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Short Communication

On Lax–Oleinhik–Type Formulas for Weak Solutions to Scalar Conservation Laws

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1. Introduction

We consider in this note the Cauchy problem for scalar conservation laws and look for explicit *weak* solutions. Namely, let us deal with an *unknown* function $u(x,t)$ by an analytical way. The equation governing u is a single conservation law, completed by an initial datum

$$\begin{aligned} &u_t + (f(t,u))_x = 0, \quad x \in \mathbb{R},\ t > 0 \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1.1}$$

Exactly, the first equation in (1.1) has the form of a first-order quasi-linear partial differential equation. As it is well-known that smooth solutions are not available even if the flux f and the initial datum u_0 are very "qualified" functions, people therefore turn to study various kinds of generalized solutions. So we recall here the definition of *weak solutions*, which are sometimes known to be *integral solutions*.

Definition 1.1. *A function $u = u(x,t), x \in \mathbb{R},\ t \geq 0$ will be called a weak solution of the Cauchy problem (1.1) if $u(.,.), f(.,u(.,.)) \in L^1_{loc}(\mathbb{R}\times[0,+\infty))$, and if, for any test function $\phi \in C_0^\infty(\mathbb{R}\times[0,+\infty))$, it holds*

$$\begin{aligned} &\int\int_{\mathbb{R}\times[0,+\infty)} \Big(u(x,t)\frac{\partial\phi(x,t)}{\partial t} + f(t,u(x,t))\frac{\partial\phi(x,t)}{\partial x}\Big)dxdt \\ &+ \int_{\mathbb{R}} u_0(x)\phi(x,0)dx = 0. \end{aligned} \tag{1.2}$$

Our goal here is to look for explicit weak solutions of Problem (1.1), relying on solutions of corresponding Cauchy problem for Hamilton–Jacobi equations: the cost function of a problem of calculus of variations can explicitly provide *Lipschitz solutions* to Hamilton–Jacobi equations. Recall also that a Lipschitz solution means a locally Lipschitz function satisfying a partial differential equation almost everywhere in the domain under consideration. More precisely,

Definition 1.2. *A function $u = u(x,t)$, $x \in \mathbb{R}, t \geq 0$, is called a Lipschitz solution of the Cauchy problem with the Cauchy datum $u_0 = u_0(x)$, $x \in \mathbb{R}$, for a partial differential equation of first-order*

$$F(x,t,u,u_x) = 0, \quad u \in \mathcal{O}, \tag{1.3}$$

where $\mathcal{O}$ is open in $\mathbb{R}$ and the function F is assumed to be continuous on $\mathbb{R} \times [0,+\infty) \times \mathcal{O} \times \mathbb{R}$, if $u \in C(\mathbb{R} \times [0,+\infty)) \cap W^{1,\infty}_{loc}(\mathbb{R} \times (0,+\infty))$, u satisfies the equation (1.3) almost everywhere and

$$u(x,0) = u_0(x), \quad \forall x \in \mathbb{R}.$$

Concretely, it is well-known that, if f is Lipschitz continuous and g is convex in $\mathbb{R}$, then the Hopf–Lax formula (see [4])

$$v(x,t) = (g^* + tf)^*(x), \quad x \in \mathbb{R},\ t \geq 0, \tag{1.4}$$

where the notation $*$ stands for the usual Fenchel transform

$$h^*(x) = \sup_{y \in \mathbb{R}}\{x.y - h(y)\}, \quad x \in \mathbb{R},$$

provides a Lipschitz solution of the following Cauchy problem for Hamilton–Jacobi equation

$$\begin{aligned} &v_t + f(v_x) = 0, \quad x \in \mathbb{R},\ t > 0, \\ &v(x,0) = g(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1.5}$$

The Hopf–Lax formula and its generalizations have been interested by many mathematicians: Lax, Oleinhik, Hopf, Bardi, Evans, Ishii, Lions, Barron, Jensen, Liu, Van,... and their colleagues and students. If we take *formally* the derivative in x in (1.5), we get

$$\begin{aligned} &v_{xt} + (f(v_x))_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ &v_x(x,0) = g'(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1.6}$$

Therefore, if v is a classical solution of (1.5), and g is differentiable with $g' = u_0$, then we see by (1.4) that the function

$$u(x,t) := v_x(x,t) = \frac{\partial}{\partial x}(g^* + tf)^*(x), \quad x \in \mathbb{R},\ t \geq 0, \tag{1.7}$$

is a solution of Problem (1.1). This motivates us to construct explicit solutions of Problem (1.1) via the Hopf–Lax formula (1.4). We call them Lax–Oleinhik-type formulas. Notice that the function v in (1.4) is no longer expected to be smooth. Here, the function u in (1.7) will then be proved to be a weak solution of Problem (1.1) under relaxed conditions so that it is applicable to quite wide classes of fluxes f and initial data u_0. In particular, the set of f would cover functions having the Caratheodory property, and the set of u_0 would contain a certain class of functions with bounded variations or functions belonging to some certain dual spaces of Sobolev spaces. These explicit solutions would then be able to be candidates for *entropy solutions*, though the question whether they are really *admissible* under *entropy criteria* is out of the scope of this paper.

Founding the Lax–Oleinhik–type formulas for solutions of conservation laws has been interested by several authors. To see this and its motivation, the reader is referred to [2-7] and the references therein.

2. Representation of Weak Solutions

Let us make out the hypotheses to be assumed throughout this paper.

Set

$$g(x) = \int_0^x u_0(y)dy, \quad x \in \mathbb{R}.$$

(A.1) The flux $f = f(t,x)$ is continuous in $\{(t,x) : t \in (0,+\infty) \setminus G, x \in \mathbb{R}\}$ for some closed set $G \subset \mathbb{R}$ whose Lebesgue measure is 0. Moreover, to each $N \in (0,+\infty)$ there corresponds a function $g_N = g_N(t)$ in $L^\infty_{\text{loc}}(\mathbb{R})$ such that

$$\sup_{|x|\leq N} |f(t,x)| \leq g_N(t) \quad \text{for almost all} \quad t \in (0,+\infty).$$

(A.2) For every bounded subset $V \in \mathbb{R} \times [0,+\infty)$, there exists a positive number $N(V)$ so that, for $(t,x) \in V$, $|y| > N(V)$

$$x.y - g^*(y) - \textstyle\int_0^t f(\tau,y)d\tau < \max_{|z|\leq N(V)} \{x.z - g^*(z) - \textstyle\int_0^t f(\tau,z)d\tau\}.$$

(B.1) The function $u_0 \in L^1_{loc}(\mathbb{R})$ is nondecreasing on $\mathbb{R}$.

Theorem 2.1. *Under the hypotheses* (A.1), (A.2), *and* (B.1), *the function*

$$u(t,x) := \frac{\partial}{\partial x}\Big(g^*(.) + \int_0^t f(\tau,.)d\tau\Big)^*(x), \quad x \in \mathbb{R}, \ t \geq 0,$$

determines a weak solution of Problem (1.1).

Functions with bounded variations are often used in many applications. Next we will present the explicit formula of weak solutions to the problem (1.1) where the initial data are functions with bounded variations. It is well-known that a function with bounded variations can be expressed as a difference of two non-decreasing functions. Let the initial datum u_0 have the form $u_0 = u_1 - u_2$. Precisely, we need the following assumptions:

(A.3) For any bounded set $V \subset \mathbb{R} \times [0,+\infty)$ and $E \subset \mathbb{R}$ there exists a number $N(V,E) > 0$ so that

$$\varphi_\alpha(x,t,p) < \max_{|q|\leq N(V,E)} \varphi_\alpha(x,t,q) \ \text{for} \ (x,t) \in V, \ \alpha \in E, \ |p| > N(V,E),$$

where
$$\varphi_\alpha(x,t,p) \stackrel{\text{def}}{=} \langle p,x\rangle - \left(\int_0^x u_1(y)dy\right)^*(p+\alpha) - \int_0^t f(\tau,p)d\tau.$$

(A.4) The set
$$L_\alpha(x,t) := \{p \in \mathbb{R} \,:\, \varphi_\alpha(x,t,p) = \max_{q\in\mathbb{R}} \varphi_\alpha(x,t,q)\}$$
consists of only one point for $(x,t) \in \mathbb{R}\times(0,+\infty)\setminus\mathcal{Q}$ where $\mathcal{Q}$ is a certain closed set whose 2-dimensional Lebesgue measure is 0 and is independent of $\alpha \in \mathbb{R}$, for every α.

(B.2) The initial function u_0 is given by a difference of two nondecreasing functions:
$$u_0 = u_1 - u_2,$$
where $u_i \in L^1_{loc}(\mathbb{R}\times[0,+\infty)), i = 1,2$, is nondecreasing and $u_2(\pm\infty)$ are finite.

Theorem 2.2. *Under the assumptions* (A.1), (A.3), (A.4), *and* (B.2), *the function*
$$u(t,x) = \frac{\partial}{\partial x}\min_{\alpha\in\mathcal{O}}\left\{\left(\int_0^x u_2(y)dy\right)^*(\alpha) + \max_{p\in\mathbb{R}}\varphi_\alpha(t,x,p)\right\} \quad \textit{on } \mathbb{R}\times[0,+\infty),$$
determines a weak solution of the Cauchy problem (1.1), *where the derivative in x may be understood in the sense of distributions and*
$$\mathcal{O} := \operatorname{dom}\left(\int_0^x u_2(y)dy\right)^*.$$

In the following we will generalize Theorem 2.1 to the case where the initial datum is a (weak) derivative of a minimum of a family of functions in $\mathbb{R}$. More precisely, we assume:

(B.3) Let $\varphi_\alpha, \alpha \in I$, be a family of continuous functions in $\mathbb{R}$ such that:
– For any φ_α, there exists a Lipschitz solution $v_\alpha = v_\alpha(x,t)$ of the Cauchy problem for the same Hamilton–Jacobi equation
$$\begin{aligned} \partial v/\partial t + f(t, \partial v/\partial x) &= 0 \qquad x\in\mathbb{R},\ t>0 \\ v(x,0) &= \varphi_\alpha(x) \qquad x\in\mathbb{R}. \end{aligned} \tag{2.1}$$
– For any bounded subset $V \in \mathbb{R}\times[0,+\infty)$, there are a set $W(V)\subset V$ whose Lebesgue measure is 0, a nonnegative number $M(V)$, and a subset $J(V)$ of I such that all the functions $v_\alpha = v_\alpha(x,t)$ for $\alpha \in J(V)$ are Lipschitz continuous in V with a common Lipschitz constant $M(V)$ and satisfy the equation (2.1) at every point of $V\setminus W(V)$ and that
$$\inf_{\alpha\in I} v_\alpha(x,t) = \min_{\alpha\in J(V)} v_\alpha(x,t)$$
for $(x,t)\in V$.

So we have the following proposition.

Proposition 2.3. *Under the hypothesis* (B.3), *let the initial datum u_0 take the form*

$$u_0(x) = \frac{\partial}{\partial x}\inf_{\alpha\in I}\varphi_\alpha(x), \quad x \in \mathbb{R}.$$

Then the function

$$u(x,t) := \frac{\partial}{\partial x}\inf_{\alpha\in I} v_\alpha(t,x), \quad x \in \mathbb{R},\ t > 0,$$

is a weak solution of the Cauchy problem (1.1).

Now we assume the hypotheses (A.1), (A.2) with φ_α in place of the function g, and that φ_α is a convex function in $\mathbb{R}$. Then, thanks to the results of [11, 12], the Cauchy problem for Hamilton–Jacobi equations (2.1) admits a Lipschitz solution

$$v_\alpha(x,t) = \max_{p\in\mathbb{R}}\{p.x - \varphi^*_\alpha(p) - \int_0^t f(\tau,p)d\tau\}, \quad x \in \mathbb{R},\ t > 0. \tag{2.2}$$

Thus, Proposition 2.3 yields

Theorem 2.4. *Assume that the functions φ_α is finite and convex in $\mathbb{R}$ and that* (A.1) *and* (A.2) *are satisfied where g is replaced by φ_α. In addition, suppose all the hypotheses used in Proposition* 2.3. *are fulfilled Then the Cauchy problem* (1.1) *admits the following function as a weak solution*

$$u(x,t) := \frac{\partial}{\partial x}\inf_{\alpha\in I} v_\alpha(t,x), \quad x \in \mathbb{R},\ t > 0,$$

where v_α is as in (2.2).

Remark. It is worth considering the initial data given by the infimum of a family of nondecreasing functions. However, in Proposition 2.3 (and therefore in Theorem 2.4), we cannot transfer the sign of derivative through the infimum. In fact, if we may exchange the order of these two operations, we would get completely different results. For example, it holds

$$\frac{\partial}{\partial x}\inf\{x,0\} = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases}$$

while

$$\inf\{x', 0'\} = 0.$$

So we let the problem of finding Lax–Oleinhik–type formulas for the Cauchy problem (1.1) with the intimal data being the infimum of a family of nondecreasing functions be open.

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