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Short Communication

On Pseudo-Buchsbaum Modules

Nguyen Tu Cuong and Nguyen Thi Hong Loan
Institute of Mathematics, P. O. Box 631, Bo Ho, Hanoi, Vietnam

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1. Introduction

Throughout this note, R denotes a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and M a finitely generated R-module with $\dim M = d \geq 1$. Let $\underline{x} = (x_1, \ldots, x_d)$ be a system of parameters (s.o.p. for short) of M. Consider the difference between the multiplicity and the length

$$J_M(\underline{x}) = e(\underline{x}; M) - l(M/Q_M(\underline{x})),$$

where $Q_M(\underline{x}) = \bigcup_{t>0} ((x_1^{t+1}, \ldots, x_d^{t+1})M : x_1^t \ldots x_d^t)$ is a submodule of M. It should be mentioned that $J_M(\underline{x})$ gives a lot of informations on the structure of M. For example, if M is a Cohen-Macaulay module, $Q_M(\underline{x}) = (x_1, \ldots, x_d)M$ by [8], therefore $J_M(\underline{x}) = 0$ for all s.o.p. $\underline{x}$ of M. Furthermore, it is known by [2] that $l(M/Q_M(\underline{x}))$ is just the length of generalized fractions defined in [10]. Therefore by [4], if M is a generalized Cohen-Macaulay module, then sup $J_M(\underline{x}) < \infty$, where $\underline{x}$ runs through all s.o.p. of M. In [2] we also showed that if M is a Buchsbaum module, then $J_M(\underline{x})$ takes constant value for all s.o.p $\underline{x}$ of M. Unfortunately, the converse is not true in general. So in [3], we defined a class of *pseudo-Buchsbaum* modules M, in which $J_M(\underline{x})$ is a constant for every s.o.p. $\underline{x}$. The purpose of this short note is to communicate results on pseudo-Buchsbaum modules, whose detailed proofs are given in [3].

2. Pseudo-Buchsbaum Modules

We begin with the following definition.

Definition 2.1. *A R-module M is called a pseudo-Buchsbaum module if $J_M(\underline{x})$ takes constant value for every s.o.p. $\underline{x}$ of M. R is called a pseudo-Buchsbaum ring if it is a pseudo-Buchsbaum module as a module over itself.*

Recall that for each system of parameters $\underline{x} = (x_1, \dots, x_d)$ of M and $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, the difference between multiplicities and lengths

$$J_M(\underline{x}(\underline{n})) = n_1 ... n_d e(\underline{x}; M) - l(M/Q_M(\underline{x}(\underline{n}))),$$

can be considered as a function in $\underline{n}$, where $\underline{x}(\underline{n}) = (x_1^{n_1}, \dots, x_d^{n_d})$. Then it is natural to ask whether $J_M(\underline{x}(\underline{n}))$ is a polynomial for $\underline{n}$ large enough ($\underline{n} \gg 0$ for short)? One has shown in [5] that this function is not a polynomial in $\underline{n}$ for $\underline{n} \gg 0$ in general. But Minh and the first author in [4] showed that this function $J_M(\underline{x}(\underline{n}))$ is non-negative and bounded above by a polynomial of degree $\leq d-2$. Moreover, the least degree of all polynomials in $\underline{n}$ bounding above the function $J_M(\underline{x}(\underline{n}))$ is independent of the choice of system of parameters $\underline{x}$. This numerical invariant is denoted by $pf(M)$. For the convenience, we stipulate that the degree of the zero-polynomial is equal to $-\infty$. Now we recall two notions introduced in [5] as follows: A module M is said to be a *pseudo Cohen-Macaulay* (p.CM for short) or *pseudo generalized Cohen–Macaulay* (p.g.CM for short) if $pf(M) = -\infty$ or $pf(M) \leq 0$, respectively. Then by definition, p.CM modules are pseudo-Buchsbaum and pseudo-Buchsbaum modules are p.g.CM. However, the converse of these statements are not true in general.

From now on, let $0 = \cap N_i$, be a reduced primary decomposition of 0 in M, where N_i is $\mathfrak{p}_i$-primary. Then we set $U_M(0) = \bigcap_{\dim R/\mathfrak{p}_j = d} N_j$, $\overline{M} = \widehat{M}/U_{\widehat{M}}(0)$, where $\widehat{M}$ is the $\mathfrak{m}$-adic completion of M and $J(M) := \sum_{i=1}^{d-1} \binom{d-1}{i-1} l(H^i_{\mathfrak{m}}(M))$. Note that this invariant $J(M)$ may be infinity. But one proved in [5] that M is a pseudo Cohen–Macaulay or pseudo generalized Cohen–Macaulay if and only if $\overline{M}$ is a Cohen–Macaulay or generalized Cohen–Macaulay over the $\mathfrak{m}$-adic completion $\widehat{R}$ of R, respectively, therefore $J(\overline{M}) < \infty$. Moreover, for pseudo-Buchsbaum modules we have the following

Lemma 2.2. *Let M be a pseudo-Buchsbaum module. Then*

$$J_M(\underline{x}) = J(\overline{M}),$$

for every s.o.p. $\underline{x}$ of M.

The following results are basic properties on pseudo-Buchsbaum modules.

Proposition 2.3. *The following statements are true.*

(i) *M is a pseudo-Buchsbaum module if and only if so is $M/H^0_{\mathfrak{m}}(M)$.*

(ii) *If M is a pseudo-Buchsbaum module and $\underline{x} = (x_1, \dots, x_d)$ is reducing s.o.p. on M, then $M/(x_1, \dots, x_i)M$ is a pseudo-Buchsbaum for $i = 1, \dots, d$.*

Proposition 2.4. *M is a pseudo-Buchsbaum module if and only if the $\mathfrak{m}$-adic completion $\widehat{M}$ of M is a pseudo-Buchsbaum module over $\widehat{R}$.*

The concept of *polynomial type* $\mathfrak{p}(M)$ introduced in [1] plays an important role for our studying of pseudo-Buchsbaum modules.

Proposition 2.5. *Let M be a pseudo-Buchsbaum module. Then $\mathfrak{m}H^i_{\mathfrak{m}}(M) = 0$ for $i = \mathfrak{p}(M)+1, \ldots, d-1$, where $\mathfrak{p}(M)$ is the polynomial type of the module M.*

3. The Main Result and Corollaries

The following characterization for pseudo-Buchsbaum modules is the main result of this note.

Theorem 3.1. *Keep all notations in the previous section. Then M is a pseudo-Buchsbaum module if and only if $\overline{M}$ is a Buchsbaum module over the completion $\widehat{R}$.*

In order to prove Theorem 3.1 we had to use a characterization of p.g.CM module in [5], Theorem 2.3 about the monomial property of a u.s.d-sequence in [7] and the following lemmas.

Lemma 3.2. *For every s.o.p. $\underline{x} = (x_1, \ldots, x_d)$ on M,*

$$J_M(\underline{x}) \leq \sum_{i=1}^{d-1} \binom{d-1}{i-1} l(H^i_{\mathfrak{m}}(M)).$$

Lemma 3.3. *The following statements are equivalent:*

(i) *$M/H^0_{\mathfrak{m}}(M)$ is a Buchsbaum module.*

(ii) *M is genralized Cohen–Macaulay and pseudo-Buchsbaum.*

Lemma 3.4. *If M is a pseudo generalized Cohen–Macaulay module, then $J_M(\underline{x}(\underline{n})) = J(\overline{M})$ for $n \gg 0$ and every s.o.p. $\underline{x}$ of M.*

By Theorem 3.1, we see that, the class of pseudo-Buchsbaum modules stricly contains the class of Buchsbaum modules. Moreover, there exists a pseudo-Buchsbaum module M which does not need to be a g.CM-module. On the other hand, there exist g. CM modules which are not pseudo-Buchsbaum modules. The following examples illustrate this.

Example. (1) Let k be a field and X_1, X_2, X_3, X_4 indeterminates. Take

$$A := k[[X_1, \ldots, X_4]]/(X_1, X_2) \cap (X_3, X_4) \cap ({X_1}^2, X_2, X_3).$$

It is easy to see that, A is a pseudo-Buchsbaum ring ($J_A(\underline{x}) = 1$ for every s.o.p. $\underline{x}$ of A) but A is not a g.CM ring.

(2) Let k be a field and $X_1, \ldots, X_n$ indeterminates ($n \geq 2$). Set $R = k[[X_1, \ldots, X_n]]$ and $M = ({X_1}^2, X_2, \ldots, X_n)R$. We have the exact sequence

$$0 \to M \to R \to R/(X_1^2, X_2, \ldots, X_n)R \to 0.$$

Since R is a CM ring and from exact sequence above we have $H^i_{\mathfrak{m}}(M) = 0$, for $i \neq 1, n$ and $H^1_{\mathfrak{m}}(M) \cong R/({X_1}^2, X_2, \ldots, X_n)R$. Therefore M is a g. CM module.

On the other hand, as $mH^1_m(M) \neq 0$, M is not a Buchsbaum module. Moreover, $U_M(0) = 0$, hence $M/U_M(0)$ is not a Buchsbaum module which implies by Theorem 3.1 that M is not a pseudo-Buchsbaum module.

Theorem 3.1 has many consequences. First we note that, the submodule $Q_R(\underline{x})$ is also used for studying the monomial conjecture of Hochster which can be described as follows (see [9]): if $\underline{x} = (x_1, \ldots, x_r)$ is a system of parameters for $R(r := \dim R)$, then for every integer $t \geq 0$, $(x_1 \ldots x_r)^t \notin ({x_1}^{t+1}, \ldots, {x_r}^{t+1})R$. This is equivalent to saying that $R \neq Q_R(\underline{x})$ for every system of parameters $\underline{x}$ of R, i. e., $l(R/Q_R(\underline{x})) \neq 0$. Hochster proved in [9] that this monomial conjecture is true for high powers of system of parameters. If R is a Buchsbaum ring, R satisfies the monomial conjecture (see [6]). Therefore Theorem 3.1 leads to the following consequence.

Corollary 3.5 *If R is a pseudo-Buchsbaum ring then R satisfies the monomial conjecture.*

Next, we are interested in the Buchsbaum property of the canonical module of a pseudo-Buchsbaum module.

Corollary 3.6 *Let M denote a pseudo-Buchsbaum module which has a canonical module K_M. Then K_M is a Buchsbaum module.*

If R is a pseudo-Buchsbaum ring, then $J_R(\underline{y}) = J(\overline{R}) := \sum_{i=1}^{d-1} \binom{d-1}{i-1} l(H^i_m(\overline{R}))$, for every s.o.p. $\underline{y} = (y_1, \ldots, y_r)$ of R ($r :=$ dim R), where $\overline{R} = \widehat{R}/U_{\widehat{R}}(0)$. Hence, by Corollary 3.5 we have

$$e(\underline{y}; R) \geq 1 + J(\overline{R})$$

for every s.o.p $\underline{y}$ of R. It follows that $e(R) \geq 1 + J(\overline{R})$. Combining the results of Yoshida about linearly maximal Buchsbaum modules in [12] with Theorem 3.1 we can easly prove the following consequence.

Corollary 3.7. *Let R be a pseudo-Buchsbaum ring which has a canonical module K_R. The following conditions are equivalent:*

(i) $e(R) = 1 + J(\overline{R})$.

(ii) *K_R is a linear maximal Buchsbaum module.*

Moreover, we know that, if M is a pseudo-Buchsbaum module, then

$$J_M(\underline{x}) = J(\overline{M})$$

for all s.o.p. $\underline{x}$ of M. Hence

$$e(\underline{x}; M) \geq J(\overline{M}).$$

for all s.o.p $\underline{x}$ of M. Therefore $e(M) \geq J(\overline{M})$. In the case the equality holds, we get the following result.

Corollary 3.8. *Suppose that* $\dim M = \dim R$. *Then the following conditions are equivalent:*

(i) *M is a pseudo-Buchsbaum module and*

$$e(M) = J(\overline{M});$$

(ii) *$\overline{M}$ is a linear maximal Buchsbaum $\widehat{R}$-module and*

$$\mu_{\widehat{R}}(\overline{M}) = \sum_{i=0}^{d-1} \binom{d}{i} l(H^i_{\mathfrak{m}}(\overline{M})),$$

where $\mu_{\widehat{R}}(\overline{M})$ denotes the minimal number of generators for $\overline{M}$.

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