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A New Sparse Set Topology

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Abstract. In this paper we introduce the concept of sparse sets in a metric space as a generalization of sets having density zero. We then generate a topology with the help of these sets called sparse set topology, and investigate certain properties of this topology.

1. Introduction

The idea of sparse sets was first introduced by Sarkhel and De [13] in the real number space as a generalization of sets having upper outer density zero. Subsequently Chakraborty and Lahiri [3, 4] generalized the concept to a topological group, taking one of the equivalent conditions of sparse sets (see [13, Theorem 3.1]) as the definition of sparse sets. In [4] a topology was also generated with the help of sparse sets, whose properties were investigated. Now in both the above cases the density function, to which the notion of sparse sets is closely related, always lies between 0 and 1.

In this paper we consider the situation in a metric space. In a metric space, using a specified class of subsets, Eames [5] defined the density function which happens to exceed 1 sometimes. In fact his density function lies between 0 and ∞. Recently Lahiri and Das [8] studied the properties of this density function and the corresponding density topology. Here we first study the feasibility of certain definitions of sparse sets, valid for a topological group, in a metric space and consequently introduce a suitable definition of sparse set so that it generalizes the notion of sets having density zero, which we do in Sec. 3 of the paper. In Sec. 4 we prove certain properties of the sparse sets. Finally, in Sec. 5, we introduce the concept of sparse set topology with the help of these sets and investigate various properties of this topology. Our nature of study does not appear to be analogous to the known methods because of the uncertain nature of the density function and our sparse set topology appears to be different from that of [4] in many respects.

2. Preliminaries

Let (X, ρ) be a metric space. Let C be a class of closed sets from (X, ρ) and τ a non-negative real valued function on C. We assume that the empty set ϕ and all the singleton sets are in C, finite union of members of C is in C and that $\tau(I) = 0$ if and only if I contains at most one point. For each $A \subset X$, let $\mu(A)$, $0 \leq \mu(A) \leq \infty$, be defined by

$$\mu(A) = \lim_{\epsilon \to 0+} \Big\{ \inf \sum_{n=1}^{\infty} \tau(I(n)) \Big\}$$

where the infimum is taken over all possible countable collections of sets $I(n)$ from C such that $A \subset \bigcup_{n=1}^{\infty} I(n)$ and the diameter of $I(n)$, $\operatorname{diam}(I(n)) < \epsilon$ for all n. As in Eames [5] we assume that such a countable collection of sets from C exists for each set A and every $\epsilon > 0$. Then μ is an outer measure function [11, p. 35]. A set A is measurable if $\mu(B) = \mu(A \cap B) + \mu(A^c \cap B)$ for every set $B \in C$ where c stands for the complement. All Borel sets of (X, ρ) are measurable (cf. 11, pp. 102–106). For every set A there is a measurable set B called a measurable cover of A such that $A \subset B$ and $\mu(A) = \mu(B)$ [11, pp. 107–108] and so μ is a regular outer measure function.

Definition A. [5] *Let $A \subset X$ and $p \in X$. Then the number $D(A, p)$, $0 \leq D(A, p) \leq \infty$, called the density of A at p, is defined by*

$$D(A, p) = \lim_{\epsilon \to 0+} \Big\{ \sup \frac{\mu(A \cap I)}{\tau(I)} \Big\},$$

where the supremum is taken over all sets I from C such that $p \in I$ and the $\operatorname{diam}(I) < \epsilon$. Also when $\tau(I) = 0$ or ∞, we take $\mu(A \cap I)/\tau(I) = 0$.

In [5] it is proved that if the sets in C satisfy certain regularity conditions and $\mu(A)$ is finite, then
(i) $D(A, p) = 1$ for almost all points of A,
(ii) $D(A, p) = 0$ for almost all points of A^c if and only if A is measurable.

Throughout we assume that the regularity conditions as given in [5] are satisfied so that (i) and (ii) hold.

Definition B. [8] *Let $D = \{U \subset X;\ D(U^c, x) = 0$ for all $x \in U\}$. Then D is a topology on X called the density topology (d-topology for short) and thus (X, D) is a topological space. Sets in D are called d-open.*

We quote below the results from [8] which will be needed in future.

Lemma A. *The set function $D(\,.\,, p)$ for a fixed $p \in X$ is monotone nondecreasing and finitely subadditive.*

Lemma B. *If E, F are measurable and $E \cap F = \phi$, then $D(E \cup F, p) = D(E, p) + D(F, p)$ for almost all $p \in X$.*

Theorem A. *$\mu(E) = 0$ if and only if E is d-closed and discrete.*

By sets we shall always mean subsets of X unless otherwise mentioned and as in [8], in our discussions, we treat only sets having finite measure.

3. Sparse Sets in a Metric Space

In this section our aim is to find a suitable definition of sparse set in a metric space so that it generalizes the notion of sets having density zero.

We first consider the definition of sparse sets in a topological group as given by Chakraborty and Lahiri [3, 4].

Definition C. *A set $E \subset G$ is said to be sparse at $x \in G$ if for every set $F \subset G$ with $\overline{D}^*(F,x) < 1$, we have $\overline{D}^*(E \cup F, x) < 1$, where G is a compact Hausdorff topological group with regular Haar measure m and outer measure m^* as defined in* [1, 3, 4]. *The details of the definition and properties of the upper and lower outer density functions $\overline{D}^*$ and $\underline{D}^*$ in G can be found in* [1], *also to some extent in* [3, 4].

Now the above definition of sparse sets does not seem to be suitable for a metric space, because unlike a topological group (where $\overline{D}^*(G,x) = 1$ for all $x \in G$), the highest value of the density function at any point of a metric space is not necessarily 1. In fact it can be any number from 0 to ∞.

We shall next consider another equivalent definition of sparse sets in a topological group which will be established in Lemma 2.

We shall make use of the following results of Chakraborty and Lahiri [4] in Lemmas 1 and 2.

Lemma C. *If E is a measurable cover of a set $A \subset G$, then for any $x \in G$*

$$\overline{D}^*(E,x) = \overline{D}^*(A,x) \text{ and } \underline{D}^*(E,x) = \underline{D}^*(A,x).$$

Lemma D. *If E is measurable then for any $x \in G$, $\overline{D}^*(E,x) + \underline{D}^*(E^c,x) = 1$.*

Also as in [4] we assume that every set in G has measurable cover, where the sense of measurable cover is to be interpreted in accordance with ([4], see also [12]).

We now prove the following two results.

Lemma 1. *$E \subset G$ is sparse at $x \in G$ if and only if for any measurable set $F \subset G$ with $\overline{D}^*(F,x) < 1$ we have $\overline{D}^*(A \cup F, x) < 1$ where A is any measurable cover of E.*

Proof. Let E be sparse at x and let F be a measurable set with $\overline{D}^*(F,x) < 1$. Then from Definition C, $\overline{D}^*(E \cup F, x) < 1$. Let A be any measurable cover of E. Then $A \cup F$ is also a measurable cover of $E \cup F$ and so by Lemma C, $\overline{D}^*(A \cup F, x) = \overline{D}^*(E \cup F, x) < 1$.

Conversely let the given condition hold. Let $F_1 \subset G$ be such that $\overline{D}^*(F_1,x) < 1$. Let F be a measurable cover of F_1. Then by Lemma C, $\overline{D}^*(F,x) =$

$\overline{D}^*(F_1, x) < 1$. Let A be a measurable cover of E. Now by the given condition, $\overline{D}^*(A \cup F, x) < 1$. Since $A \cup F$ is a measurable cover of $E \cup F_1$, by Lemma C, $\overline{D}^*(E \cup F_1, x) = \overline{D}^*(A \cup F, x) < 1$. Hence E is sparse at x.

Lemma 2. *$E \subset G$ is sparse at $x \in G$ if and only if for any measurable set $F \subset G$ with $\underline{D}^*(F^c, x) > 0$ we have $\underline{D}^*((A \cup F)^c, x) > 0$ where A is any measurable cover of E.*

The proof immediately follows from Lemma 1 and Lemma D.

Now in accordance with Lemma 2, we consider the following definition of sparse set in a metric space.

Definition 1. *A subset E of X is said to be sparse at x if for every measurable set $F \subset X$ with $D(F^c, x) > 0$ we have $D((A \cup F)^c, x) > 0$, where A is any measurable cover of E.*

The following property of the density function will be used in our next result.

Lemma 3. *If A is a measurable cover of $E \subset X$ then for any $x \in X$, $D(E, x) = D(A, x)$.*

The proof is omitted.

Theorem 1. *$E \subset X$ is sparse at x if and only if $D(E, x) = 0$.*

Proof. First let $D(E, x) = 0$ and let $F \subset X$ be a measurable set such that $D(F^c, x) > 0$. If possible let $D((A \cup F)^c, x) = 0$ for a measurable cover A of E. Then by Lemma A,

$$\begin{aligned} D(F^c, x) &= D\big[(A \cup F)^c \cup (A - F), x\big] \\ &\leq D\big((A \cup F)^c, x\big) + D(A - F, x) \\ &\leq D\big((A \cup F)^c, x\big) + D(A, x) \\ &= 0 + 0 = 0, \end{aligned}$$

(Since by Lemma 3, $D(E, x) = D(A, x)$), i.e. $D(F^c, x) = 0$, a contradiction. Hence $D\big((A \cup F)^c, x\big) > 0$ for any measurable cover A of E and so E is sparse at x.

Conversely let E be sparse at x. If possible let $D(E, x) > 0$. Let A be a measurable cover of E. Then by Lemma 3, $D(E, x) = D(A, x) = D\big((A^c)^c, x\big) > 0$. Now since E is sparse at x, from Definition 1, $D\big((A \cup A^c)^c, x\big) = D(\phi, x) > 0$ which is impossible. Hence $D(E, x) = 0$.

Remark 1. Theorem 1 shows that the above definition of sparse sets (Definition 1) does not at all generalize the concept of a set with density zero, which is our main purpose. However we have the following interesting result which gives a new characterisation of the density topology in a metric space.

Theorem 2. *A set $U \subset X$ is d-open if and only if for all points $x \in U$, U^c satisfies the condition of Definition 1.*

Let us again consider the earlier definition of sparse sets, i.e, Definition C. We note that in view of Lemma A, $D(X,x)$ is the highest value of the density function at any point x, though it may be any number from 0 to ∞. With this in mind we now define a sparse set in a metric space as follows.

Definition 2. A set $E \subset X$ is said to be sparse at a point $x \in X$ where $D(X,x) \neq 0$ if for every set $F \subset X$ with $D(F,x) < D(X,x)$ we have $D(E \cup F,x) < D(X,x)$. If $x \in X$ is such that $D(X,x) = 0$, then any set $E \subset X$ is said to be sparse at x. The collection of all sparse sets at x will be denoted by $S(x)$.

Note 1. In particular if the density function for a metric space is bounded in $[0,1]$ and $D(X,x) = 1$ for all $x \in X$, then Definition 2 coincides with Definition C.

In the remaining part of this paper, by sparse sets we shall always mean sets satisfying the conditions of Definition 2.

4. Properties of Sparse Sets

The following lemma shows that the notion of sparse sets is a generalization of the notion of sets having density zero.

Lemma 4. *If $D(E,x) = 0$ then $E \in S(x)$.*

Proof. Let $F \subset X$ be such that $D(F,x) < D(X,x)$. Then by Lemma A,

$$D(E \cup F,x) \leq D(E,x) + D(F,x) < D(X,x).$$

So $E \in S(x)$.

Lemma 5. *If $A \subset E$ and $E \in S(x)$, then $A \in S(x)$.*

Lemma 6. *If $E_1, E_2 \in S(x)$ then $E_1 \cup E_2 \in S(x)$.*

The proofs of Lemmas 5 and 6 run parallel to those of Lemmas 4 and 5 in [4] and so are omitted.

Lemma 7. *$S(x)$ is a ring.*

The proof follows from Lemmas 5 and 6.

Lemma 8. *If $E \in S(x)$ and A is a measurable cover of E then $A \in S(x)$.*

Proof. If $D(X,x) = 0$ then there is nothing to prove. Let $F \subset X$ be such that $D(F,x) < D(X,x)$ where $D(X,x) \neq 0$. Since $E \in S(x)$, $D(E \cup F,x) < D(X,x)$. Let F_1 be a measurable cover of F. Then clearly $A \cup F_1$ is a measurable cover of $E \cup F$ and so by Lemma 3, $D(A \cup F_1,x) = D(E \cup F,x) < D(X,x)$. Since

$A \cup F \subset A \cup F_1$, by Lemma A, $D(A \cup F, x) \leq D(A \cup F_1, x) < D(X, x)$. This proves the lemma.

Theorem 3. (cf. [4, Theorem 1]) *If $E \in S(x)$ and $F \subset X$ is such that $D(F, x) = 0$ then $D((E \cup F)^c, x) = D(X, x)$.*

Proof. If $D(X, x) = 0$ then there is nothing to prove. So let $D(X, x) \neq 0$. By Lemma A,

$$D(X, x) \leq D(F, x) + D(F^c, x) = D(F^c, x) \leq D(X, x)$$

i.e.

$$D(F^c, x) = D(X, x).$$

Now

$$D(E \cup (E \cup F)^c, x) = D(E \cup F^c, x) = D(X, x)$$

(by Lemma A and since $D(F^c, x) = D(X, x)$).

Then $D((E \cup F)^c, x) = D(X, x)$ for if $D((E \cup F)^c, x) < D(X, x)$ then, because $E \in S(x)$, $D(E \cup (E \cup F)^c, x) < D(X, x)$ which is not the case. Hence the theorem.

Corollary 1. *If $E \in S(x)$ then $D(E^c, x) = D(X, x)$.*

Proof. Since $D(\phi, x) = 0$, by Theorem 1,

$$D(E^c, x) = D((E \cup \phi)^c, x) = D(X, x).$$

5. Sparse Set Topology

Definition 3. *Let $\tau = \{E;\ E \subset X$ and $E^c \in S(x)$ for all $x \in E\}$.*

Theorem 4. *(X, τ) is a topological space.*

Proof. Obviously $\phi, X \in \tau$. Let $E_i \in \tau$ for $i \in \Delta$ where Δ is an index set. Let $E = \bigcup_{i \in \Delta} E_i$ and $x \in E$. Then $x \in E_i$ for some $i \in \Delta$. Since $E_i \in \tau$, $E_i^c \in S(x)$. Since $E^c \subset E_i^c$, by Lemma 4, $E^c \in S(x)$. Since $x \in E$ is arbitrary, $E \in \tau$. Finally let $E_1, E_2 \in \tau$ and $x \in E_1 \cap E_2$. Since $x \in E_j$ and $E_j \in \tau$, $E_j^c \in S(x)$ for $j = 1, 2$. Then by Lemma 5, $(E_1 \cap E_2)^c = E_1^c \cup E_2^c \in S(x)$. Since $x \in E_1 \cap E_2$ is arbitrary, $E_1 \cap E_2 \in \tau$. Hence τ is a topology on X and (X, τ) is a topological space.

The topology thus obtained will be called the sparse set topology (s-topology for short). The sets open (closed) in τ are called s-open (s-closed).

Theorem 5. *The s-topology is finer than the d-topology.*

Proof. Let U be d-open. Then $D(U^c, x) = 0$ for all $x \in U$ and so by Lemma 4, $U^c \in S(x)$ for all $x \in U$ and so U is s-open.

Theorem 6. (cf. [4, Lemma 9]) *If E is s-open and $x \in E$. Then $D(E,x) = D(X,x)$.*

Proof. Since $E \in \tau$ and $x \in E$, $E^c \in S(x)$ and so by Corollary 1, $D(E,x) = D((E^c)^c, x) = D(X,x)$.

Theorem 7. (cf. [4. Lemma 8]) *If for some $x \in X$ with $D(X,x) \neq 0$, $D(E^c,x) < D(X,x)$ then x is a s-limit point of E.*

Proof. Suppose x is not a s-limit point of E. Then there is a s-open set V containing x such that $V \cap (E - \{x\}) = \phi$, i.e. $V \subset (E - \{x\})^c = E^c \cup \{x\}$. Since $\mu(\{x\}) = 0$, by Lemma A,

$$D(V,x) \leq D(E^c \cup \{x\}, x) \leq D(E^c,x) + D(\{x\},x) = D(E^c,x).$$

Again from Theorem 6, $D(V,x) = D(X,x)$ and so $D(X,x) = D(V,x) < D(E^c,x)$, a contradiction. Hence x is a s-limit point of E.

Theorem 8. *For a measurable set E, $\mu(E) = 0$ if and only if E is closed and discrete in s-topology.*

Proof. First assume that $\mu(E) = 0$. Then by Theorem A, E is discrete and closed in d-topology. Since s-topology is finer than d-topology, E is closed and discrete in s-topology.

Conversely let E be closed and discrete in s-topology. Then by Theorem 8, $D(E^c,x) = D(X,x)$ for all $x \in X$. Now by Lemma B,

$$D(E,x) + D(E^c,x) = D(X,x)$$

for all $x \in X - N$ where $\mu(N) = 0$. Thus $D(E,x) = 0$ for all $x \in X - N$. Let $E_1 = \{x \in E;\ D(E,x) = 1\}$ and $E_2 = E - E_1$. Then by (i), $\mu(E_2) = 0$. Also we must have $E_1 \subset N$, i.e. $\mu(E_1) = 0$. Since $E = E_1 \cup E_2$, $\mu(E) = 0$. This proves the theorem.

Theorem 9. (cf. [4, Theorem 7]) *(X,τ) is not first countable if it contains at least one point x with $D(X,x) > 0$.*

Proof. First assume that there is a point $x \in X$ with $D(X,x) > 0$. We shall show that there is no countable base at x. On the contrary assume that $\{B_1, B_2, B_3, ...\}$ is a countable base at x. We note that each B_i is uncountable for if B_i is countable then $\mu(B_i) = 0$ and so $D(B_i,x) = 0 \neq D(X,x)$ which is impossible in view of Theorem 6.

We select $x_1 \neq x$ from B_1, $x_2 \neq x, x_1$ from B_2 etc. Let $A = \{x_1, x_2, x_3, ...\}$. Then $\mu(A) = 0$ and so A is s-closed by Theorem 8.

Let U be any s-open set containing x. Then $U - A$ is also a s-open set containing x. Clearly no B_i is contained in $U - A$. This shows that $\{B_1, B_2 ...\}$ is not a countable base at x. Hence (X,τ) is not first countable.

We shall show that the above mentioned condition is essential. If $D(X,x) = 0$ for all $x \in X$, then $D(\{x\}^c, x) = 0$ for all $x \in X$ by Lemma A which implies that $\{x\}$ is d-open and so s-open (by Theorem 5) for all $x \in X$. Then evidently (X, τ) is first countable.

Theorem 10. (cf. [4, Theorem 8]) *If $\mu(E) = 0$, then E is nowhere dense in the s-topology provided E does not contain any point x with $D(X, x) = 0$.*

Proof. If E does not contain any point x with $D(X, x) = 0$, then the proof runs parallel to that of Theorem 8 in [4]. We shall only show that the given condition is essential. If $D(X, x) = 0$ for some $x \in E$, then as in Theorem 9, $\{x\}$ is s-open and so s-interior of E is not empty which implies that E is not nowhere dense.

Theorem 11. *Compact sets in the s-topology are finite.*

Proof. Let A be a countably infinite set. Since $\mu(\{x\}) = 0$ for all $x \in X$, $\mu(A) = 0$. Now for any $y \in A$, $\mu(A - \{y\}) = 0$ and so by Theorem 8, $A - \{y\}$ is s-closed which implies that $A^c \cup \{y\}$ is s-open. Thus $\{A^c \cup \{y\};\ y \in A\}$ is a-open cover of A which has no finite subcover. Hence A is not s-compact.

References

1. S.K. Berberian, *Measure and Integration*, Chelsea Publishing Company, Bronx, New York, 1965.
2. S. Chakraborty and B. K. Lahiri, Density topology in a topological group, *Indian J. Pure Appl. Math.* **15** (1984) 753–764.
3. S. Chakraborty and B.K. Lahiri, Proximal density topology in a topological group, *Bull. Inst. Math. Acad. Sinica* **15** (1985) 257–272.
4. S. Chakraborty and B.K. Lahiri, Sparse set topology, *Bull. Cal. Math. Soc.* **82** (1990) 349–356.
5. W. Eames, Local property of measurable sets, *Canad. J. Math.* **12** (1960) 632–640.
6. C. Goffman and D. Waterman, Approximately continuous transformations, *Proc. Amer. Math. Soc.* **12** (1961) 116–121.
7. C. Goffman, C.J. Neugebauer, and T. Nishiura, Density topology and approximate continuity, *Duke Math. J.* **28** (1961) 497–503.
8. B.K. Lahiri and Pratulananda Das, Density topology in a metric space, *J. Ind. Math. Soc.* **65** (1998) 107–117.
9. B.K. Lahiri and Pratulananda Das, Density and the space of approximately continuous mappings, *Vietnam J. Math.* **27** (1999) 123–130.
10. N.F.G. Martin, A topology for certain measure spaces, *Trans. Amer. Math. Soc.* **112** (1964) 1–18.
11. M.E. Munroe, *Introduction to Measure and Integration*, Cambridge, Mass., 1953.
12. P.K. Saha and B.K. Lahiri, Density topology in Romanovski spaces, *J. Ind. Math. Soc.* **54** (1989) 65–84.
13. D.N. Sarkhel and A.K. De, The proximally continuous integrals, *J. Aust. Math. Soc.* (series A) **31** (1981) 26–45.