Vietnam Journal of Mathematics **30**:1 (2002) 97-102

Vietnam Journal of MATHEMATICS

Short Communication

Homotopy of Configuration Spaces*

Nguyen Viet Dung
Institute of Mathematics, P. O. Box 631, Bo Ho, Hanoi, Vietnam

Received December 14, 2001

1. Let M be a topological manifold and G a finite group, acting freely on M. For an integer $m \geq 0$, let $Gq_1, \ldots, Gq_m$ be m distinct orbits of G, $q_i \in M$ for $1 \leq i \leq m$. Set $Q_m = \cup_{i=1}^m Gq_i$.

Definition 1. (see [2]) *The G-equivariant configuration space of G-manifold M, denoted by $F_{m,n}(M;G)$, is the subspace of the Cartesian product $\underset{n}{\times} M$, consisting of all n-tuples of points in $M \backslash Q_m$ lying on n pairwise distinct orbits*

$$F_{m,n}(M;G) = \{(x_1, \ldots, x_m) | x_i \in M \backslash Q_m \text{ and } Gx_i \cap Gx_j = \emptyset \text{ if } i \neq j\}.$$

The case when G is the unit group, $F_{m,n}(M)$ were first studied by Fox and Neuwrith in [3]. We consider two special cases.

First case. Let $M = \mathbb{R}^q$, $m = 0$ and G is the unit group. Then the space $F_{0,n}(M;G)$ is usually denoted by $F_n(\mathbb{R}^q)$ or $F_A(\mathbb{R}^q, n)$ and is closely related to symmetric groups, the Weyl group of type A. We call it the configuration space of type A. Indeed, the symmetric group Σ_n, acts freely on $F_A(\mathbb{R}^q, n)$ by permutations.

Second case. Let $M = \mathbb{R}^q \backslash \{0\}$, $m = 0$ and $G = \mathbb{Z}_2$ operating on M by antipodal action. The configuration $F_{0,n}(M;G)$ then is denoted by $F_B(\mathbb{R}^q, n)$. The semi-direct product $W_n = \Sigma_n \tilde{\times} \mathbb{Z}_2^n$, the Weyl group of type B, acts on $\underset{n}{\times} \mathbb{R}^q$ and therefore on $F_B(\mathbb{R}^q, n)$ by permutations and antipodes. We call $F_B(\mathbb{R}^q, n)$ the configuration space of type B. This space is recently studied in [6].

The configuration space $F_A(\mathbb{R}^q, n)$ has been studied by many authors [3], [4], [5]. In this paper we will carry out studies on the homotopy of configuration

* This work was supported in part by the National Basic Program in Natural Sciences, Vietnam.

spaces of type B, $F_B(\mathbb{R}^q, n)$.

Consider the quotient space M/G. We then can define the configuration space $F_{m,n}(M/G)$ by taking the unit group as action group and using the set $\bar{Q}_m = \{\bar{q}_i, q_i \in Q_m\}$ instead of the set Q_m. Here $\bar{x}$ denotes the image of $x \in M$ by the natural projection $M \longrightarrow M/G$.

It is easy to see that

$$F_{m,n}(M;G)/G \longrightarrow F_{m,n}(M/G)$$
$$[(p_1, \ldots, p_n)] \mapsto (\bar{p}_1, \ldots, \bar{p}_n)$$

is a homeomorphism and we can identify these two spaces. Therefore, the map

$$F_{m,n}(M;G) \longrightarrow F_{m,n}(M/G)$$
$$(x_1, \ldots, x_n) \mapsto (\bar{x}_1, \ldots, \bar{x}_n)$$

is a regular covering with discrete fiber G. The exact homotopy sequence of this covering implies

$$\pi_i(F_{m,n}(M;G)) \cong \pi_i(F_{m,n}(M/G)),$$

for $i \geq 2$.

From this and the Theorem 2 of [3] we get

Theorem 2. *We have*

$$\pi_i(F_{1,n}(M;G)) \cong \bigoplus_{k=0}^{n-1}(M \backslash Q_k)$$

for $i \geq 2$. Moreover, if the fiber bundle

$$\pi : F_{0,n}(M/G)) \longrightarrow M/G$$
$$(y_1, \ldots, y_n) \mapsto y_1$$

accepts a cross section then we have

$$\pi_i(F_{0,n}(M;G)) \cong \bigoplus_{k=0}^{n-1}(M \backslash Q_k),$$

for $i \geq 2$.

Now it is easy to see that the fiber bundle

$$F_{0,n}(\mathbb{R}^q \backslash \{0\}; \mathbb{Z}_2) \longrightarrow (\mathbb{R}^q \backslash \{0\})/(\mathbb{Z}_2)$$

accepts a cross section. So, according to Theorem 2 for $i \geq 2$ we have

$$\pi_i(F_B(\mathbb{R}^q, n)) \cong \bigoplus_{k=0}^{n-1} \pi_i((\mathbb{R}^q \backslash \{0\}) \backslash Q_k).$$

Moreover, $\{\mathbb{R}^q \backslash \{0\}\} \backslash Q_k$ is obviously homotopy equivalent to $\underbrace{\mathbb{S}^{q-1} \vee \cdots \vee \mathbb{S}^{q-1}}_{2k+1 \text{ times}}$. So, we have

Theorem 3. *We have*

$$\pi_i(F_B(\mathbb{R}^q,n)) \cong \bigoplus_{k=0}^{n-1} \pi_i(\underbrace{\mathbb{S}^{q-1}\vee\cdots\vee\mathbb{S}^{q-1}}_{2k+1 times})$$

for $i \geq 2$.

When $q \geq 3$ the space $F_B(\mathbb{R}^q,n)$ is 1-connected. The exact homotopy sequence of the universal covering

$$F_B(\mathbb{R}^q,n) \longrightarrow F_B(\mathbb{R}^q,n)/W_n,$$

with fiber W_n gives us

Corollary 4. *For* $q \geq 3$ *we have* $\pi_i(F_B(\mathbb{R}^q,n)/W_n) \cong W_n$.

Let consider the canonical inclusion $\mathbb{R}^q \overset{\subseteq}{\to} \mathbb{R}^{q+1}$. It induces the inclusion $F_B(\mathbb{R}^q,n) \overset{\subseteq}{\to} F_B(\mathbb{R}^{q+1},n)$. Put $F_B(\mathbb{R}^\infty,n) = \lim_{q\to\infty} F_B(\mathbb{R}^q,n)$. As a consequence of Theorem 3 and Corollary 4 we have

Corollary 5. $F_B(\mathbb{R}^\infty,n)$ *is an Eilenberg–Mac Lane space* $K(W_n,1)$.

2. It is well known that if a topological manifold M of dimension n is equipped with a cellular decomposition $\mathcal{C}_M$ then its fundamental group can be computed via the 2-skeleton $\mathcal{C}_M^{(2)}$. In an earlier work [1] we have suggested the dual method computing the group $\pi_1(M)$ via the so-called 2-codimensional skeleton of $\mathcal{C}_M$. Precisely, the group $\pi_1(M)$ can be computed via the n-cells, $(n-1)$-cells and $(n-2)$-cells of $\mathcal{C}_M$.

In order to compute the group $\pi_1(F_B(\mathbb{R}^q,n)/W_n)$ we introduce the notion of Nakamura decomposition of type B of $\times_n\mathbb{R}^q$ (see e.g. [2]). Suppose that $\mathbb{R}^q$ is ordered by lexicographic order. Then each point $a \in \times_n\mathbb{R}^q$ can always be written as $w.(a_1,\dots,a_n)$, where w is a certain element of the group W_n, $a_i \in \mathbb{R}^q$ satisfying $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Such a sequence $(a_1,\dots,a_n)$ gives rise to a sequence of integer $(r_1,r_2,\dots,r_n;r_{n+1})$ with $r_1 = q$, $0 \leq r_i \leq q$ for all i, where r_i is defined by

$$\begin{cases} a_i^j = a_{i+1}^j & \text{if } j \leq \rho_i \\ a_i^{\rho_i+1} > a_{i+1}^{\rho_i+1} & \text{if } \rho_i < q, \end{cases} \qquad (*)$$

and $\rho_i = q - r_i$.

Conversely, given a sequence of integer $(r_1,r_2,\dots,r_n;r_{n+1})$ with $r_1 = q$, $0 \leq r_i \leq q$ for all i, let denote by $\alpha = (r_1,r_2,\dots,r_n;r_{n+1})$ the subset of $\times_n\mathbb{R}^q$ consisting of all $(a_1,\dots,a_n)$ satisfying $(*)$. Put

$$\mathcal{C}(q,n) = \{w.\alpha \mid w \in W_n,\ \alpha = (r_1,r_2,\dots,r_n;r_{n+1}) \text{ with } r_1 = q,\ 0 \leq r_i \leq q\}.$$

It is easy to see that each $w.\alpha$ is homeomorphic to a disc of dimension $|w.\alpha| = \sum_{i=1}^{n+1} r_i - q$ and that $\mathcal{C}(q,n)$ is a cellular decomposition of $\times_n\mathbb{R}^q$. The space

$F_B(\mathbb{R}^q, n)$ is then a W_n-invariant sub cellular space of $\overset{n}{\times}\mathbb{R}^q$. It consists of all cells of the form $w.(r_1, r_2, \dots, r_n; r_{n+1})$ with $r_i \neq 0$ for all i. Apply the result of [1] to the space $F_B(\mathbb{R}^q, n)$ together with the above cellular decomposition we have

Theorem 6. (i) *The fundamental group $\pi_1(F_B(\mathbb{R}^2, n)/W_n)$ admits a presentation with generators $g_1, g_2, \dots, g_{n-1}, h$ and defining relations*

(a) $g_i.g_j.g_i^{-1}.g_j^{-1} = 1$ if $|i-j| \geq 2$

$g_i.h.g_i^{-1}.h^{-1} = 1$ if $n - i \geq 2$

(b) $g_i.g_{i+1}.g_i.g_{i+1}^{-1}.g_i^{-1}.g_{i+1}^{-1} = 1$

(c) $(g_{n-1}.h)^2.(h.g_{n-1})^{-2} = 1$.

(ii) *The fundamental group $\pi_1(F_B(\mathbb{R}^q, n)/W_n)$ with $q \geq 3$ admits a presentation with generators $g_1, g_2, \dots, g_{n-1}, h$ and defining relations* (a), (b), (c) *given in the part* (i) *and*

(d) $g_i^2 = 1$ *and* $h^2 = 1$.

Combining the second part of this theorem and the Corollary 5 we get

Corollary 7. *The Weyl group of type B, W_n admits a presentation with generators $g_1, g_2, \dots, g_{n-1}, h$ and defining relations* (a), (b), (c) *and* (d).

3. The relation between configuration spaces of type A and configuration spaces of type B is established by a family of maps $f_{h,k}$ defined below.

Denote by $\hat{\mathbb{R}}$ the set of positive real numbers. Then we have the configuration space

$$F_A(\hat{\mathbb{R}}^q, n) = \{(x_1, \dots, x_n) \mid x_i \in \hat{\mathbb{R}}^q,\ x_i \neq x_j \text{ if } i \neq j\}.$$

Obviously, $\hat{\mathbb{R}} \approx \mathbb{R}$ and we can identify $F_A(\hat{\mathbb{R}}^q, n)$ with $F_A(\mathbb{R}^q, n)$. Let consider the unit sphere $\mathbb{S}^k \subset \mathbb{R}^{k+1}$. Here an element $x \in \mathbb{R}^{k+1}$ is written in the form

$$\begin{pmatrix} x^k \\ x^{k-1} \\ \vdots \\ x^0 \end{pmatrix}$$

Identifying $F_A(\mathbb{R}^q, n)$ and $F_A(\hat{\mathbb{R}}^q, n)$ we have

Definition 8. *For each pair of integers $h > 0$, $k \geq 0$ we define a Σ_n-equivariant map*

$$\begin{aligned} \bar{f}_{h,k}:\ F_A(\mathbb{R}^h, n) \times (\mathbb{S}^k/\mathbb{Z}_2)^n &\longrightarrow F_B(\mathbb{R}^{h+k}, n)/(\mathbb{Z}_2)^n \\ (a_1, \dots, a_n) \times (\bar{x}_1, \dots, \bar{x}_n) &\mapsto [(b_1, \dots, b_n)] \end{aligned}$$

where $b_i \in \mathbb{R}^{h+k}$, $1 \leq i \leq n$ *is the vector*

$$\begin{pmatrix} x_i^k \\ x_i^{k-1} \\ \vdots \\ x_i^0 \\ a_i^1 \\ \vdots \\ a_i^h \end{pmatrix}$$

From the definition we have the following commutative diagram .

$$\begin{array}{ccc} F_A(\mathbb{R}^h,n)\times(\mathbb{S}^k/\mathbb{Z}_2)^n & \xrightarrow{\bar{f}_{h,k}} & F_B(\mathbb{R}^{h+k},n)/\mathbb{Z}_2^n \\ \downarrow i_A\times i'_A & & \downarrow i'_{h,k} \\ F_A(\mathbb{R}^{h+1},n)\times(\mathbb{S}^{k+1}/\mathbb{Z}_2)^n & \xrightarrow{\bar{f}_{h+1,k+1}} & F_B(\mathbb{R}^{h+1+k+1},n)/\mathbb{Z}_2^n \end{array}$$

Therefore we can define the Σ_n-equivariant map

$$\bar{f}_{\infty,\infty}: F_A(\mathbb{R}^\infty,n)\times(\mathbb{S}^\infty/\mathbb{Z}_2)^n \longrightarrow F_B(\mathbb{R}^\infty,n)/\mathbb{Z}_2^n.$$

By the Σ_n-equivariant property, the maps $\bar{f}_{h,k}$ induce maps

$$f_{h,k}: F_A(\mathbb{R}^h,n)\underset{\Sigma_n}{\times}(\mathbb{S}^k/\mathbb{Z}_2)^n \longrightarrow F_B(\mathbb{R}^{h+k},n)/W_n. \tag{1}$$

Similarly, the map $\bar{f}_{\infty,\infty}$ induces a map

$$f_{\infty,\infty}: F_A(\mathbb{R}^\infty,n)\underset{\Sigma_n}{\times}(\mathbb{S}^\infty/\mathbb{Z}_2)^n \longrightarrow F_B(\mathbb{R}^\infty,n)/W_n.$$

The group $\pi_1(F_B(\mathbb{R}^{h+k},n)/W_n)$ has been computed in Theorem 6. The fundamental group of the left hand side of (1) can also be computed in the same way. We can check that the maps $f_{h,k}$ send bijectively the generators and defining relations of $\pi_1(F_A(\mathbb{R}^h,n)\underset{\Sigma_n}{\times}(\mathbb{S}^k/\mathbb{Z}_2)^n)$ to those of $\pi_1(F_B(\mathbb{R}^{h+k},n)/W_n)$. Therefore we get

Theorem 9. *For* $h,k\geq 3$ *the maps* $f_{h,k}$ *induce isomorphism on fundamental groups*

$$(f_{h,k})_\#: \pi_1(F_A(\mathbb{R}^h,n)\times(\mathbb{S}^k/\mathbb{Z}_2)^n)\cong\pi_1(F_B(\mathbb{R}^{h+k},n)/W_n).$$

Moreover, $F_B(\mathbb{R}^\infty,n)$ is a $K(\pi,1)$ space. So, we have

Corollary 10. *The map* $f_{\infty,\infty}$ *is a homotopy equivalence.*

References

1. Nguyen Viet Dung, The fundamental groups of the spaces of regular orbits of affine Weyl groups, *Topology* **22** (1983) 425–435.
2. Nguyen Viet Dung, *Topology of configuration spaces of type B*, Ph. D. Dissertation, Institute of Mathematics, Hanoi, 1987.
3. E. Fadell and L. Neuwirth, Configuration spaces, *Math. Scand.* **10** (1962) 111–118.
4. Nguyen H. V. Hung, The mod 2 equivariant cohomology algebras of configuration spaces, *Acta Math. Vietnam.* **7** (1982) 95–100.
5. M. Yoshida, *Hypergeometric Functions, My Love*, Wiesbaden: Vieweg., 1997, 292 p.
6. G. Ziegler, Orbits configuration spaces, preprint, Berlin, 2000.