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Fractal Langevin Equation

Tran Hung Thao and Tran Trong Nguyen

Institute of Mathematics, P.O. Box 631, Bo Ho, Hanoi, Vietnam

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Abstract. A generalization of the classical Langevin equation is introduced for describing a linear dynamical system of particles driven by a fractional process. An approximate fractal equation are considered and an approximate solution is found. The L^2-convergence to the exact solution is investigated and an estimation of error is studied as well.

1. Introduction

The classical Langevin equation is one of best-known equations in physics describing the motion of a linear dynamical system of particles perturbed by some white noise. It is of the form

$$dX_t = -bX_t dt + \sigma dW_t, \quad 0 \le t \le T, \tag{1.1}$$

where b and σ are some constants, W_t is a Brownian motion. The solution of (1.1) is known as an Ornstein-Uhlenbeck process:

$$X_t = X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)} dW_s, \quad 0 \le t \le T. \tag{1.2}$$

One of assumptions under which the equation (1.1) was established is that the position of a particle at a moment depends only on its previous position and not on that of long time before. In fact, the solution (1.2) is of Markov property that expresses a loss-memory evolution of the system [2, 4].

However, under certain situations, the state of the system can influence on its long-range behavior. It is the case of motions in a fractal medium on which some absolutely continuous limiting distribution are supported. Here the system can not be driven by an ordinary Brownian motion W_t. And one thinks of perturbations expressed by a fractional Brownian motion B_t^H that exhibits a long term dependence between system states.

Recall that a fractional Brownian motion (fBm) of index $H \in (0,1)$ is a centered Gaussian process $(W_t^H,\ t \geq 0)$ having the covariance given by

$$R(s,t) = k_H.\frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}), \tag{1.3}$$

where

$$k_H = \frac{\Gamma(2-2H)\cos\pi H}{\pi H(1-2H)}, \quad \text{(refer to [3])}.$$

In the case where $H = 1/2$, W_t^H is a standard Brownian motion. And one can propose the following model

$$dX_t = -bX_t dt + \sigma dB_t^H \tag{1.4}$$

as a generalized Langevin equation for the fractal case. However, since W_t^H is neither a Markov process nor a semimartingale for $H \neq 1/2$, the usual stochastic calculus can not be applied to solve (1.4). On the other hand, it is known that for $H < 1/2$ the fBm B_t^H has a representation as follows:

$$B_t^H = \frac{1}{\Gamma(1-\alpha)}\Big[Z_t + \int_0^t (t-s)^{-\alpha} dW_s\Big], \tag{1.5}$$

where $(W_s,\ s \geq 0)$ is a standard Brownian motion, $\alpha = 1/2 - H \in (0, 1/2)$, and $Z_t = \int_{-\infty}^0 [(t-s)^{-\alpha} - (-s)^{-\alpha}] dW_s$. Since Z_t is of absolutely continuous trajectories, the long-range property of B_t^H is essentially expressed by the term $\int_0^t (t-s)^{-\alpha} dW_t$ that will be denoted by B_t from now on.

So in this paper we consider the following model

$$dX_t = -bX_t dt + \sigma dB_t, \quad 0 \leq t \leq T, \tag{1.6}$$

where

$$B_t = \int_0^t (t-s)^{-\alpha} dW_s, \quad 0 < \alpha < \frac{1}{2}, \tag{1.7}$$

as a fractal Langevin equation. A solution of (1.6) is defined as a stochastic process satisfying the following relation

$$X_t = -b\int_0^t X_s ds + \sigma B_t. \tag{1.6'}$$

And the formal writing dB_t can only be considered and calculated as a differential in this context. We will give an approximation solution of (1.6) by substituting the fractional process B_t by a family of process (B_t^ε) defined below and converging to B_t as $\varepsilon \to 0$.

2. Approximate Perturbations

For every $\varepsilon > 0$ we define

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{-\alpha} dW_s, \quad 0 < \alpha < \frac{1}{2}, \tag{2.1}$$

then by the Itô formula we can see that

$$dB_t^\varepsilon = \Big(-\int_0^t \alpha(t-s+\varepsilon)^{-\alpha-1}dW_s\Big)dt + \varepsilon^{-\alpha}dW_t. \tag{2.2}$$

So $(B_t^\varepsilon,\ t \geq 0)$ is a semimartingale.

Theorem 1. *B_t^ε converges to B_t in $L^2(\Omega)$ when ε tends to 0. This convergence is uniform with respect to $t \in [0,T]$.*

Proof. From the following elementary formula for every function $f \in C^1(\mathbb{R})$

$$f(u+h) - f(u) = hf'(u+\theta h), \quad 0 \leq \theta \leq 1,$$

we have

$$\begin{aligned}|(t-s+\varepsilon)^{-\alpha} - (t-s)^{-\alpha}| &\leq \alpha\varepsilon \sup_{0\leq\theta\leq 1}|(t-s+\theta\varepsilon)^{-\alpha-1}| \\ &= \alpha\varepsilon(t-s)^{-\alpha-1}, \quad 0 < s < t. \end{aligned} \tag{2.3}$$

Taking into account of the isometry in the Itô theory of stochastic integration we can see that

$$\begin{aligned} E|B_t^\varepsilon - B_t|^2 = \ & E\Big|\int_0^t \big[(t-s+\varepsilon)^{-\alpha} - (t-s)^{-\alpha}\big]dW_s\Big|^2 \\ = & \int_0^t \big|(t-s+\varepsilon)^{-\alpha} - (t-s)^{-\alpha}\big|^2 ds \\ = & \int_0^{t-\varepsilon} \big|(t-s+\varepsilon)^{-\alpha} - (t-s)^{-\alpha}\big|^2 ds \\ & + \int_{t-\varepsilon}^t \big|(t-s+\varepsilon)^{-\alpha} - (t-s)^{-\alpha}\big|^2 ds. \end{aligned} \tag{2.4}$$

The evaluation (2.3) applied to the first term of the right hand side of (2.4) will give us:

$$\int_0^{t-\varepsilon} \big|(t-s+\varepsilon)^{-\alpha} - (t-s)^{-\alpha}\big|^2 ds \leq \alpha^2\varepsilon^2 \int_0^{t-\varepsilon} (t-s)^{-2\alpha-2}ds. \tag{2.5}$$

For the second term of the right hand side of (2.4) we have

$$\int_{t-\varepsilon}^t \big|(t-s+\varepsilon)^{-\alpha} - (t-s)^{-\alpha}\big|^2 ds \leq \int_{t-\varepsilon}^t (t-s)^{-2\alpha}ds. \tag{2.6}$$

It follows from (2.4), (2.5), (2.6) that

$$\begin{aligned} \|B_t^\varepsilon - B_t\|^2 \leq \ & \alpha^2\varepsilon^2 \int_0^t (t-s)^{-2\alpha-2}ds + \int_{t-\varepsilon}^t (t-s)^{-2\alpha}ds \\ & \leq C(\alpha)\varepsilon^{1-2\alpha}, \end{aligned} \tag{2.7}$$

where $\|\cdot\|$ denotes the norm in $L^2(\Omega)$, and the coefficient $C(\alpha)$ depends only on α.

Therefore

$$\sup_{0\le t\le T}\|B_t^\varepsilon - B_t\| \le K(\alpha)\varepsilon^{\frac{1}{2}-\alpha} \to 0 \text{ as } \varepsilon\to 0,$$

where $0<\alpha<\frac{1}{2}$ and $K(\alpha)=\sqrt{C(\alpha)}$.

So B_t^ε converges to B_t in $L^2(\Omega)$ uniformly with respect to $t\in[0,T]$. ■

3. Approximate Fractal Langevin Equations

Now instead of (1.6) we consider the following equation

$$dX_t = -bX_t dt + \sigma dB_t^\varepsilon. \tag{3.1}$$

Combining (2.2) and (3.1) yields

$$dX_t = -[bX_t + \sigma\varphi(t)]dt + \varepsilon^{-\alpha}\sigma dW_t, \tag{3.2}$$

where

$$\varphi(t) = \int_0^t \alpha(t-s+\varepsilon)^{-\alpha-1}dW_s\,. \tag{3.3}$$

And we try to find a solution for (3.2), (3.3).

The equation (3.2) can be splitted into two equations:

$$dX_1(t) = -bX_1(t)dt + \varepsilon^{-\alpha}\sigma dW_t, \tag{3.4}$$

and

$$dX_2(t) = -bX_2(t)dt - \sigma\varphi(t)dt. \tag{3.5}$$

The solution of (3.2) will be $X_t = X_1(t)+X_2(t)$. We see that (3.4) is a classical Langevin equation whose solution is an Ornstein-Uhlenbeck process:

$$X_1(t) = X_1^0 e^{-bt} + \sigma\varepsilon^{-\alpha}\int_0^t e^{-b(t-s)}dW_s, \tag{3.6}$$

where X_1^0 is the initial value of $X_1(t) : X_1^0 = X_1(0)$ that is supposed to be a random variable independent of $(W_t,\ 0\le t\le T)$.

The equation (3.5) is an ordinary differential equation for every fixed ω and its solution is:

$$X_2(t) = X_2^0 e^{-bt} - \sigma\int_0^t e^{-b(t-s)}\varphi(s)ds, \tag{3.7}$$

where $X_2^0 = X_2(0)$ independent of $(W_t,\ 0\le t\le T)$, and $X_1^0+X_2^0$ is the initial value of $X_t : X_0 = X_1^0 + X_2^0$.

The process $\varphi(t)$ appeared in (3.7) can be simulated as follows. We have

$$\begin{aligned}\varphi(t) &= \alpha\int_0^t (t-s+\varepsilon)^{-\alpha-1}dW_s\\ &\approx \alpha\sum_{k=0}^{N-1}(t-k\frac{t}{N}+\varepsilon)^{-\alpha-1}[W((k+1)\frac{t}{N}) - W(k\frac{t}{N})],\end{aligned} \tag{3.8}$$

where N is the number of equal subintervals in a partition of $[0,t]$. The law of the family $\{W((k+1)\frac{t}{N}) - W(k\frac{t}{N}),\ 0 \le k \le N-1\}$ is identical to that of a family of independent centered gaussian variables of variance $\frac{t}{N}$. By a simulation we can replace $W((k+1)\frac{t}{N}) - W(k\frac{t}{N})$ by $g_k\sqrt{\frac{t}{N}}$ where (g_k) is a sequence of independent centered gaussian variables. It follows that

$$\begin{aligned}\varphi(t) &= \alpha\int_0^t (t-s+\varepsilon)^{-\alpha-1}dW_s \\ &\approx \alpha\sqrt{\frac{t}{N}}\sum_{k=0}^{N-1}(t-k\frac{t}{N}+\varepsilon)^{-\alpha-1}g_k.\end{aligned} \tag{3.9}$$

Thus we have

Theorem 2. *The solution of the equation*

$$dX_t = -bX_tdt + \sigma dB_t^\varepsilon$$

can be expressed by

$$X_t = X_0e^{-bt} + \sigma\varepsilon^{-\alpha}\int_0^t e^{-b(t-s)}dW_s - \sigma\int_0^t e^{-b(t-s)}\varphi(s)ds, \tag{3.10}$$

where $\varphi(t)$ can be simulated by

$$\varphi(t) \approx \alpha\sqrt{\frac{t}{N}}\sum_{k=0}^{N-1}(t-k\frac{t}{N}+\varepsilon)^{-\alpha-1}g_k,$$

with sufficiently large N and a sequence of independent centered gaussian variables (g_k).

Remark

(i) We see from the proof of Theorem 2 that $X_t = X_1(t) + X_2(t)$, where $X_1(t)$ is a loss memory process while $X_2(t)$ exhibits a long memory motion.

(ii) *Expectation of the solution of* (3.1). Denote the mathematical expectation and the variance of X_0 respectively by a and C^2, we see that:

$$E[X_0e^{-bt}] = e^{-bt}E[X_0] = ae^{-bt},$$

$$E\Big[\int_0^t e^{-b(t-s)}dW_s\Big] = 0,$$

$$E[\varphi(t)] = E\Big[\int_0^t \alpha(t-s+\varepsilon)^{-\alpha-1}dW_s\Big] = 0.$$

Then

$$E(X_t) = ae^{-bt}. \tag{3.11}$$

(iii) *Variance of the solution of* (3.1). In the following calculation, the isometry in Itô's integration is used.

Noticing that X_0 is independent of (W_t) and that (3.10) can be rewritten as

$$X_t = X_0 e^{-bt} + h\int_0^t e^{bs}dW_s + k\int_0^t e^{bs}\varphi(s)ds, \tag{3.12}$$

where $h = \varepsilon^{-\alpha}\sigma e^{-bt}$ and $k = -\sigma e^{-bt}$, we obtain

$$\begin{aligned} D[X_t] &= D\big[X_0 e^{-bt}\big] + D\Big[h\int_0^t e^{bs}dW_s + k\int_0^t e^{bs}\varphi(s)ds\Big] \\ &= C^2 e^{-2bt} + D[hA + kB], \end{aligned}$$

where $A =: \int_0^t e^{bs}dW_s$ and $B =: \int_0^t e^{bs}\varphi(s)ds$.
Since $E[A] = 0$ and $E[B] = 0$ we have

$$\begin{aligned} D[hA + kB] &= E\big[(hA + kB)^2\big] \\ &= h^2E[A^2] + k^2E[B^2] + 2hkE[AB]. \end{aligned} \tag{3.13}$$

Now we see that

$$E[A^2] = E\Big[\Big(\int_0^t e^{bs}dW_s\Big)^2\Big] = \int_0^t e^{2bs}ds = \frac{1}{2b}(e^{2bt} - 1), \tag{3.14}$$

$$\begin{aligned} E[B^2] &= E\Big[\Big(\int_0^t e^{bs}\Big(\int_0^s \alpha(t-u+\varepsilon)^{-\alpha-1}dW_u\Big)ds\Big)^2\Big] \\ &= E\Big[\Big(\int_0^t (\alpha(t-u+\varepsilon)^{-\alpha-1}\int_u^t e^{bs}ds)dW_u\Big)^2\Big] \\ &= \int_0^t \alpha^2(t-u+\varepsilon)^{-2\alpha-2}\Big(\int_u^t e^{bs}ds\Big)^2du \\ &= \int_0^t \frac{\alpha^2}{b^2}(e^{bt} - e^{bu})^2(t-u+\varepsilon)^{-2\alpha-2}du, \end{aligned}$$

or

$$E[B^2] = \frac{\alpha^2}{b^2}\int_0^t (e^{bt} - e^{bu})^2(t-u+\varepsilon)^{-2\alpha-2}du, \tag{3.15}$$

$$\begin{aligned} AB &= \Big(\int_0^t e^{bs}dW_s\Big)\Big(\int_0^t e^{bs}\Big(\int_0^s \alpha(t-u+\varepsilon)^{-\alpha-1}dW_u\Big)ds\Big) \\ &= \Big(\int_0^t e^{bu}dW_u\Big)\int_0^t \frac{\alpha}{b}(e^{bt} - e^{bu})(t-u+\varepsilon)^{-\alpha-1}dW_u, \end{aligned}$$

$$E[AB] = \frac{\alpha}{b}\int_0^t e^{bu}(e^{bt} - e^{bu})(t-u+\varepsilon)^{-\alpha-1}du. \tag{3.16}$$

Combining relations (3.13), (3.14), (3.15) and (3.16) yields:

$$D[hA+kB]^2 = h^2.\frac{1}{2b}(e^{2bt}-1) + k^2.\frac{\alpha^2}{b^2}\int_0^t (e^{bt}-e^{bu})^2(t-u+\varepsilon)^{-2\alpha-2}du$$
$$+ 2hk\frac{\alpha}{b}\int_0^t e^{bu}(e^{bt}-e^{bu})(t-u+\varepsilon)^{-\alpha-1}du$$
$$= \frac{1}{2b}\varepsilon^{-2\alpha}\sigma^2 e^{-2bt}(e^{2bt}-1)$$
$$+ \sigma^2.e^{-2bt}.\frac{\alpha^2}{b^2}\int_0^t (e^{bt}-e^{bu})^2(t-u+\varepsilon)^{-2\alpha-2}du$$
$$- 2\varepsilon^{-\alpha}\sigma^2 e^{-2bt}\frac{\alpha}{b}\int_0^t e^{bu}(e^{bt}-e^{bu})(t-u+\varepsilon)^{-\alpha-1}du,$$

or

$$D[hA+kB] = \sigma^2 e^{-2bt}\Bigg[\frac{1}{2b}\varepsilon^{-2\alpha}(e^{2bt}-1) + \frac{\alpha^2}{b^2}\int_0^t (e^{bt}-e^{bu})^2(t-u+\varepsilon)^{-2\alpha-2}du$$
$$- \frac{2\alpha\varepsilon^{-\alpha}}{b}\int_0^t e^{bu}(e^{bt}-e^{bu})(t-u+\varepsilon)^{-\alpha-1}du\Bigg]. \tag{3.17}$$

Finally we find that

$$D[X_t] =$$
$$\sigma^2 e^{-2bt}\Bigg[\frac{C^2}{\sigma^2} + \frac{1}{2b}\varepsilon^{-2\alpha}(e^{2bt}-1) + \frac{\alpha^2}{b^2}\int_0^t (e^{bt}-e^{bu})^2(t-u+\varepsilon)^{-2\alpha-2}du$$
$$- \frac{2\alpha\varepsilon^{-\alpha}}{b}\int_0^t e^{bu}(e^{bt}-e^{bu})(t-u+\varepsilon)^{-\alpha-1}du\Bigg]. \tag{3.18}$$

4. Convergence

A natural question should arise: Is it true that the solution of (3.1) can be considered as an approximate solution of (1.6)?

Suppose that X_t and X_t^ε are solutions of (1.6) and (3.1), respectively:

$$dX_t = -bX_t dt + \sigma dB_t, \quad 0 \le t \le T, \tag{4.1}$$

$$dX_t^\varepsilon = -bX_t^\varepsilon dt + \sigma dB_t^\varepsilon, \quad 0 \le t \le T. \tag{4.2}$$

Now the convergence of X_t^ε to X_t as $\varepsilon \to 0$ can be shown as below

Theorem 3. *X_t^ε converges to X_t in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$.*

Proof. We have

$$X_t - X_t^\varepsilon = -b\int_0^t (X_s - X_s^\varepsilon)ds + \sigma(B_t - B_t^\varepsilon), \tag{4.3}$$

then

$$\|X_t - X_t^\varepsilon\| \le \|b\int_0^t (X_s - X_s^\varepsilon)ds\| + \sigma\|B_t - B_t^\varepsilon\|$$

or

$$\|X_t - X_t^\varepsilon\| \le b\int_0^t \|X_s - X_s^\varepsilon\|ds + \sigma K(\alpha)\varepsilon^{\frac{1}{2}-\alpha}, \ 0 \le t \le T, \tag{4.4}$$

where $\|\cdot\|$ denotes the norm in $L^2(\Omega)$.

A standard application of Gronwall's lemma starting from (4.4) will give us:

$$\|X_t - X_t^\varepsilon\| \le \sigma K(\alpha)\varepsilon^{\frac{1}{2}-\alpha}e^{bt}. \tag{4.5}$$

It follows that

$$\sup_{0\le t\le T} \|X_t - X_t^\varepsilon\| \le \sigma K(\alpha)\varepsilon^{\frac{1}{2}-\alpha}e^{bT}, \quad \text{as} \quad \varepsilon \to 0, \tag{4.6}$$

in assuming $b > 0$ without loss of generality.

So $X_t^\varepsilon \to X_t$ in $L^2(\Omega)$ uniformly with respect to t. ∎

Remark. The inequality in (4.6) expresses also an estimation for square mean error between X_t^ε and X_t.

5. Conclusion

As referred in the Introduction, the fractal Langevin equation describes more precisely the motion of particles in a fractal medium. And we have given an approximate solution of this equation with an arbitrary exactitude. This is an attempt to study long memory systems that represent various phenomena in the natural world. Also we have overcome difficulties frequently met in considering a system perturbed by a fractional Brownian motion without invoking hard tools of mathematics such as Malliavin Calculus that is not easy for numerics (see [1, 3]).

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