$|y| = \max\{|\gamma| : |a_\gamma| = |f|_{(1\ldots,1)}\}.$$

The set of z in $\mathbf{C}_p$ with $|z| \leq 1$ forms a closed subring of $\mathbf{C}_p$. We denote this subring by $\mathcal{O}$ (called the ring of integers of $\mathbf{C}_p$), and the set of z with $|z| < 1$ forms a maximal ideal I in $\mathcal{O}$. We denote the residue class field $\mathcal{O}/I$ by $\widehat{\mathbf{C}}_p$. Note that since $\mathbf{C}_p$ is algebraically closed, so is $\widehat{\mathbf{C}}_p$, and in particular $\widehat{\mathbf{C}}_p$ cannot be a finite field.

Given an element w in $\mathcal{O}$, we denote its equivalence class in $\widehat{\mathbf{C}}_p$ by $\widehat{w}$.

Define $\widehat{f}$ by

$$\widehat{f} = \sum_{|\gamma|=0}^{\infty} \frac{\widehat{a_\gamma}}{a_y} z^\gamma.$$

Since f is entire, $\left|\frac{a_\gamma}{a_y}\right| < 1$ for all but finitely many γ, and thus $\widehat{f}$ is a polynomial in m variables with coefficients in $\widehat{\mathbf{C}}_p$. Since

$$\left|\frac{a_y}{a_y}\right| = 1,$$

$\widehat{f}$ is not the zero polynomial.

Lemma 2.3. *Let $f_s(z_{(m)}) = \sum_{|\gamma|=0}^{\infty} a_\gamma^s z^\gamma, s = 1, \ldots, q,$ be q non-zero entire functions on $\mathbf{C}_p^m$. Then for any $D_{r_{(m)}}$ in $\mathbf{C}_p^m$ $(D_{<r_{(m)}>} \neq \emptyset)$ there exists $u = u_{(m)} \in D_{r_{(m)}}$ such that*

$$|f_s(u_{(m)})| = |f_s|_{r_{(m)}}, s = 1, \ldots, q.$$

Proof. First we prove that if $r_{(m)} = (1, \ldots, 1)$, then there exists $w = w_{(m)} \in D^m$ such that

$$|f_s(w)| = \max_{0 \le |\gamma| < \infty} |a_\gamma^s|, s = 1, \ldots, q. \tag{2.1}$$

For each $s = 1, \ldots, q$, choose $y_s = (y_1^s, \ldots, y_m^s)$ such that

$$|y_s| = \max\{|\gamma| : |a_\gamma^s| = |f|_{(1,\ldots,1)}\}.$$

Set

$$\mathcal{M} = \{\widehat{f}_s, s = 1, \ldots, q\}.$$

Since $\widehat{f}_s$ is not the zero polynomial, so is $\prod_{s=1}^{q} \hat{f}_s$.

Let $w = w_{(m)} \in D^m$ be such that $\widehat{w}$ is not a solution of $\prod_{s=1}^{q} \hat{f}_s$.

Set

$$\frac{f_s(w)}{a_{y_s}} = b_s, \quad s = 1, \ldots, q.$$

We have

$$\widehat{b_s} = \widehat{f}_s(\widehat{w}).$$

Since $\widehat{w}$ is not a solution of all $\widehat{f}_s$,

$$b_s \notin I.$$

Thus

$$\left|\frac{f_s(w)}{a_{y_s}}\right| = 1.$$

Hence, $|f_s(w)| = |a_{y_s}|$.

Now let $x_1, \ldots, x_m \in \mathbf{C}_p$ such that $|x_i| = r_i$.

Consider the following transformations of $\mathbf{C}_p^m$:

$$\varphi(z_{(m)}) = (x_1 z_1, \ldots, x_m z_m).$$

Set

$$x = (x_1, \ldots, x_m).$$

We have

$$\varphi(D^m) = D_{r_{(m)}},$$

and

$$f_s \circ \varphi(z_{(m)}) = \sum_{|\gamma|=0}^{\infty} (a_\gamma^s x^\gamma) z^\gamma$$

are non-zero entire functions on $\mathbf{C}_p^m$.

By (2.1) there exists $w = w_{(m)}$ such that

$$\begin{aligned}\left|f_s \circ \varphi(w)\right| &= \max_{0\le|\gamma|<\infty} \left|a_\gamma^s x^\gamma\right| = \max_{0\le|\gamma|<\infty} \left|a_\gamma^s\right|\left|x_1\right|^{\gamma_1}\cdots\left|x_m\right|^{\gamma_m} \\ &= \max_{0\le|\gamma|<\infty} \left|a_\gamma^s\right| r^\gamma = \left|f_s\right|_{r_{(m)}}.\end{aligned}$$

Set $u = \varphi(w)$. Then $u \in D_{r_{(m)}}$ and $\left|f_s(u)\right| = \left|f_s\right|_{r_{(m)}}$, $s = 1, \ldots, q$. ∎

Lemma 2.4. *Let $f_s(z_{(m)}), s = 1, 2, \ldots, q$, be q non-zero holomorphic functions on $D_{r_{(m)}}$. Then there exists $u = u_{(m)} \in D_{r_{(m)}}$ such that*

$$|f_s(u)| = |f_s|_{r_{(m)}}, s = 1, 2, \ldots, q.$$

Proof. Let

$$f_s = \sum_{|\gamma|=0}^{\infty} a_\gamma^s z^\gamma.$$

For each $s = 1, 2, \ldots, q$, we set

$$k_s = \max_{0\le|\gamma|<\infty} \left\{|\gamma| : |a_\gamma^s| r^\gamma = \left|f_s\right|_{r_{(m)}}\right\}.$$

Then

$$P_s = \sum_{0\le|\gamma|\le k_s} a_\gamma^s z^\gamma,\ s = 1, \ldots, q,$$

are non-zero entire functions on $\mathbf{C}_p^m$.

By Lemma 2.3, there exists $u_{(m)} = (u_1, \ldots, u_m) \in D_{r_{(m)}}$ with $|u_i| = r_i$ such that

$$\left|P_s(u_{(m)})\right| = \left|P_s\right|_{r_{(m)}},\ s = 1, \ldots, q.$$

Moreover,

$$\left|P_s\right|_{r_{(m)}} = \left|f_s\right|_{r_{(m)}}, \left|P_s(u_{(m)})\right| = \left|f_s(u_{(m)})\right|,\ s = 1, \ldots, q.$$

Thus

$$\left|f_s(u_{(m)})\right| = |f_s|_{r_{(m)}},\ s = 1, \ldots, q. \quad ∎$$

As an immediate consequence of Lemma 2.4 we have

Corollary 2.5. *Let $f(z_{(m)})$ be a non-zero holomorphic function on $D_{r_{(m)}}$. Then*

$$\max_{u\in D_{r_{(m)}}} |f(u)| = |f|_{r_{(m)}}.$$

3. p-Adic Poisson–Jensen Formula in Several Variables

Let f be a non-zero holomorphic function on $D_{r_{(m)}}$.
Write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}(\widehat{z_i}) z_i^k, \quad i = 1, 2, \ldots, m.$$

Let

$$n_{i,f}(0,0) = \min\{k : f_{i,k}(\widehat{z_i}) \not\equiv 0\}.$$

For a fixed i $(i = 1, \ldots, m)$ we set for simplicity

$$n_{i,f}(0,0) = \ell, \ k_1 = n^-_{i,f}(t_{(m)}), \ k_2 = n^+_{i,f}(t_{(m)}).$$

Then there exist multi-indices $\gamma = (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(t_{(m)})$ and $\mu = (\mu_1, \ldots, \mu_i, \ldots, \mu_m) \in I_f(t_{(m)})$ such that $\gamma_i = k_1, \mu_i = k_2$.
We consider the following holomorphic functions on $D_{r_{(m)}}$

$$f_\ell(z_{(m)}) = f_{i,\ell}(\widehat{z_i}) z_i^\ell,$$

$$f_{k_1}(z_{(m)}) = f_{i,k_1}(\widehat{z_i}) z_i^{k_1},$$

$$f_{k_2}(z_{(m)}) = f_{i,k_2}(\widehat{z_i}) z_i^{k_2}.$$

The functions are not identically zero.

Set

$$U_i = \{u = u_{(m)} \in D_{r_{(m)}} : |f_\ell(u)| = |f_\ell|_{r_{(m)}}, |f(u)| = |f|_{r_{(m)}}, \\ |f_{k_1}(u)| = |f_{k_1}|_{r_{(m)}}, |f_{k_2}(u)| = |f_{k_2}|_{r_{(m)}}\},$$

where $i = 1, \ldots, m$.
By Lemma 2.4, U_i is a non-empty set. For each $u \in U_i$, set

$$f_{i,u}(z) = \sum_{k=0}^{\infty} f_{i,k}(\widehat{u_i}) z_i^k, \quad z = z_i \in D_{r_i}.$$

The following theorem shows that we can use the Weierstrass Preparation Theorem [5] to count zeros by s licing with a generic line through the point u.

Theorem 3.1. *Let $f(z_{(m)})$ be a holomorphic function on $D_{r_{(m)}}$. Assume that $f(z_{(m)})$ is not identically zero. Then for each $i = 1, \ldots, m$, and for all $u \in U_i$, we have*

1) $H_f(t_{(m)}) = H_{f_{i,u}}(t_i)$,
2) $n^-_{i,f}(t_{(m)})$ is equal to the number of zeros of $f_{i,u}$ in D_{r_i},
3) $n^-_{i,f}(t_{(m)}) - n^+_{i,f}(t_{(m)})$ *is equal to the number of zeros of $f_{i,u}$ at $v(z) = t_i$.*

Proof. Set $k_3 = n^-_{f_{i,u}}(t_i), k_4 = n^+_{f_{i,u}}(t_i)$. Since

$$|f|_{r_{(m)}} = |a_\gamma| r_1^{\gamma_1} \ldots r_i^{k_1} \ldots r_m^{\gamma_m} = |a_\mu| r_1^{\mu_1} \ldots r_i^{k_2} \ldots r_m^{\mu_m} \\ = |f_{k_1}|_{r_{(m)}} = |f_{k_2}|_{r_{(m)}} = |f(u_{(m)})|,$$

we obtain

$$|f_{i,k_1}(\widehat{u_i})|r_i^{k_1} = |f|_{r_{(m)}} = |f_{i,k_2}(\widehat{u_i})|r_i^{k_2} = |f(u_{(m)})|.$$

On the other hand, we have

$$|f_{i,k_2}(\widehat{u_i})|r_i^{k_2} = |f_{i,k_1}(\widehat{u_i})|r_i^{k_1} \leq |f_{i,u}|_{r_i} \leq |f|_{r_{(m)}}.$$

Hence

$$|f_{i,k_2}(\widehat{u_i})|r_i^{k_2} = |f_{i,u}|_{r_i} = |f_{i,k_1}(\widehat{u_i})|r_i^{k_1}.$$

From this it follows that $k_1 \leq k_3$ and $k_4 \leq k_2$. Now we consider j such that

$$|f_{i,j}(\widehat{u_i})|r_i^j = |f_{i,u}|_{r_i}.$$

Then there exists $\eta = (\eta_1, \ldots, \eta_i, \ldots, \eta_m)$ with $\eta_i = j$ such that

$$|f(u_{(m)})| = |f_{i,u}(u_i)| \leq |f_{i,u}|_{r_i} = |f_{i,j}(\widehat{u_i})|r_i^j \leq |a_\eta|r^\eta \leq |f|_{r_{(m)}}.$$

Since

$$|f(u_{(m)})| = |f|_{r_{(m)}},$$

we have

$$|a_y|r^\eta = |f|_{r_{(m)}}.$$

Hence $k_2 \leq j \leq k_1$. From this it follows that $k_4 \geq k_2$ and $k_3 \leq k_1$. Since $k_1 \leq k_3$ and $k_2 \geq k_4$, so $k_2 = k_4$ and $k_1 = k_3$. By Lemma 2.4 and Theorem 2.2, we see that $H_f(t_{(m)}) = H_{f_{i,u}}(t_i)$, and $n_{i,f}^-(t_{(m)})$ is equal to the number of zeros of $f_{i,u}$ in D_{r_i} and $n_{i,f}^-(t_{(m)}) - n_{i,f}^+(t_{(m)})$ is equal to the number of zeros of $f_{i,u}$ at $v(z) = t_i$.

Theorem 3.1 is proved. ■

For each $i = 1, \ldots, m$, from Theorem 3.1 we see that $n_{i,f}(0,0) = n_{f_{i,u}}(0,0)$ for all $u \in U_i$.

Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Define $n_{i,f}(0, r_{(m)})$ to be the number of zeros with absolute value $\leq r_i$ of the one-variable function $f_{i,u}(z)$.

Theorem 3.1 tells us that

$$n_{i,f}(0, r_{(m)}) = n_{i,f}^-(t_{(m)}).$$

For a an element of $\mathbf{C}_p$ and f a holomorphic function on $D_{r_{(m)}}$, which is not identically equal to a, define

$$n_{i,f}(a, r_{(m)}) = n_{i,f-a}(0, r_{(m)}),\ n_{i,f}(a, 0) = n_{i,f-a}(0,0), \quad i = 1, \ldots, m.$$

We fix real numbers $\rho_1, \ldots, \rho_m$ with $0 < \rho_i \leq r_i$, $i = 1, \ldots, m$.

For each $x \in \mathbf{R}$, we set

$$A_i(x) = (\rho_1, \ldots, \rho_{i-1}, x, r_{i+1}, \ldots, r_m),\ i = 1, \ldots, m.$$

Define the counting function $N_f(a, t_{(m)})$ by

$$N_f(a, t_{(m)}) = \frac{1}{\ln p} \sum_{k=1}^{m} \int_{\rho_k}^{r_k} \frac{n_{k,f}(a, A_k(x))}{x} dx.$$

If $a = 0$, then set $N_f(t_{(m)}) = N_f(0, t_{(m)})$.

For each $t \in \mathbf{R}$, set

$$T_i(t) = (c_1, \ldots, c_{i-1}, t, t_{i+1}, \ldots, t_m),$$

where

$$c_i = -\log \rho_i, i = 1, \ldots, m.$$

Theorem 3.2. (*p*-adic Poisson–Jensen Formula in several variables) *Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Then*

$$H_f^+(t_{(m)}) - H_f^+(c_{(m)}) = N_f(t_{(m)}).$$

Proof. Let

$$f = \sum_{k=0}^{\infty} f_{1,k}(\widehat{z_1}) z_1^k.$$

Set

$$\ell = n_{1,f}(0,0), \quad a = \log \left| f_{1,\ell}(\widehat{z_1}) \right|_{\widehat{r_1}},$$

$$M = \frac{1}{\ln p} \int_0^{r_1} \frac{n_{1,f}(0, A_1(x)) - \ell}{x} dx + \ell \log r_1,$$

$$M_1 = \frac{1}{\ln p} \int_0^{\rho_1} \frac{n_{1,f}(0, A_1(x)) - \ell}{x} dx + \ell \log \rho_1,$$

$$M_2 = \frac{1}{\ln p} \int_{\rho_1}^{r_1} \frac{n_{1,f}(0, A_1(x)) - \ell}{x} dx + \ell \log \frac{r_1}{\rho_1},$$

$$M_3 = \frac{1}{\ln p} \int_{\rho_1}^{r_1} \frac{n_{1,f}(0, A_1(x))}{x} dx,$$

$$\Gamma = \{T_1(t) : (n_{1,f}^- \circ T_1(t) - n_{1,f}^+ \circ T_1(t)) \neq 0, \ t \geq t_1\}.$$

We will prove

$$H_f^+(t_{(m)}) - H_f^+ \circ T_1(c_1) = M_3. \tag{3.1}$$

To show (3.1) first we prove the following

$$H_f^+(t_{(m)}) - a = M. \tag{3.2}$$

Case 1. $\ell = 0$.

Then

$$M = \frac{1}{\ln p}\int_0^{r_1} \frac{n_{1,f}(0, A_1(x))}{x}dx.$$

If $\Gamma = \emptyset$, then $H_f^+(t_{(m)}) = a$ and $M = 0$. Therefore

$$H_f^+(t_{(m)}) - a = M.$$

If $\Gamma \neq \emptyset$, then Γ is a finite set. Suppose that Γ contains n elements

$$y^{(1)} = T_1(t^{(1)}),$$
$$\dots$$
$$y^{(n)} = T_1(t^{(n)}),$$

where $t_1 \leq t^{(1)} < t^{(2)} < \cdots < t^{(n)}$.

Set $b_i = p^{-t^{(i)}}$, $i = 1, 2, \ldots, n$, $s = n_{1,f}(0, r_{(m)})$, $s_1 = n_{1,f}(0, A_1(b_2))$, $a_1 = |f_{1,s}(\widehat{z_1})|_{\widehat{r_1}}$, $a_2 = \log|f|_{r_{(m,1)}(b_1)}$, $a_3 = \log|f|_{r_{(m,1)}(b_2)}$, $a_4 = |f_{1,s_1}(\widehat{z_1})|_{\widehat{r_1}}$.

Then $0 < b_n < b_{n-1} < \cdots < b_1 \leq r_1$.

We will prove (3.2) by induction on n.

Case $n = 1$.

If $b_1 = r_1$, then $n_{1,f}(0, A_1(x)) = 0$, $0 < x < r_1$. Moreover, by the continuity of $H_f^+ \circ T_1(t)$, we obtain (3.2).

Consider $b_1 < r_1$. We have

$$M = s\big(\log r_1 - \log b_1\big) = \log(a_1 r_1^s) - \log(a_1 b_1^s).$$

Since $b_1 < r_1$ and $n = 1$,

$$H_f^+(t_{(m)}) = \log(a_1 r_1^s).$$

Furthermore, $T_1(t) \notin \Gamma$ with $t > t^{(1)}$ and $H_f^+ \circ T_1(t)$ is continuous.

Thus

$$\log\left(a_1 b_1^s\right) = a.$$

Hence (3.2) follows. So (3.2) is proved in this case.

Now we will prove (3.2) for any n.

Case $b_1 < r_1$.

Then $0 < b_n < b_{n-1} \cdots < b_1 < r_1$ and $t_1 < t^{(1)} < \cdots < t^{(n)}$. Apply the induction hypothesis,

$$\frac{1}{\ln p}\int_0^{b_1} \frac{n_{1,f}(0, A_1(x))}{x}dx = a_2 - a.$$

Thus

$$M = a_2 - a + \frac{1}{\ln p}\int_{b_1}^{r_1} \frac{n_{1,f}(0, A_1(x))}{x}dx.$$

On the other hand,

$$\frac{1}{\ln p}\int_{b_1}^{r_1}\frac{n_{1,f}(0,A_1(x))}{x}dx = s(\log r_1 - \log b_1)$$

$$= \log(a_1 r_1^s) - \log(a_1 b_1^s),$$

$$a_2 = \log(a_1 b_1^s). \tag{3.3}$$

Since $T_1(t) \notin \Gamma$ with $t_1 \le t < t^{(1)}$,

$$H_f^+(t_{(m)}) = \log(a_1 r_1^s). \tag{3.4}$$

By (3.3) and (3.4),

$$M = H_f^+(t_{(m)}) - a.$$

Case $b_1 = r_1$.

Then $0 < b_n < \cdots < b_2 < b_1 = r_1$ and $t_1 = t^{(1)} < \cdots < t^{(n)}$.

Apply the induction hypothesis,

$$\frac{1}{\ln p}\int_{0}^{b_2}\frac{n_{1,f}(0,A_1(x))}{x}dx = a_3 - a.$$

Thus

$$M = a_3 - a + \frac{1}{\ln p}\int_{b_2}^{b_1}\frac{n_{1,f}(0,A_1(x))}{x}dx. \tag{3.5}$$

Moreover, $n_{1,f}(0,A_1(x)) = s_1$ with $b_2 \le x < b_1$, and

$$\frac{1}{\ln p}\int_{b_2}^{b_1}\frac{n_{1,f}(0,A_1(x))}{x}dx = s_1(\log b_1 - \log b_2) = \log(a_4 b_1^{s_1}) - \log(a_4 b_2^{s_1}),$$

$$a_3 = \log(a_4 b_2^{s_1}).$$

Since $T_1(t) \notin \Gamma$ with $t^{(1)} < t < t^{(2)}$, and by the continuity of $H_f^+ \circ T_1(t)$,

$$H_f^+(t_m) = \log(a_4 b_1^{s_1}). \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$M = H_f^+(t_m) - a.$$

Case $\ell \neq 0$.

Then $f = f_1 f_2$ with $f_1 = z_1^{\ell}$.

We have

$$n_{1,f_2}(0,0) = 0,$$

$$n_{1,f}(0,0) = \ell,\ n_{1,f}(0,A_1(x)) = n_{1,f_2}(0,A_1(x)) + \ell,$$

$$H_f^+(t_{(m)}) = H_{f_1}^+(t_{(m)}) + H_{f_2}^+(t_{(m)}) = \ell\log r_1 + H_{f_2}^+(t_{(m)}).$$

By case $\ell = 0$,

$$\frac{1}{\ln p}\int_0^{r_1}\frac{n_{1,f}(0, A_1(x))}{x}dx = H^+_{f_2}(t_{(m)}) - a.$$

Thus

$$M = H^+_{f_2}(t_{(m)}) - a + \ell \log r_1 = H^+_f(t_{(m)}) - a.$$

Similarly we obtain

$$M_1 = H^+_f \circ T_1(c_1) - a. \tag{3.7}$$

We have

$$M = M_1 + M_2, \quad M_3 = M_2.$$

Apply (3.2) and (3.7),

$$M_3 = M - M_1 = H^+_f(t_{(m)}) - H^+_f \circ T_1(c_1).$$

Similarly we have

$$H^+_f \circ T_{i-1}(c_{i-1}) - H^+_f \circ T_i(c_i) = \frac{1}{\ln p}\int_{\rho_i}^{r_i}\frac{n_{1,f}(0, A_1(x))}{x}dx$$

$$\text{for } i = 2, \ldots, m. \tag{3.8}$$

Apply (3.8),

$$H^+_f(t_{(m)}) - H^+_f \circ T_m(c_m) = H^+_f(t_{(m)}) - H^+_f \circ T_1(c_1) + H^+_f \circ T_1(c_1) - \ldots$$
$$+ H^+_f \circ T_{m-1}(c_{m-1}) - H^+_f \circ T_m(c_m),$$

we obtain

$$H^+_f(t_{(m)}) - H^+_f(c_{(m)}) = N_f(t_{(m)}).$$ ■

Acknowledgements. The author would like to thank Professor Ha Huy Khoai for suggesting these problems to him.

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