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On the Noetherian Dimension of Artinian Modules*

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Abstract. Some new results of Noetherian dimension of Artinian modules are given and several properties of Noetherian dimension of local cohomology modules of finitely generated modules are shown. After giving an example of an Artinian module A which has Krull dimension strictly larger than its Noetherian dimension, we present some sufficient conditions on Artinian modules A such that the both notions of dimension of A are the same.

1. Introduction

The concept of Krull dimension for Artinian modules was introduced by Roberts [12]. Kirby [7] changed the terminology of Roberts and referred to Noetherian dimension (N-dim) to avoid any confusion with Krull dimension defined for finitely generated modules. In this note we use the terminology of Kirby [7]. Many properties of Noetherian dimension of Artinian modules have been given in [3, 7, 12, 17]. The purpose of this note is to study Noetherian dimension of Artinian modules. Expecially, we prove some results on Noetherian dimension of local cohomology modules when these modules are Artinian.

This paper is divided into 4 sections. In Sec. 2 we will give some preliminary results on Noetherian dimension of Artinian modules. Let R be a commutative Noetherian ring, A an Artinian R-module and M a finitely generated R-module. Recall that the Krull dimension of A, denoted by $\dim_R A$, is the Krull dimension of the Noetherian ring $R/\operatorname{Ann}_R A$. We prove in Proposition 2.5 that $\text{N-dim}_R A \leq \dim_R A$ and the equality holds if R is a complete local ring (Corollary 2.6). Note that if $(R, \mathfrak{m})$ is a local ring then the local cohomology modules

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$H^i_{\mathfrak{m}}(M)$ and $H^{\dim_R M}_{\alpha}(M)$ are Artinian for any integer i and any ideal α of R. In Sec. 3 we give some results of Noetherian dimension of local cohomology modules. We prove (see Theorem 3.1) that, for an integer t and an ideal α of R, if $H^i_{\alpha}(M)$ is Artinian for all $i = 1, \dots, t$ then $\text{N-dim}_R\big(H^i_{\alpha}(M)\big) \le i$ for all $i =, 1, \dots, t$. An immediate consequence of this theorem is $\text{N-dim}_R\big(H^i_{\mathfrak{m}}(M)\big) \le i$ for all $i = 0, 1, \dots, \dim_R M$ (Corollary 3.2). We also give a necessary condition for the Matlis dual of the local cohomology module to be a finitely generated R-module (Proposition 3.3). For an ideal α of R, we show in Theorem 3.5 that if the Artinian module $H^{\dim_R M}_{\alpha}(M)$ is non-zero then $\text{N-dim}_R\big(H^{\dim_R M}_{\alpha}(M)\big) = \dim_R M$. In particular, $\text{N-dim}_R\big(H^{\dim_R M}_{\mathfrak{m}}(M)\big) = \dim_R M$ (Corollary 3.6). We present in Sec. 4 some relations between $\dim_R A$ and $\text{N-dim}_R A$. Note that there are Artinian modules A over a Noetherian local ring R for which $\text{N-dim}_R A < \dim_R A$ (Example 4.1). The notion "magnitude" of modules (mag) was defined by Yassemi [19]. It should be mentioned that $\text{mag}_R A = \dim_R A$. Therefore, Example 4.1 claims that Theorem 2.10 of [19] is false. So we give a sufficient condition for Artinian modules A to have $\text{N-dim}_R A = \dim_R A$ (Proposition 4.6).

2. Preliminaries

Throughout this paper we assume that R is a commutative Noetherian ring and A is an Artinian R-module.

We recall first notion of N-dim by using the terminology of Kirby [7].

Definition 2.1. *The Noetherian dimension of A, denoted by $\text{N-dim}_R A$, is defined inductively as follows: when $A = 0$, put $\text{N-dim}_R A = -1$. Then by induction, for an integer $d \ge 0$, we put $\text{N-dim}_R A = d$ if $\text{N-dim}_R A < d$ is false and for every ascending sequence $A_0 \subseteq A_1 \subseteq \dots$ of submodules of A, there exists n_0 such that $\text{N-dim}_R(A_n/A_{n+1}) < d$ for all $n > n_0$. Therefore $\text{N-dim}_R A = 0$ if and only if A is a non-zero Noetherian module.*

From Definition 2.1 we obtain the following lemma (see [7, 2.3]).

Lemma 2.2. *Let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be an exact sequence of Artinian R-modules. Then we have*

$$\text{N-dim}_R A = \max\{\text{N-dim}_R A', \text{N-dim}_R A''\}.$$

Remark 1. Suppose that $(R, \mathfrak{m})$ is local and $A \ne 0$.

(i) Kirby [5] has shown that $\ell_R(0 : \mathfrak{m}^n)_A$ is a polynomial with rational coefficients when n large enough ($n \gg 0$). Roberts [12] has then proved that

$$\begin{aligned}\text{N-dim}_R A &= \deg(\ell_R(0 : \mathfrak{m}^n)_A)\\ &= \inf\{t \ge 0 : \exists x_1, \dots, x_t \in \mathfrak{m} : \ell_R(0 : (x_1, \dots, x_t)R)_A < \infty\}.\end{aligned}$$

Then a system $\{x_1, \dots, x_d\}$ of ($d = \text{N-dim}_R A$) elements in $\mathfrak{m}$ is called a *system of parameters* of A if $\ell_R(0 : (x_1, \dots, x_d)R)_A < \infty$.

(ii) Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Then A has a natural structure as an $\widehat{R}$-module as follows (see [15, 1.11, 1.12]): let $(x_n) \in \widehat{R}$, where $x_n \in R$, and let

$u \in A$. Then we get $u.\mathfrak{m}^n = 0$ for $n \gg 0$. Therefore $x_n.u$ is constant for $n \gg 0$. So we defined $(x_n).u = x_n.u$ for $n \gg 0$. With this structure, a subset of A is an R-submodule if and only if it is an $\widehat{R}$-submodule. Therefore we have

$$\text{N-dim}_R A = \text{N-dim}_{\widehat{R}} A.$$

The following lemma, which is often used in this paper, is derived from Remark 1, (i).

Lemma 2.3. *Suppose that $(R, \mathfrak{m})$ is local and x is an element in $\mathfrak{m}$. Then we have*

$$\text{N-dim}_R(0 : x)_A \geq \text{N-dim}_R A - 1.$$

Moreover, if $\text{N-dim}_R A > 0$ *then* $\text{N-dim}_R(0 : x)_A = \text{N-dim}_R A - 1$ *if and only if x is a parameter element of A.*

The theory of secondary representation is in some sense dual to that of primary decomposition. An R-module $S \neq 0$ is said to be *secondary* if for any $x \in R$, the multiplication by x on S is either surjective or nilpotent. The radical of the annihilator of S is then a prime ideal $\mathfrak{p}$ and we say that S is $\mathfrak{p}$-secondary.

An R-module S is said to be representable if it has a *minimal secondary representation*, i.e. it has an expression $S = S_1 + S_2 + \ldots + S_n$ of $\mathfrak{p}_i$-secondary modules S_i, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are all distinct and $S_i \not\subseteq S_1 + S_{i-1} + S_{i+1} + \ldots + S_n$ for all $i = 1, \ldots, n$. Then the set $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ is independent of the choice of minimal representation of S. This set is denoted by $\text{Att}_R S$ and called the set of attached prime ideals of S (see [6] and [8] for more details).

Recall that the *Krull dimension* of A, denoted by $\dim_R A$, is the Krull dimension of the Noetherian ring $R/\text{Ann}_R A$ (see [9]). For the convenience, we stipulate that $\dim_R A = -1$ if $A = 0$. Note that every Artinian module A is representable and the set of minimal prime ideals of $\text{Ann}_R A$ is just the set of minimal elements of $\text{Att}_R A$ (see [8]). Therefore $\dim_R A$ is the supremum of $\dim R/\mathfrak{p}$, where $\mathfrak{p}$ runs over $\text{Att}_R A$.

Proposition 2.4. *The following statements are true*
(i) $\text{N-dim}_R A = 0$ *if and only if* $\dim_R A = 0$.
In this case, the length of A is finite and the ring $R/\text{Ann}_R A$ is Artinian.
ii) $\text{N-dim}_R A \leq \dim_R A$.

Proof. (i) Suppose that $\text{N-dim}_R A = 0$. Then A is Noetherian and hence A has finite length. So $\dim_R A = 0$. Conversely, suppose that $\dim_R A = 0$. Then all prime ideals containing $\text{Ann}_R A$ are maximal. Let J be the intersection of all prime ideals contained in $\text{Att}_R A$. Hence $J^n A = 0$ for some n. Since J is the intersection of finitely many maximal ideals of R, it follows that A has finite length. Therefore $\text{N-dim}_R A = 0$.

(ii) The proof is done by induction on $d = \dim_R A$. If $d = 0$ then $\text{N-dim}_R A = 0$ by (i). Let $d > 0$. Suppose that $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are all elements of $\text{Att}_R A$ such that $d = \dim R/\mathfrak{p}_i$. Let $J(A)$ be the intersection of all elements in $\text{Supp}_R A$. Since A is

Artinian, $\mathrm{Supp}_R A$ is a finite set of maximal ideals of R (see [7, 2.5]). Therefore we can choose an element $x \in J(A)$ such that $x \notin \mathfrak{p}_i$ for all $i = 1, \ldots, k$. Hence $\dim_R(0 : xR)_A \leq d - 1$. Hence $\text{N-dim}_R(0 : xR)_A \leq d - 1$ by the induction hypothesis. Therefore we get by [2, 2.6] that $\text{N-dim}_R A \leq d$. ∎

Suppose $(R, \mathfrak{m})$ is a local ring. Denote by E the injective hull of $R/\mathfrak{m}$. Note that $\mathrm{Hom}_R(M; E)$, the Matlis dual of a finitely generated R-module M, is an Artinian R-module, but $\mathrm{Hom}_R(A; E)$ is not necessary a finitely generated R-module. However, if R is complete then the Matlis dual of an Artinian R-module is a finitely generated R-module (see [1, 10.2]).

Corollary 2.5 *If $(R, \mathfrak{m})$ is a complete local ring then* $\text{N-dim}_R A = \dim_R A$.

Proof. By Proposition 2.4, (ii), we have only to show that $\text{N-dim}_R A \geq \dim_R A$. We prove this by induction on $d = \text{N-dim}_R A$. If $d = 0$ then $\dim_R A = 0$ by Proposition 2.4, (i). Let $d > 0$. Let $x \in \mathfrak{m}$ be a parameter element of A. Then $\text{N-dim}_R(0 : x)_A = d - 1$. By induction hypothesis we have $\dim_R(0 : x)_A \leq d - 1$. Since R is complete, $\mathrm{Hom}_R((0 : x)_A; E)$ is a finitely generated R-module. Therefore we obtain

$$\begin{aligned} d - 1 \geq \dim_R(0 : x)_A &= \dim_R\big(\mathrm{Hom}_R((0 : x)_A; E)\big) \\ &= \dim_R\big(\mathrm{Hom}_R(A; E)/x\,\mathrm{Hom}_R(A; E)\big) \\ &\geq \dim_R\big(\mathrm{Hom}_R(A; E)\big) - 1 \\ &= \dim_R A - 1 \end{aligned}$$

as required. ∎

Note that $\text{N-dim}_R A$ is always finite by [7, 2.6]. However, when R is non-local, $\dim_R A$ may be infinite. Here is an example.

Example 2.6. *There exists an Artinian module A over a Noetherian non-local ring for which* $\dim_R A = \infty$:

Let $T = K[x_1, \ldots, x_n, \ldots]$ be the polynomial ring of infinitely many invariants $x_1, x_2, \ldots, x_n, \ldots$ over a field K. Let $m_1, \ldots, m_n, \ldots$ be positive integers such that $m_i - m_{i-1} < m_{i+1} - m_i$ for all i. Let $\mathfrak{p}_i$ be the prime ideal of T generated by all x_j such that $m_i \leq j < m_{i+1}$. Let S be the intersection of complements of all $\mathfrak{p}_i$ in T and $R = T_S$. Then we have by [10, A1, Example 1] that R is a Noetherian ring and $\dim R = \infty$. Let A be the injective hull of $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. Then A is Artinian and we can check that $\text{N-dim}_R A = \mathrm{ht}(\mathfrak{m})$. It is easily seen that R is a domain. Hence $\mathrm{Ann}_R A = 0$ by [16, p. 108]. Therefore $\dim_R A = \dim R = \infty$ as required.

3. Noetherian Dimension of Local Cohomology Modules

In this section we assume that $(R, \mathfrak{m})$ is a Noetherian local ring and M is a finitely generated R-module with $\dim_R N = d$.

Theorem 3.1. *Let $t \geq 0$ be a positive integer and $\mathfrak{a}$ an ideal of R. Assume that the local cohomology modules $H^i_{\mathfrak{a}}(M)$ are Artinian for $i = 1, \ldots, t$. Then we have*

$$\text{N-dim}_R \left(H^i_{\mathfrak{a}}(M)\right) \leq i$$

for $i = 0, 1, \ldots, t$.

Proof. We prove this theorem by induction on d. Let $d = 0$. Since $H^0_{\mathfrak{a}}(M)$ is Noetherian, $\text{N-dim}_R(H^0_{\mathfrak{a}}(M) \leq 0$. Assume that $d \geq t > 0$. By the Prime Avoidance Theorem, we can choose an element $x \in \mathfrak{m}$ such that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}_R M \setminus \{\mathfrak{m}\}$. From the exact sequence

$$0 \longrightarrow M/(0 : xR)_M \xrightarrow{.x} M \longrightarrow M/xM \longrightarrow 0$$

by noticing that $(0 : xR)_M$ has finite length by the choice of x, we get the following exact sequences

$$0 \longrightarrow H^i_{\mathfrak{a}}(M)/xH^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/xM) \longrightarrow (0 : xR)_{H^{i+1}_{\mathfrak{a}}(M)} \longrightarrow 0$$

for all $i \geq 1$ and

$$0 \longrightarrow H^0_{\mathfrak{a}}(M/(0 : xR)_M) \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow H^0_{\mathfrak{a}}(M/xM) \longrightarrow (0 : xR)_{H^1_{\mathfrak{a}}(M)} \longrightarrow 0.$$

(Note that x may be not in $\mathfrak{a}$. So $H^0_{\mathfrak{a}}(M/(0 : xR)_M)$ may be non-zero.) Therefore $H^i_{\mathfrak{a}}(M/xM)$ is Artinian for $i = 1, \ldots, t-1$. By applying the induction hypothesis, we get

$$\text{N-dim}_R \left(H^i_{\mathfrak{a}}(M/xM)\right) \leq i$$

for all $i = 0, 1, \ldots, t-1$. So we have by Lemma 2.2 that

$$\text{N-dim}_R \left((0 : xR)_{H^{i+1}_{\mathfrak{a}}(M)}\right) \leq \text{N-dim}_R \left(H^i_{\mathfrak{a}}(M/xM)\right) \leq i$$

for all $i = 0, 1, \ldots, t-1$. Therefore we obtain by Lemma 2.3 that $\text{N-dim}_R \left(H^i_{\alpha}(M)\right) \leq i$ for all $i = 0, 1, \ldots, t$. ∎

It is known that the local cohomology module $H^i_{\mathfrak{m}}(M)$ is Artinian for all i (see [1, 7.1.3]). Therefore the following result is an immediate consequence of Theorem 3.1.

Corollary 3.2. $\text{N-dim}_R \left(H^i_{\mathfrak{m}}(M)\right) \leq i$ *for all i.*

The proposition below gives a necessary condition for the Matlis dual of a local cohomology module to be a finitely generated module.

Proposition 3.3. *Let $\mathfrak{a}$ be an ideal of R and $i \geq 0$ an integer. If $H^i_{\mathfrak{a}}(M)$ is an Artinian R-module and its Matlis dual is a finitely generated R-module then*

$$\text{N-dim}_R \left(H^i_{\mathfrak{a}}(M)\right) = \dim_R \left(H^i_{\mathfrak{a}}(M)\right).$$

Proof. Let $K^i_{\mathfrak{a}}(M)$ be the Matlis dual of R-module $H^i_{\mathfrak{a}}(M)$ and $\widehat{K^i_{\alpha}(M)}$ the $\mathfrak{m}$-adic completion of $K^i_{\mathfrak{a}}(M)$. Since $K^i_{\mathfrak{a}}(M)$ is a finitely generated R-module, we have

$$\dim_R \left(K^i_{\mathfrak{a}}(M)\right) = \dim_{\widehat{R}} \left(\widehat{K^i_{\mathfrak{a}}(M)}\right).$$

Since

$$\widehat{K^i_{\mathfrak{a}}(M)} \cong K^i_{\mathfrak{a}}(M) \otimes \widehat{R} \cong K^i_{\mathfrak{a}}(M)$$

as $\widehat{R}$-modules, we get

$$\dim_R\left(K^i_\alpha(M)\right)=\dim_{\widehat{R}}\left(K^i_\alpha(M)\right)=\dim_{\widehat{R}}\left(H^i_\alpha(M)\right).$$

On the other hand, it follows by Corollary 2.5 and Remark 1, (ii) that

$$\dim_{\widehat{R}}\left(H^i_\mathfrak{a}(M)\right)=\text{N-dim}_{\widehat{R}}\left(H^i_\mathfrak{a}(M)\right)=\text{N-dim}_R\left(H^i_\mathfrak{a}(M)\right).$$

Thus, $\dim_R\left(H^i_\mathfrak{a}(M)\right)=\dim_R\left(K^i_\mathfrak{a}(M)\right)=\text{N-dim}_R\left(H^i_\mathfrak{a}(M)\right)$ as required. ■

Let $\mathfrak{a}_i = \operatorname{Ann}_R\left(H^i_\mathfrak{m}(N)\right)$ for $i=1,\dots,d$. When R admits a dualizing complex, it is not difficult to see that $\text{N-dim}\, H^i_\mathfrak{m}(M)=\dim R/\mathfrak{a}_i$ for all $i=1,...,d$. Therefore from Theorem 3.1, we obtain again the following result of [13].

Corollary 3.4. [13, 2.2.4] *If R possesses dualizing complex then*

$$\dim R/\mathfrak{a}_i \le i$$

for all $i=1,\dots,d$.

Theorem 3.5. *Let α be an ideal of R such that the Artinian R-module $H^d_\mathfrak{a}(M)$ is non-zero. Then we have*

$$\text{N-dim}_R\left(H^d_\mathfrak{a}(M)\right)=d$$

and therefore $H^d_\mathfrak{a}(M)$ is not finitely generated if $d>0$.

Proof. We may assume without any loss of generality that $\operatorname{Ann}_R M=0$. Since M is a finitely generated R-module, there exists an integer n and an exact sequence of finitely generated R-modules

$$0\longrightarrow K\longrightarrow R^n\longrightarrow M\longrightarrow 0.$$

Therefore we get an exact sequence $\left(H^d_\mathfrak{a}(R)\right)^n\longrightarrow H^d_\mathfrak{a}(M)\longrightarrow 0$. Since $\dim R=d$, it follows by [14, 3.4] that $H^d_\mathfrak{a}(R)$ is an Artinian R-module and

$$\operatorname{Att}_{\widehat{R}}\left(H^d_\alpha(R)\right)=\{\mathfrak{p}\in \operatorname{Spec}\widehat{R}, \dim\widehat{R}/\mathfrak{p}=d, \dim\widehat{R}/(\mathfrak{a}\widehat{R}+\mathfrak{p})=0\}.$$

Therefore $H^d_\mathfrak{a}(M)$ is an Artinian R-module and

$$\operatorname{Att}_{\widehat{R}}((H^d_\mathfrak{a}(M))\subseteq\{\mathfrak{p}\in \operatorname{Spec}\widehat{R}, \dim\widehat{R}/\mathfrak{p}=d, \dim\widehat{R}/(\alpha+\mathfrak{p})=0\}.$$

Since $H^d_\mathfrak{a}(M)\neq 0$, it follows that $\operatorname{Att}_{\widehat{R}}((H^d_\alpha(M))\neq\emptyset$. Thus we get by Remark 1, (ii) and Corollary 2.5 that

$$\text{N-dim}_R\left(H^d_\mathfrak{a}(M)\right)=\text{N-dim}_{\widehat{R}}\left(H^d_\mathfrak{a}(M)\right)=\dim_{\widehat{R}}\left(H^d_\mathfrak{a}(M)\right)=d.$$

The rest of the theorem follows from the fact that $H^d_\mathfrak{a}(M)$ is finitely generated if and only if $\text{N-dim}_R\left(H^d_\mathfrak{a}(M)\right)=0$. ■

It is known that if $M\neq 0$ then $H^d_\mathfrak{m}(M)\neq 0$. Therefore we have the following immediate consequence.

Corollary 3.6. $\text{N-dim}_R\left(H^d_\mathfrak{m}(M)\right)=d$.

4. Relations Between Krull Dimension and Noetherian Dimension

We begin by giving an Artinian module A over a Noetherian local ring $(R,\mathfrak{m})$ for which $\text{N-dim}_R A < \dim_R A$.

Example 4.1. *There exists an Artinian module A over a Noetherian local ring $(R,\mathfrak{m})$ such that $\text{N-dim}_R A < \dim_R A$.*

Proof. Denote by $(R,\mathfrak{m})$ the Noetherian local domain of dimension 2 constructed by Ferrand and Raynaud in [4] for which the $\mathfrak{m}$-adic completion $\widehat{R}$ of R has an associated prime $\mathfrak{q}$ of dimension 1 (see also [10, A1, Example 2]). Since $\mathfrak{q} \in \operatorname{Ass}\widehat{R}$, $\dim \widehat{R}/\mathfrak{q} = 1$ with notice that

$$H^1_{\mathfrak{m}}(R) \cong H^1_{\widehat{\mathfrak{m}}}(\widehat{R})$$

as $\widehat{R}$-modules, we get by [1, 11.3.3] that $\mathfrak{q} \in \operatorname{Att}_{\widehat{R}}\left(H^1_{\mathfrak{m}}(R)\right)$. Since $\mathfrak{q} \in \operatorname{Ass}\widehat{R}$, we have $\mathfrak{q} \cap R = 0$. Therefore we have

$$\operatorname{Ann}_R\left(H^1_{\mathfrak{m}}(R)\right) = \operatorname{Ann}_{\widehat{R}}\left(H^1_{\mathfrak{m}}(R)\right) \cap R \subseteq \mathfrak{q} \cap R = 0.$$

Hence $\dim_R\left(H^1_{\mathfrak{m}}(R)\right) = 2$. On the other hand, it follows by Theorem 3.1 and Proposition 2.4, (i) that $\text{N-dim}_R\left(H^1_{\mathfrak{m}}(R)\right) = 1$. Thus, for the Artinian R-module $A = H^1_{\mathfrak{m}}(R)$, we get $\text{N-dim}_R A = 1 < 2 = \dim_R A$ as required. ∎

Remark 2. Yassemi has defined in [18] the set of coassociated prime ideals (Coass) for an arbitrary module. Then he defined in [19] the notion of magnitude of a module N, denoted by $\operatorname{mag} N$, as the supremum of $\dim R/\mathfrak{p}$ where $\mathfrak{p}$ runs over $\operatorname{Coass}_R N$. If $N = 0$ we put $\operatorname{mag}_R N = -1$. Note that for an Artinian module A, $\operatorname{Coass}_R A$ is just the set $\operatorname{Att}_R A$. So $\operatorname{mag}_R A$ is nothing else but $\dim_R A$. Therefore Example 4.1 shows that Theorem 2.10 of [19] is false in general. We also show below in Example 4.3 with the same module A as in Example 4.1, that there exists an element $x \in \mathfrak{m}$ such that $(0 : xR)_A$ has finite length. It implies that

$$\operatorname{mag}_R\left((0 : xR)_A\right) = 0 < 1 = \operatorname{mag}_R A - 1.$$

Therefore, Theorem 2.4 of [19] is false and this shows that the proof of [19, 2.10] is incorrect.

In oder to answer the question when is $\text{N-dim}_R A = \dim_R A$, we need the following definition.

Definition 4.2. *Denote by $V(\operatorname{Ann}_R A)$ the set of all prime ideals of R containing $\operatorname{Ann}_R A$. We say that A satisfies the condition $(*)$ if $\operatorname{Ann}_R\left((0 : \mathfrak{p})_A\right) = \mathfrak{p}$ for all $\mathfrak{p} \in V(\operatorname{Ann}_R A)$.*

It should be mentioned that, for any finitely generated R-module M, we always have $\operatorname{Ann}_R(M/\mathfrak{p}M) = \mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of R containing $\operatorname{Ann}_R M$. However, the dual result for Artinian modules A is not true in general, i.e. there exists an Artinian R-module which does not satisfy the condition $(*)$.

Example 4.3. *An Artinian module A over a Noetherian local ring $(R,\mathfrak{m})$ for which A does not satisfy the condition $(*)$:*

Let R and $A = H^1_{\mathfrak{m}}(R)$ as in Example 4.1. Take a prime ideal $\mathfrak{p}$ satisfying $\mathfrak{p} \neq 0$ and $\mathfrak{p} \neq \mathfrak{m}$. Since $\operatorname{Ann}_R A = 0$, we get $\mathfrak{p} \in V(\operatorname{Ann}_R A)$. Let $0 \neq x \in \mathfrak{p}$. Consider the exact sequence

$$0 \longrightarrow R \xrightarrow{.x} R \longrightarrow R/xR \longrightarrow 0.$$

This gives the derived exact sequence of local cohomology modules

$$0 \longrightarrow H^0_{\mathfrak{m}}(R/xR) \longrightarrow H^1_{\mathfrak{m}}(R) \xrightarrow{.x} H^1_{\mathfrak{m}}(R).$$

Hence $H^0_{\mathfrak{m}}(R/xR) \cong (0 : xR)_{H^1_{\mathfrak{m}}(R)}$ and $(0 : xR)_{H^1_{\mathfrak{m}}(R)}$ has finite length. So $(0 : \mathfrak{p})_A$ has finite length, thus $\operatorname{Ann}_R\big((0 : \mathfrak{p})_A\big) \neq \mathfrak{p}$. It means that A does not satisfy the condition $(*)$.

However, the following result shows that there exist still many Artinian modules which satisfy the condition $(*)$.

Lemma 4.4. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and A an Artinian R-module. If R is complete or A contains a submodule, which is isomorphic to the injective hull of $R/\mathfrak{m}$, then A satisfy the condition $(*)$.*

Proof. Suppose that R is complete. Then $\operatorname{Hom}_R(A;E)$ is a finitely generated R-module. Let $\mathfrak{p} \in V(\operatorname{Ann}_R A)$. It follows that $\mathfrak{p} \in \operatorname{Supp}\big(\operatorname{Hom}_R(A;E)\big)$. Therefore

$$\begin{aligned}\operatorname{Ann}_R(0:\mathfrak{p})_A &= \operatorname{Ann}_R\big(\operatorname{Hom}_R((0:\mathfrak{p})_A;E)\big)\\ &= \operatorname{Ann}_R\big(\operatorname{Hom}_R(A;E)/\mathfrak{p}\operatorname{Hom}_R(A;E)\big) = \mathfrak{p}.\end{aligned}$$

Thus A satisfies the condition $(*)$.

Suppose that A contains a submodule which is isomorphic to E. Let $\mathfrak{p} \in V(\operatorname{Ann}_R A)$. Then we have by [10, 4.1] that

$$\operatorname{Ass}_{R_{\mathfrak{p}}}\big(\operatorname{Hom}_R(R_{\mathfrak{p}};A)\big) \supseteq \operatorname{Ass}_{R_{\mathfrak{p}}}\big(\operatorname{Hom}_R(R_{\mathfrak{p}};E(R/\mathfrak{m}))\big) = \{\mathfrak{q}R_{\mathfrak{p}} : \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Therefore $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}}\big(\operatorname{Hom}_R(R_{\mathfrak{p}};A)\big)$. It follows that

$$(0 : \mathfrak{p}R_{\mathfrak{p}})_{\operatorname{Hom}_R(R_{\mathfrak{p}};A)} \neq 0.$$

Hence $\operatorname{Hom}_R\big(R_{\mathfrak{p}};(0:\mathfrak{p})_A\big) \neq 0$. So we get by [10, p. 130] that $\mathfrak{p} \supseteq \operatorname{Ann}_R\big((0 : \mathfrak{p})_A\big)$. Therefore $\mathfrak{p} = \operatorname{Ann}_R\big((0:\mathfrak{p})_A\big)$ as required. ∎

The following result shows that the condition (*) is sufficient for an Artinian module A to have $\operatorname{N-dim}_R A = \dim_R A$.

Proposition 4.5. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and A an Artinian R-module. Suppose that A satisfies the condition (*). Then we have*

$$\operatorname{N-dim}_R A = \dim_R A.$$

Proof. Let $\mathfrak{a}$ be an arbitrary ideal of R. Clearly, we have

$$\operatorname{rad}\big(\operatorname{Ann}_R(0:\mathfrak{a})_A\big) \supseteq \operatorname{rad}\big(\alpha + \operatorname{Ann}_R A\big).$$

It follows by the hypothesis that $\mathrm{Ann}_R\big((0:\mathfrak{a})_A\big) \subseteq \mathrm{Ann}_R\big((0:\mathfrak{p})_A\big) = \mathfrak{p}$ for all prime ideals $\mathfrak{p}$ contaning $\mathfrak{a} + \mathrm{Ann}_R A$. Therefore $\mathrm{rad}\big(\mathrm{Ann}_R(0:\mathfrak{a})_A\big) = \mathrm{rad}\big(\mathfrak{a} + \mathrm{Ann}_R A\big)$. Let $\text{N-dim}_R A = d$. Then there exist by Remark 1, (i) elements $x_1, \dots, x_d \in \mathfrak{m}$ such that $\ell_R(0:(x_1,\dots,x_d)R)_A < \infty$. Applying the fact above to $\mathfrak{a} = (x_1, \dots, x_d)R$, we get

$$\begin{aligned} 0 &= \dim_R\big((0:(x_1,\dots,x_d)R)_A\big) \\ &= \dim\big(R/((x_1,\dots,x_d)R + \mathrm{Ann}_R A)\big) \geq \dim_R A - d. \end{aligned}$$

Therefore $\dim_R A \leq d$. Now the assertion follows by Proposition 2.4, (ii). ∎

Note that the converse of Proposition 4.5 is not true. Here is an example.

Example 4.6. *There exists an Artinian module A over a Noetherian local ring $(R,\mathfrak{m})$ for which $\text{N-dim}_R A = \dim_R A$, but A does not satisfy the condition $(*)$.*

Proof. Assume first that there exist Artinian modules A' and A'' over a Noetherian local ring R for which the following conditions are satisfied:

(i) $\text{N-dim}_R A' = \dim_R A' > \dim_R A'' > \text{N-dim}_R A''$.

(ii) There exists $\mathfrak{p} \in V(\mathrm{Ann}_R A'')$ and $\mathfrak{p} \notin V(\mathrm{Ann}_R A')$ for which $\mathrm{Ann}_R\big((0:\mathfrak{p})_{A''}\big) \neq \mathfrak{p}$.

Let $A = A' \oplus A''$. Then A is Artinian, $\dim_R A = \text{N-dim}_R A$ and $\mathfrak{p} \in V(\mathrm{Ann}_R A)$. We have

$$\mathrm{Ann}_R\big((0:\mathfrak{p})_A\big) = \mathrm{Ann}_R\big((0:\mathfrak{p})_{A'}\big) \cap \mathrm{Ann}_R\big((0:\mathfrak{p})_{A''}\big) \neq \mathfrak{p}.$$

It means that A does not satisfy the condition $(*)$.

Now, we have only to show the existence of modules A' and A'' as above. Let R be as in Example 4.1. Let $S = R[[x_1, \dots, x_t]]$, with $t \geq 3$, the formal power series ring of t variants $x_1, \dots, x_t$ over R. Let A' be the formal power inverse series ring $k[[x_1^{-1}, \dots, x_t^{-1}]]$ over the field $k = R/\mathfrak{m}$ which was defined as in [5] and [12]. Then A' is an Artinian S-module and $\text{N-dim}_S A' = t$. Since $\mathrm{Ann}_S A' = \mathfrak{m}.S$, it follows that $\dim_S A' = \dim\big(k[[x_1, \dots, x_t]]\big) = t$. Let A'' be the local cohomology module $H^1_{\mathfrak{m}}(R)$ as S-module with the product $x_i.A'' = 0$ for all $i = 1, \dots, t$. Then a subset of A'' is an R-submodule of A'' if and only if it is a S-submodule of A''. Therefore A'' is also an Artinian S-module and $\dim_S A'' = 2$, $\text{N-dim}_S A'' = 1$. It is clear that $\mathrm{Ann}_S A'' = (x_1, \dots, x_t)S$. Let $\mathfrak{p}$ be a non-maximal prime ideal of S for which $\mathfrak{p}$ strictly contains $\mathrm{Ann}_S A''$. Then $\mathfrak{p} \notin V(\mathrm{Ann}_S A')$. By computing similarly to Example 4.3, we get $\mathrm{Ann}_S\big((0:\mathfrak{p})_{A''}\big) \neq \mathfrak{p}$ as required.

Corollary 4.7. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and A an Artinian R-module. Denote by $\widehat{R}$ the $\mathfrak{m}$-adic completion of R. Then we have*

$$\text{N-dim}_R A = \dim_{\widehat{R}} A.$$

Proof. By Lemma 4.4, A satisfies the condition $(*)$ considering as $\widehat{R}$-module. Therefore we get by Proposition 4.5 and Remark 1, (ii) that $\text{N-dim}_R A = \text{N-dim}_{\widehat{R}} A$. ∎

References

1. M. P. Brodmann and R. Y. Sharp, *Local Cohomology: an Algebraic Introduction with Geometric Applications*, Cambridge University Press, 1998.
2. N. T. Cuong and L. T. Nhan, Dimension, multiplicity and Hilbert function of Artinian modules, *East-West J. Math.* **2** (1999) 179–196.
3. L. H. Denizler and R. Y. Sharp, Co-Cohen–Macaulay modules over commutative rings, *Glasgow Math. J.* **38** (1996) 359–366.
4. D. Ferrand and M. Raynaud, Fibres formelles d'un anneau local Noetherian, *Ann. Sci. École Norm. Sup.* **3** (1970) 295–311.
5. D. Kirby, Artinian modules and Hilbert polynomials, *Quart. J. Math. Oxford* **24** (1973) 47–57.
6. D. Kirby, Coprimary decomposition of Artinian modules, *J. London Math. Soc.* **6** (1973) 571–576.
7. D. Kirby, Dimension and length of Artinian modules, *Quart. J. Math. Oxford* **41** (1990) 419–429.
8. I. G. Macdonald, Secondary representation of modules over a commutative ring, *Symposia Mathematica* **11** (1973) 23–43.
9. H. Matsumura, *Commutative ring theory,* Cambridge University Press, 1986.
10. M. Nagata, *Local Rings*, Interscience, New York, 1962.
11. A. Ooishi, Matlis duality and the width of a module, *Hiroshima Math. J.* **6** (1976) 573–587.
12. R. N. Roberts, Krull dimension for Artinian modules over quasi-local commutative rings, *Quart. J. Math. Oxford* **26** (1975) 269–273.
13. P. Schenzel, Dualisierende Komplexe in der lokalen Algebra und Buchsbaum Ringe, *Lecture Notes in Math.* **907**, Springer-Verlag, Berlin - Heidelberg - New York, 1982.
14. R. Y. Sharp, On the attached prime ideals of certain Artinian local cohomology modules, *Edinburgh Math. Soc.* **24** (1981) 9–14.
15. R. Y. Sharp, A method for the study of Artinian modules with an application to asymptotic behaviour, *In: Commutative Algebra* (Math. Siennces Research Inst. Publ. No. 15, Spinger-Verlag, 1989), 177–195.
16. D. W. Sharpe and P. Vámos, *Injective Modules,* University Prees Cambridge, 1972.
17. Z. Tang and H. Zakeri, Co-Cohen–Macaulay modules and modules of generalized fractions, *Comm. Algebra* **22** (1994) 2173–2204.
18. S. Yassemi, Coassociated primes, *Comm. Algebra* **23** (1995) 1473–1498.
19. S. Yassemi, Magnitute of modules, *Comm. Algebra* **23** (1995) 3993–4008.