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On a Class of Locally Supersoluble Groups

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Abstract. This article identifies a class of locally supersoluble groups, called HCA-groups. This class contains the cyclic-by-abelian groups, as well as the nilpotent groups. The main result states that the product of a normal HCA-subgroup and a subnormal locally supersoluble subgroup is always locally supersoluble.

1. Introduction

It is a well-known fact that the product of two normal supersoluble subgroups need not be supersoluble. In [1], Beidleman and Smith showed that if $G = HK$, where H is a normal finitely generated nilpotent subgroup of G and K is a subnormal supersoluble subgroup of G, then G is supersoluble. In view of this result, one might be inclined to ask: If $G = HK$, where H is a normal metacyclic subgroup of G and K is a normal supersoluble subgroup of G, does it follow that G is supersoluble? This paper answers this question in the affirmative. More generally, we shall see that if $G = HK$, where H is a normal cyclic-by-abelian subgroup of G and K is a normal locally supersoluble subgroup of G, then G is locally supersoluble. Furthermore, we will show that the aforementioned property holds true for a wider class of locally supersoluble groups called HCA-groups.

Definition 1.1. *A group G is said to be an HCA-group if G contains a normal nilpotent subgroup N such that G/N' is cyclic-by-abelian.*

The main result of this article is the following:

Theorem 3.4. *If $G = HK$, where H is a normal HCA-subgroup of G and K is a subnormal locally supersoluble subgroup of G, then G is locally supersoluble.*

In the next section, we shall briefly examine some basic properties of HCA-groups. The Fitting subgroup and the derived subgroup of a group G will be denoted by $\mathrm{Fit}(G)$ and G', respectively.

2. HCA-groups

Theorem 2.1.
(i) *Nilpotent groups and cyclic-by-abelian groups are HCA-groups.*
(ii) *HCA-groups are nilpotent-by-abelian and nilpotent-by-finite.*
(iii) *HCA-groups are hypercyclic and hence locally supersoluble.*

Proof.
(i) is obvious.
(ii) To prove that an HCA-group G is nilpotent-by-abelian, let $N \lhd G$ with N nilpotent and G/N' cyclic-by-abelian. Then G'/N' is cyclic. By Fitting's Theorem ([3, Theorem 1.6.1] or [5, Theorem 2.18] or [6, 5.2.8]), $G'N/N' = (G'/N')(N/N')$ is nilpotent. It follows from a well-known result of Hall in [2] (or [5, Theorem 2.27] or [6, 5.2.10]) that $G'N$, and consequently G', is nilpotent.

Now we show that an HCA-group is nilpotent-by-finite. Suppose G has a normal nilpotent subgroup N where G/N' is cyclic-by-abelian. Then G'/N' is cyclic. Put $C = C_G(G'/N')$. Then $N \leq C$ and G/C is finite. Note that N_{ab} is a poly-trivial C-module. Hence by a result of Robinson in [4] N is contained in some (finite) term of the upper central series of C and so C is nilpotent.
(iii) Let $N \lhd G$ with N nilpotent and G/N' cyclic-by-abelian. Then N/N' has an ascending G-invariant series with cyclic factors. By results in [4] G is hypercyclic. The proof is complete. ∎

Corollary 2.2. *If G is an HCA-group,* $\mathrm{Fit}(G)$ *is nilpotent.*
(This follows from (ii).)

Remark. The class of finite HCA-groups lies strictly between nilpotent groups and supersoluble groups. The following example gives an HCA-group that is neither nilpotent nor cyclic-by-abelian.

Let G be the subgroup of $GL(3,3)$ which is generated by the matrices

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The group $N = \langle a, b \rangle$ has order 27 and is a normal nilpotent subgroup of G. Since $G' \simeq C_3 \times C_3$ and $N' \simeq C_3$, $(G/N')'$ is cyclic of order 3. Therefore G/N' is cyclic-by-abelian and G is an HCA-group. The subgroup $\langle x \rangle$ is a Sylow 2-subgroup of G that is not normal in G, hence G is not nilpotent.

We now illustrate a supersoluble group that is not an HCA-group. Let A be a non-cyclic abelian group of odd order, let $\langle t \rangle$ have order 2 and put $G = \langle t \rangle \times A$

where $a^t = a^{-1}, (a \in A)$. Then G is supersoluble. Suppose that N is a normal nilpotent subgroup of G and G/N' is cyclic-by-abelian. Then $N \leq \mathrm{Fit}(G) = A$ and so G is cyclic-by-abelian. This is wrong since $G' = A$.

Theorem 2.3. *Let N be a normal nilpotent subgroup of a group G. If G/N' is an HCA-group, then G is an HCA-group.*

Proof. G/N' has a normal nilpotent subgroup M/N' such that $(G/N')/(M/N')'$ is cyclic-by-abelian. By Fitting's Theorem $MN/N' = (M/N')(N/N')$ is nilpotent. It now follows from a result of Hall that MN is nilpotent. Since

$$\frac{G}{(MN)'} \simeq \frac{G/N'}{(MN)'/N'} \simeq \frac{(G/N')/(M/N')'}{((MN)'/N')/(M/N')'}$$

is cyclic-by-abelian, G is an HCA-group. The proof is complete. ■

Since the class of HCA-groups is closed with respect to the formation of quotients, we see that a nilpotent-by-abelian group G is an HCA-group if and only if G/G'' is an HCA-group.

Lemma 2.4. *Suppose G is a cyclic-by-abelian group and G' is not finite. Then either G is nilpotent or $G/\mathrm{Fit}(G)$ is cyclic of order 2.*

Proof. Assume G is not nilpotent. Because G' is infinite cyclic $|G : C_G(G')| \leq 2$. If $C_G(G') = G$, then $[G, G'] = 1$ and G is nilpotent. Thus $|G : C_G(G')| = 2$. Since $C_G(G') \leq \mathrm{Fit}(G)$ and since $\mathrm{Fit}(G)$ is nilpotent, we must have $C_G(G') = \mathrm{Fit}(G)$. This completes the proof. ■

For example, let G be the infinite dihedral group. Since G' is infinite cyclic and G is not nilpotent, we have $G/\mathrm{Fit}(G)$ is cyclic of order 2.

Theorem 2.5. *If G is an HCA-group then either $G'/(\mathrm{Fit}(G))'$ is finite or $G/\mathrm{Fit}(G)$ is cyclic of order 2.*

Proof. Write $F = \mathrm{Fit}(G)$. By hypothesis G contains a normal nilpotent subgroup N such that G/N', and consequently G/F', is cyclic-by-abelian. Assume $G'/F' = (G/F')'$ is not finite. By Lemma 2.4, either G/F' is nilpotent or $(G/F')/\mathrm{Fit}(G/F')$ is cyclic of order 2. If G/F' were nilpotent then G would be nilpotent. Consequently G'/F' would be the trivial group, a contradiction. Therefore $(G/F')/\mathrm{Fit}(G/F')$ is cyclic of order 2. Since

$$G/F \simeq (G/F')/(F/F') = (G/F')/\mathrm{Fit}(G/F')$$

the result now follows. The proof is complete.

3. The Main Result

The primary aim of this section is to establish Theorem 3.4. We start by citing a result of Beidleman and Smith in [1] which states that if $G = HK$, where H is a normal finitely generated nilpotent subgroup of G and K is a subnormal supersoluble subgroup of G, then G is supersoluble.

Lemma 3.1. *If $G = HK$, where H is a normal locally nilpotent subgroup of G and K is a normal locally supersoluble subgroup of G, then G is locally supersoluble.*

Proof. Take a finitely generated subgroup of G, say $\langle h_1k_1, \ldots, h_mk_m\rangle$, $h_i \in H$, $k_i \in K$, which needs to be shown as being supersoluble. Consider $X = \langle h_1, \ldots, h_m\rangle$ and $Y = \langle k_1, \ldots, k_m\rangle$, and set $Z = \langle X, Y\rangle$. Clearly, it suffices to show Z is supersoluble. By imitating the calculations in the proof of the celebrated Hirsch-Plotkin Theorem given in [6, 12.1.2], we see that $Z = \langle X, Y\rangle = X^YY^X$ is the product of a normal finitely generated nilpotent subgroup X^Y and a normal supersoluble subgroup Y^X. It now follows from the above mentioned result of Beidleman and Smith that Z is supersoluble and this completes the proof. ■

Corollary 3.2. *If $G = HK$, where H is a normal locally nilpotent subgroup of G and K is a subnormal locally supersoluble subgroup of G, then G is locally supersoluble.*

Proof. Suppose the subnormal defect of K in G is equal to r and the proof proceeds by induction on r. The result is clear when $r = 1$. Therefore suppose that $r \geq 2$ and assume the usual hypothesis of induction. We see that $K^G = (H \cap K^G)K$ is the product of a normal locally nilpotent subgroup and a subnormal locally supersoluble subgroup. Since the subnormal defect of K in K^G is $r - 1$, the inductive hypothesis implies that K^G is locally supersoluble. Therefore, $G = HK^G$ is locally supersoluble by the preceding lemma; and the proof is complete. ■

Lemma 3.3. *If $G = HK$, where H is a normal cyclic-by-abelian subgroup of G and K is a subnormal locally supersoluble subgroup of G, then G is locally supersoluble.*

Proof. By Corollary 3.2, $H'K$ is locally supersoluble. Now, $G/H' = (H/H')(H'K/H')$ is locally supersoluble and H' is cyclic. Hence G is locally supersoluble. This completes the proof. ■

Now we can prove the main result of this paper.

Theorem 3.4. *If $G = HK$, where H is a normal HCA-subgroup of G and K is a subnormal locally supersoluble subgroup of G, then G is locally supersoluble.*

Proof. Let $N \lhd H$ where N is nilpotent and H/N' is cyclic-by-abelian. Then

$N \leq F = \text{Fit}(H) \lhd G$. Hence $G/F' = (H/F')(KF'/F')$ is locally supersoluble by Lemma 3.3. Since F is nilpotent, it follows from a result of Robinson in [4] (or [5, page 57]) that G is locally supersoluble. The proof is complete. ■

Note that, in any group G, the main result implies that the product of all normal HCA-subgroups of G is a characteristic locally supersoluble subgroup (which contains the Fitting subgroup of G). One can see this by using transfinite induction and the fact that the union of an ascending chain of locally supersoluble groups is locally supersoluble. Finally, it is worth noting that the product of a subnormal metacyclic subgroup and a normal supersoluble subgroup need not be supersoluble. For example, consider the nonsupersoluble group G of order 72 which is described in [6, Exercise 6 p. 152]. One can verify that G possesses a normal supersoluble subgroup K of order 36 and a subnormal subgroup H isomorphic to the symmetric group of order 6 such that K does not contain H; and this implies $G = HK$.

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