Vietnam Journal of Mathematics **30:3** (2002) 201–238

Vietnam Journal
of
MATHEMATICS

Survey

Constructive Methods of Optimization of Dynamical Systems*

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Received July 20, 2001

Abstract. A new approach for solving of the optimal synthesis problem is suggested. This approach, based on principles of the adaptive method of linear programming [10] and constructive methods of optimization, has been developed in Minsk since 90 s. In the framework of this approach in the paper we suggest algorithms of optimization of linear control systems with intermediate state constraints, piecewise linear control systems, and a nonlinear control system. Problems of constructing open-loop and closed-loop solutions are under consideration. Results are illustrated by two examples of control for piecewise linear and nonlinear oscillating systems.

1. Introduction

The first papers dealing with the statement and solutions of the optimal synthesis problem appeared fifty years ago. The problem was solved by engineers, experts in automatic control [5, 18]. Optimal controls obtained by engineers were discontinuous and took only boundary values. That differed them from the known solutions of classical calculus of variations problems solved in other applications in 30–40 s. The solutions of the latter problems allowed one to show potential possibilities of static and dynamic systems. The optimal synthesis problem differs fundamentally from calculus of variations problems. In the

* This work has been done in the framework of State Programme of Fundamental Research "Mathematical Structures 22" under partial support of Belarusian Republican Foundation of Basic Research (grant F99R–002).

terms of control theory, the solutions of calculus of variations problems are called optimal open-loop solutions. Such kind of optimal (open-loop) solutions are not possible to use in real control processes. Real control systems are designed, as a rule, according to a feedback principle. Constructing optimal feedback controls is generally considered as the synthesis of optimal systems. Open-loop controls are functions of time, whereas optimal feedback controls (closed-loop controls) are functions of phase states of control systems, they also are called positional controls.

In 50s, by analogy with calculus of variations mathematicians formulated problems of optimal open - loop control. Further development of the theory of optimal control proceeded according to the way of calculus of variations. A basic result of this development is the Pontryagin maximum principle [22]. The second basic result in mathematical theory of optimal control represents the Bellman dynamic programming [2]. Unlike the maximum principle dealing with optimal open-loop controls, dynamic programming gives a solution to an optimal control problem in the form of optimal feedback. Therefore it does not make sense to compare these two fundamental results. For a long time mathematicians considered dynamic programming having no sufficient basis due to its heuristic nature. Recently strong definitions of the Bellman equations and its solutions have been found by methods of nonsmooth analysis. It means that one can believe that dynamic programming has become rigorously justified theory. But it offers a little help to engineers in effective implementation of dynamic programming for the synthesis of optimal systems because its principal difficulty, known as "curse of dimension", had not been overcome and evidently can not be overcome in principle. To sum up the analysis of contemporary state of the theory of optimal control one can assume that the problem of optimal synthesis stated 50 years ago remains unsolved although outstanding results in very different fields of the theory have been obtained.

The purpose of the paper is to present results on the solution of optimal synthesis problem obtained in Minsk (Belarus) during the last years. These results are based on constructive methods of optimization [7, 11, 12, 20, 21] and a new concept of the solution of the optimal synthesis problem [14-16]. The covered areas are: optimal open-loop and closed-loop controls for linear dynamical systems with intermediate state constraints, for piecewise linear and nonlinear dynamical systems. For optimization of nonlinear control systems, a procedure of asymptotic correction is suggested. Very important problems such as optimal observation, identification, output feedback control, dual optimization, robustness will be considered in a separate paper.

2. Problem Statement

Let X be a bounded set of R^n, $T = [0, t^*]$ be a control interval, $h = t^*/N$ be a quantization period, N be an integer, $T_h = \{0, h, ..., t^* - h\}$. A function $u(t)$, $t \in T$, is said to be a discrete control if $u(t) = u(kh)$, $t \in [kh, (k+1)h[$, $k = 0, \dots, N-1$.

On the set X in the class of discrete controls we consider the following problem

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = f(x) + bu, x(0) = x_0, \tag{1}$$

$$x(t^*) \in X^* = \{x \in R^n : Hx = g\}, \quad |u(t)| \leq 1, \ t \in T.$$

Here $c, b \in R^n$, $g \in R^m$, $H \in R^{m \times n}$, rank $H = m < n$, $x = x(t)$ is an n-vector of the state of the system at an instant t; $u = u(t)$ is a value of a scalar control; $f(x)$, $x \in X$, is an n-vector–function infinitely differentiable in int$X$.

An accessible control $u(t)$, $t \in T$, is said to be an admissible (open-loop) control if $|u(t)| \leq 1$, $t \in T$, and the corresponding trajectory of system (1) satisfies the endpoint constraint $x(t^*) \in X^*$. An admissible control $u^0(t)$, $t \in T$, is said to be an open-loop solution (optimal open-loop control) of problem (1) if it gives the maximal value to the performance index: $c'x^0(t^*) = \max c'x(t^*)$. Here the maximum is calculated on all the admissible controls, $x^0(t)$, $t \in T$, and is an optimal trajectory.

To introduce the concept of the positional solution (optimal feedback) of problem (1) we assume that the state of the control system is known not only at the initial instant $t = 0$ but also at each current instant $\tau \in T_h$ of the control process. Under this assumption we imbed problem (1) into the family of problems

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = f(x) + bu, \ x(\tau) = z, \tag{2}$$

$$x(t^*) \in X^* = \{x \in R^n : Hx = g\}, \quad |u(t)| \leq 1, \ t \in T^\tau = [\tau, t^*],$$

depending on a scalar $\tau \in T_h$ and an n-vector z.

Let $u^0(t|\tau, z)$, $t \in T^\tau$, be an optimal open-loop control of problem (2), X_τ be a set of all $z \in X$ for which there exists an open-loop solution to (2) at a fixed τ.

Definition 1. *A function*

$$u^0(\tau, z) = u^0(\tau|\tau, z), \quad z \in X_\tau, \ \tau \in T_h, \tag{3}$$

is said to be an optimal (discrete) feedback.

According to the introduced definitions, open-loop $u^0(t)$, $t \in T$, and closed-loop $u^0(t, x)$, $x \in X_t$, $t \in T_h$, controls are determined by a priori information on problem (1). So "theoretically" they might be constructed before the beginning of the actual control process. Calculating these solutions analytically (in the explicit and closed, formula form) for problem (1), as a rule, is impossible. Numerical constructing open-loop and closed-loop solutions is based on their tabulation. This operation does not cause any problems when one constructs open-loop solutions even to dynamical system (1) of high order, but tabulation of positional solutions for solving problem (1) with quite high accuracy leads to "the curse of dimension" [2], which seems not to be overcome in the nearest future.

In [15] a detour of overcoming "the curse of dimension" has been suggested. Its essence consists in the following. When solving applied problems, optimal feedback (3) is constructed on the base of mathematical models (1) but is used for controlling actual systems. The latters differ from "ideal" models (1) by inaccuracies of mathematical modelling and during functioning they are affected by disturbances which could not be taken into account in advance (before the beginning of control process). Let the behaviour of an actual system closed by feedback (3) be described by the equation

$$\dot{x} = f(x) + bu^0(t,x) + w, \quad x(0) = x_0, \tag{4}$$

where $w = w(t,x)$, $x \in X$, $t \in T$, is an unknown but bounded disturbance which is a piecewise continuous n-vector function $w(t) = w(t, x(t))$, $t \in T$, along each continuous function $x = x(t)$, $t \in T$.

Under classical definition of a positional solution (with the use of piecewise continuous and measurable admissible controls) there arises a mathematical problem to understand a solution to differential equation (4) the right side of which is, as a rule, discontinuous. Because of this, the classical solution to the equation may not exist, the use of the generalized Filippov solutions does not eliminate the problem completely.

In the paper we consider a trajectory of system (4) as a trajectory of the equation

$$\dot{x} = f(x) + bu^*(t) + w(t), \; x(0) = x_0, \tag{5}$$

under the discrete control

$$u^*(t) = u^0(kh, x(kh)), \quad t \in [kh, (k+1)h[, \; k = 0, \dots, N-1. \tag{6}$$

It is clear that now the problem of existence of the classical solution does not arise.

From (5) it is seen that in any particular control process corresponding to an initial state x_0 and realizing disturbance $w(t)$, $t \in T$, the feedback (2) is not used as a whole. All one needs to know is values of its realization (6) along the continuous curve $x(t)$, $t \in T$.

If at each current instant $\tau \in T_h$ the time of calculating the value $u^*(\tau)$ does not exceed h, we say that the realization of the optimal feedback is constructed in real time.

Definition 2. *Any device able to perform this work is said to be an optimal controller for problem* (1).

Thus the problem of the synthesis of optimal feedbacks is reduced to constructing an algorithm for operating an optimal controller.

Below, an approach to constructing optimal open-loop and closed-loop solutions to problem (1) is described. We begin with general scheme of solving problem (1) and with presentation of results concerning linear control systems with intermediate state constraints to use them in the following constructions. More details for the case of linear control systems can be found in [13].

3. General Scheme of Solving the Problem

When elaborating approximate methods of solving OC problems the method of linearization [4] is often used. However, linear approximations only give satisfactory descriptions of local behaviour of nonlinear systems in a vicinity of certain trajectories, so the field of their use is limited.

One of the natural ways of expanding the field of application of linear optimization methods is the use of piecewise linear approximations. Although after this approximation the problem remains nonlinear, it allows to elaborate effective optimization methods by taking into account the specific character of a piecewise linear model.

Assume that the closure of the set X may be represented as a unification of polyhedral sets $X_1, X_2, \dots, X_p$, such that $\text{int}X_i \cap \text{int}X_j = \emptyset$, $i \neq j$. The function $f(x)$, $x \in X$, in system (1) is replaced by the function $\hat{f}(x)$, $x \in X$, linear on each set X_j, $j = 1, \dots, p$. The number $\delta = \max_{x \in \bar{X}} \|f(x) - \hat{f}(x)\| / \|f(x)\|$ is said to be an accuracy of approximation. Then system (1) can be written in the form

$$\dot{x} = \hat{f}(x) + \delta g(x) + bu, \tag{7}$$

where $g(x) = (f(x) - \hat{f}(x))/\delta$, $x \in X$, and the problem (1) can be solved in two stages:

1) solving the problem of optimization of the piecewise linear system

$$\dot{x} = \hat{f}(x) + bu, \quad x(0) = x_0;$$

2) correcting of the solution to the piecewise linear problem by asymptotic methods.

The accuracy of the solution to problem (1) is determined by both the accuracy of approximation δ and the order of the asymptotic approximation.

The simplest case is when we have quite a small δ for $p = 1$, i.e., on the set X the function $\hat{f}(x)$ is linear and the stage 1 is solving the endpoint OC problem. An effective method of constructing open-loop and closed-loop solutions to this problem has been suggested in [13]. The correction of the solution to the linear problem is performed by an asymptotic method of optimization of quasilinear systems. A method of constructing an asymptotically optimal open-loop control was proposed in [8, 19]. On its base a realization of asymptotically optimal feedback for a quasilinear system with a special nonlinearity was constructed in [9].

A generalization of these methods to the case of piecewise linear and piecewise quasilinear systems allows to construct open-loop and closed-loop solutions to problem (1) with nonlinearities $f(x)$ of a quite general form.

In elaborating details of the suggested approach, the authors proceeded along the path of successive complication of studied problems: at first, the algorithm of solving a linear endpoint OC problem was generalized to the OC problem for a linear system with intermediate state constraints; then it was supplemented with the scheme of storing and transforming an auxiliary information and with the procedure of optimization in parameters for solving the OC problem of a piecewise linear system; and at last, the asymptotic method of optimization of quasilinear systems was generalized to piecewise quasilinear control systems.

4. Optimization of Linear Control Systems with Intermediate State Constraints

OC problems with intermediate state constraints are often used [4] for solving problems with state constraints on the whole control interval. On the other hand, they also arise as auxiliary problems in nonlinear OC problems without state constraints [3]. So effective algorithms of constructing their open-loop and closed-loop solutions are of great practical importance. In this section the approach to solving linear endpoint OC problems suggested in [13] is generalized to the new, more complicated class of OC problems.

4.1. Problem Statement

In the class of discrete controls we consider an OC problem with intermediate state constraints

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \; x(t_*) = x_0, \tag{8}$$
$$g_*(s) \le H(s)x(s) \le g^*(s), \; s \in S; \quad |u(t)| \le 1, \; t \in T.$$

Here $A(t), b(t), t \in T$, are piecewise continuous $n \times n$-matrix and n-vector functions respectively; $g_*(s), g^*(s) \in R^{m(s)}$; $H(s) \in R^{m(s)\times n}$, $s \in S$; $H_{(i)}(s)$ is the ith row of $H(s)$, $i \in I(s) = \{1, \dots, m(s)\}$, $m = \sum_{s\in S} m(s)$; $S = \{s_1, \dots, s_{j^*}\} \subset T_h \cup t^*$ is a set of intermediate instants: $t_* + h < s_1 < \dots < s_{j^*} = t^*$, $|\bar{s} - s| > h$, $\bar{s}, s \in S$; $T(s) = [t_*, s[$, $s \in S$; $J = \{1, \dots, j^*\}$.

The notions of admissible $u(\cdot) = (u(t), \; t \in T)$, optimal $u^0(\cdot)$, ε-optimal $u^\varepsilon(\cdot)$ open-loop controls are introduced in standard way [16]. Each admissible control $u(\cdot)$ is accompanied by a trajectory $x(t)$, $t \in T$, and an output signal $z(s) = H(s)x(s)$, $s \in S$, of control system (8).

Let us describe a method of constructing an optimal open-loop solution to problem (8), which is a dynamic realization of the adaptive method [16] and is originally elaborated for static linear programming problems.

4.2. Support. Optimality Criterion

A support is the main tool of the adaptive method. To define the dynamic analogue of the support for problem (8) we write the latter in the equivalent functional form

$$\sum_{t\in T_h} c(t)u(t) \to \max, \; \tilde{g}_*(s) \le \sum_{t\in T_h} d(s,t)u(t) \le \tilde{g}^*(s), \; s \in S;$$
$$|u(t)| \le 1, \; t \in T_h. \tag{9}$$

Here

$$c(t) = \int_t^{t+h} \psi_c'(\vartheta)b(\vartheta)d\vartheta, \; d(s,t) = \begin{pmatrix} d_i(s,t), \\ i \in I(s) \end{pmatrix} = \begin{cases} \int_t^{t+h} G(s,\vartheta)b(\vartheta)d\vartheta, & s > t, \\ 0, & s \le t, \end{cases}$$

$$\tilde{g}_*(s) = g_*(s) - G(s,t_*)x_0, \ \tilde{g}^*(s) = g^*(s) - G(s,t_*)x_0,$$

$\psi_c(t)$, $t \in T$, is a solution to the adjoint equation

$$\dot{\psi} = -A'(t)\psi \tag{10}$$

with the initial condition

$$\psi(t^*) = c; \tag{11}$$

$G(s,t)$, $t \in T(s)$, is an $m(s)\times n$-matrix function being a solution to the equation

$$\dot{G} = -GA(t), \quad G(s,s) = H(s). \tag{12}$$

For small $h > 0$ problem (9) is a large interval linear programming problem with the specific dense stair matrix. To solve it, standard linear programming methods are ineffective. The further transformations are aimed at elaborating a "dynamic" realization of the adaptive method [16] in which the essence of the elements of problem (9) is taken into account at most.

Following [16], for every $s \in S$ we choose a subset $I_{sup}(s)$ from $I(s)$. Denote $S_{sup} = \{s \in S : I_{sup}(s) \neq \emptyset\}$, $I_{sup} = \{I_{sup}(s), s \in S_{sup}\}$. From the set T_h we choose a subset T_{sup} so that $|T_{sup}| = \sum_{s\in S_{sup}} |I_{sup}(s)|$. Compose the matrix

$$D_{sup} = D(I_{sup}, T_{sup}) = \begin{pmatrix} d_i(s,t), \ t \in T_{sup} \\ i \in I_{sup}(s), \ s \in S_{sup} \end{pmatrix}.$$

Definition 3. *A set $K_{sup} = \{I_{sup}, T_{sup}\}$ with $I_{sup} \neq \emptyset, T_{sup} \neq \emptyset$ is said to be a support if* $\det D_{sup} \neq 0$. *In the case $K_{sup} = \{I_{sup} = \emptyset, T_{sup} = \emptyset\}$ the set K_{sup} is an empty support by definition.*

A support K_{sup} is accompanied by the following elements:

(1) A function of the Lagrange multipliers $\nu(s) = \nu(I(s)|s) = (\nu_i(s), i \in I(s))$, $s \in S$. To construct this function we set $\nu_n(s) = \nu(I_n(s)|s) = 0$, $I_n(s) = I(s)\backslash I_{sup}(s)$. Support components $\nu_{sup} = (\nu_{sup}(s), s \in S_{sup})$, $\nu_{sup}(s) = \nu(I_{sup}(s)|s) = (\nu_i(s), i \in I_{sup}(s))$, $s \in S_{sup}$, are calculated as a solution to the equation

$$D'_{sup}\nu_{sup} = c_{sup},$$

where $c_{sup} = c(T_{sup}) = (c(t), t \in T_{sup})$. In the case of the empty support we set $\nu(s) = 0$, $s \in S$.

(2) A cotrajectory $\psi(t)$, $t \in T$, is a solution to adjoint system (10) with initial condition (11) and with jumps at the intermediate instants

$$\psi(s-0) = \psi(s+0) - H'(s)\nu(s), \ s \in S \ (t^* + 0 = t^*).$$

(3) A cocontrol $\Delta(t) = \int_t^{t+h} \psi'(\vartheta)b(\vartheta)d\vartheta$, $t \in T_h$ ($\Delta(t) = 0$, $t \in T_{sup}$).

(4) A pseudocontrol $\omega(t)$, $t \in T$, and a pseudooutput signal $\zeta(s)$, $s \in S$. To construct them, at first, we set nonsupport values $\omega(t)$, $t \in T_n = T_h\backslash T_{sup}$, of the pseudocontrol:

$$\omega(t) = \text{sign}\Delta(t) \text{ if } \Delta(t) \neq 0; \ \omega(t) \in [-1,1] \text{ if } \Delta(t) = 0, \ t \in T_n, \tag{13}$$

and support values $\zeta_{sup} = (\zeta_i(s), i \in I_{sup}(s), s \in S_{sup})$ of the pseudooutput signal:

$$\zeta_i(s) = g_{*i}(s) \text{ if } \nu_i(s) < 0; \ \zeta_i(s) = g_i^*(s) \text{ if } \nu_i(s) > 0;$$
$$\zeta_i(s) \in [g_{*i}(s), g_i^*(s)] \text{ if } \nu_i(s) = 0, \quad i \in I_{sup}(s), \ s \in S_{sup}.$$

In the case of the empty support we set $\zeta_{sup} = 0$. Support components $\omega_{sup} = \omega(T_{sup}) = (\omega(t), t \in T_{sup})$ of the pseudocontrol are calculated from the system

$$\sum_{t\in T_{sup}} d_i(s,t)\omega(t) + \sum_{t\in T_n} d_i(s,t)\omega(t) = \zeta_i(s) - G'_{(i)}(s,t_*)x_0, \ i \in I_{sup}(s), \ s \in S_{sup}. \tag{14}$$

Nonsupport components $\zeta_n = (\zeta_n(s), s \in S) = (\zeta_i(s), i \in I_n(s), s \in S)$ of the pseudooutput signal are calculated by the formula

$$\zeta_i(s) = G'_{(i)}(s,t_*)x_0 + \sum_{t\in T_h} d_i(s,t)\omega(t), \quad i \in I_n(s), \ s \in S.$$

(5) A pseudotrajectory æ(t), $t \in T$, is a solution to primal equation (8) with the initial state $x(t_*) = x_0$ and with the control $u(t) = \omega(t)$, $t \in T$. The equality $\zeta(s) = H(s)$æ(s), $s \in S$, holds.

(6) A quasicontrol

$$\tilde{\omega}(t) = \begin{cases} \omega(t) & \text{if } |\omega(t)| \le 1; \\ \text{sign}\omega(t) & \text{if } |\omega(t)| > 1, \ t \in T_h. \end{cases}$$

Definition 4. *A support K_{sup} is said to be regular if $\nu_i(s) \neq 0$, $i \in I_{sup}(s)$, $s \in S_{sup}$; $\Delta(t) \neq 0$, $t \in T_n$.*

Denote by $z(s)$, $s \in S$, a pseudooutput signal corresponding to the control $u(\cdot)$: $z_i(s) = H'_{(i)}(s)x(s) = G'_{(i)}(s,t_*)x_0 + \sum_{t\in T_h} d_i(s,t)u(t)$, $i \in I(s)$, $s \in S$.

In accordance with [16] a pair $\{u(\cdot), K_{sup}\}$ of an admissible control and a support is said to be a support control.

Definition 5. *A support control $\{u(\cdot), K_{sup}\}$ is said to be primally nondegenerated if $|u(t)| < 1$, $t \in T_{sup}$; $g_{*i}(s) < z_i(s) < g_i^*(s)$, $i \in I_n(s)$, $s \in S$, and dually nondegenerated if the support K_{sup} is regular.*

Definition 6. *A number*

$$\beta(u(\cdot), K_{sup}) = c'\text{æ}(t^*) - c'x(t^*)$$
$$= \sum_{t\in T_n} \Delta(t)(\omega(t) - u(t)) + \sum_{s\in S_{sup}} \nu'(s)(\zeta(s) - z(s))$$

is called a suboptimality estimate of the support control $\{u(\cdot), K_{sup}\}$.

The following statement holds.

4.3. Maximum Principle

For the optimality of an admissible control $u(t)$, $t \in T_h$, it is necessary and sufficient that such a support $K_{sup} = \{I_{sup}, T_{sup}\}$, whose accompanying elements satisfy the conditions:

(1) the maximum condition for the control:

$$\int_t^{t+h} \psi'(\vartheta)b(\vartheta)d\vartheta\; u(t) = \max_{|u|\le 1} \int_t^{t+h} \psi'(\vartheta)b(\vartheta)d\vartheta\; u, \quad t \in T_n; \tag{16}$$

(2) the transversality condition (the maximum condition for the output signal):

$$\nu'(s)z(s) = \max_{g_*(s)\le z\le g^*(s)} \nu'(s)z, \quad s \in S_{sup}, \tag{17}$$

exists.

A support K_{sup}, which identifies an optimal open-loop control, is called optimal; it is accompanied by optimal elements.

If equalities (16), (17) are satisfied with an accuracy of preassigned $\varepsilon \ge 0$, we get the ε-maximum principle.

Definition 7. *For given $\varepsilon \ge 0$, $\delta \ge 0$ an accessible control $u(t)$, $t \in T$, is said to be an $\varepsilon\delta$-solution to problem (8), if the inequalities*

$$c'x^0(t^*) - c'x(t^*) \le \varepsilon, \quad \max \|\tilde g(s|u(\cdot))\| \le \delta,$$

hold, where $\tilde g(s|u(\cdot)) = (\tilde g_i(s|u(\cdot)),\ i \in I(s))$, $s \in S$; $\tilde g_i(s|u(\cdot)) = \rho(z_i(s), [g_{*i}(s), g_i^*(s)])$; $\rho(c,[a,b]) = \max\{a-c, c-b, 0\}$.

In the case when a pseudocontrol and a pseudooutput signal constructed by K_{sup} satisfy the inequalities

$$|\omega(t)| \le 1,\ t \in T_{sup};\ \ g_{*i}(s) \le \zeta_i(s) \le g_i^*(s),\ i \in I_n(s),\ s \in S, \tag{18}$$

the pseudocontrol $\omega(t)$, $t \in T_h$, is an optimal control of problem (8). If for a given $\delta \ge 0$ a quasicontrol constructed by K_{sup} satisfies the inequality

$$\max \|\tilde g(s|\tilde\omega(\cdot))\| \le \delta, \tag{19}$$

then the quasicontrol $\tilde\omega(t)$, $t \in T_h$, is a 0δ-solution to problem (8).

For constructing optimal and suboptimal open-loop controls, the primal and dual methods have been elaborated by the authors. An iteration of the primal method [16] is a change of the "old" support control to a "new" one: $\{u(\cdot), K_{sup}\} \longrightarrow \{\bar u(\cdot), \bar K_{sup}\}$ so that $\beta(\bar u(\cdot), \bar K_{sup}) \le \beta(u(\cdot), K_{sup})$. In the dual method, the information on an admissible control is not used, and in the course of iterations, the transformation of the support is performed with the purpose to ensure inequalities (18) or (19).

Let us dwell on the dual method of constructing an optimal (suboptimal) open-loop control to problem (18), just as this method is used for optimizing piecewise linear systems and for constructing positional controls.

4.4. Dual Method

Denote $T(j) = \{s_{j-1}, \dots, s_j - h\}$, $T_{sup}(j) = T(j) \cap T_{sup}$, $T_n(j) = T(j) \setminus T_{sup}(j)$, $s_j \in S$, $j \in J$, $s_0 = t_*$. To simplify the notations we assume that

$$\Delta(t-h)\Delta(t+h) < 0 \text{ if } s_{j-1} < t < s_j - h,\ t \in T_{sup}(j); \tag{20}$$
$$\Delta(s+h) \ne 0 \text{ if } s \in T_{sup}; \quad \Delta(s-2h) \ne 0 \text{ if } s-h \in T_{sup},\ s \in S.$$

(The general case is considered in [11]).

Before performing iterations of the dual method it is necessary to prepare the required information according to Sec. 3.

If an initial support K_{sup} is non-empty, then with the use of $|T_{sup}|$ microprocessors, functioning in parallel, the work of one microprocessor for forming D_{sup}, does not exceed one integration of the primal or adjoint system on interval T.

To calculate the cocontrol we present it in the form

$$\Delta(t) = \int_t^{t+h} \Big(\psi_c'(\vartheta) - \sum_{s \in S_{sup}, s>t} \nu'(s) G(s,\vartheta)\Big) b(\vartheta) d\vartheta, \; t \in T_n .$$

If $m < n$, then using $m+1$ processors we calculate the functions $\psi_c(t)$, $G(s,t)$, $t \in T_h$, $s \in S$, $s > t$, and the values $\Delta(t)$, $t \in T_n$. If $m > n$, then for calculating $\psi_c(t)$, $G(s,t)$, $t \in T_h$, $s \in S$, it might be more economical to calculate matrices $F^{-1}(t)$, $t \in T_h : \dot{F}^{-1} = -F^{-1}A(t)$, $F^{-1}(0) = E$, by n processors. In doing so the fundamental matrix $F(s)$, $s \in S$:

$$\dot{F} = A(t)F, \quad F(0) = E,$$

has been constructed and stored in advance, in parallel with forming the matrix D_{sup}. Then

$$G(s,t) = H(s)F(s)F^{-1}(t), \qquad (21)$$
$$\psi_c(t) = c'F(s)F^{-1}(t), \; t \in T_h, \; s \in S, \; s > t.$$

Instants $t \in T_n \backslash S$ for which $\Delta(t-h)\Delta(t) < 0$, are said to be nonsupport zeros of the cocontrol. A set of nonsupport zeros of the cocontrol is denoted by $T_{no} = \bigcup_{j=1}^{j^*} T_{n0}(j)$.

While analyzing the behaviour of the cocontrol at the right end of the interval $T_j = [s_{j-1}, s_j[$ we use the notation $s_j - 0 \in T_j$, $j \in J$, and at the left end of T_{j+1} we use the notation $s_j + 0 \in T_{j+1}$. Let $T_{sn}(j) = T_{sup}(j) \cup T_{n0}(j) \cup \{s_{j-1}, s_j - 0\} = \{t_k(j), k \in K(j) \cup k(j)+1\}$, $K(j) = \{0, 1, \ldots, k(j)\}$, $T_{sn} = \bigcup_{j=1}^{j^*} T_{sn}(j)$. Denote by $T_k(j)$, $k \in K(j)$, the intervals of constant sign of the cocontrol on the set $T(j)$:

$$T_k(j) = \{t_{*k}(j) = t_k(j), t_k(j)+h, \ldots, t_k^*(j) = t_{k+1}(j) - h\}$$
$$\text{if } t_k(j) \notin T_{sup};$$
$$T_k(j) = \{t_{*k}(j) = t_k(j)+h, t_k(j)+2h, \ldots, t_k^*(j) = t_{k+1}(j) - h\}$$
$$\text{if } t_k(j) \in T_{sup}.$$

If $s_j - h \in T_{sup}(j)$, we set $T_{k(j)}(j) = \emptyset$. Introduce the numbers

$$\gamma_j = \begin{cases} \text{sign}\Delta(s_{j-1}) & \text{if } s_{j-1} \notin T_{sup}; \\ \text{sign}\Delta(s_{j-1}+h) & \text{if } s_{j-1} \in T_{sup}, \; j \in J, \end{cases}$$

and the vectors

$$p_k^j(s) = \sum_{t \in T_k(j)} d(s,t), \; k = 0, \ldots, k(j), \; j \in J; \; s_j < s, \; s \in S,$$

$$p(s)=\sum_{j\in J:s_j\leq s}\gamma_j\sum_{k=0}^{k(j)}(-1)^k p_k^j(s)+G(s,t_*)x_0,\ s\in S;$$

$$p=\begin{pmatrix} p(s), \\ s\in S \end{pmatrix},\quad p_{sup}=\begin{pmatrix} p_i(s), \\ i\in I_{sup}(s), s\in S \end{pmatrix}$$

Then equation (14) for calculating ω_{sup} is transformed to the form

$$D_{sup}\omega_{sup}=\zeta_{sup}-p_{sup}, \tag{22}$$

and the expression (15) for calculating ζ_n takes the form

$$\zeta_i(s)=p_i(s)+\sum_{t\in T_{sup}(s)}d_i(s,t)\omega(t),\ \ i\in I_n(s),\ s\in S, \tag{23}$$

where $T_{sup}(s)=T_{sup}\cap T(s)$.

Denote

$$d(t)=\begin{pmatrix} d(s,t), \\ s\in S \end{pmatrix},\quad D_{|sup|}=(d(t),\ t\in T_{sup}).$$

Assume that besides the parameters $A(t)$, $b(t)$, $t\in T$; $H(s)$, $g_*(s)$, $g^*(s)$, $s\in S$; c of the problem the following information are available at the beginning of an iteration:
1) a support K_{sup}; 2) a set of nonsupport zeros T_{n0}; 3) a matrix $D_{|sup|}$; 4) matrices $G(s,t)$, $t\in T_{sn}$, $s\in S$; $s>t$, vectors $\psi_c(t)$, $t\in T_{sn}$; 5) numbers γ_j, $j\in J$; 6) vector p; 7) support values ν_{sup} of the function of the Lagrange multipliers.

Calculate ω_{sup}, ζ_n according to (22), (23). Set $\rho_i(s)=\rho(\zeta_i(s),[g_{*i}(s),g_i^*(s)])$, $i\in I_n(s)$, $s\in S$; $\rho(t)=\rho(\omega(t),[-1;1])$, $t\in T_{sup}$. Calculate

$$\begin{aligned} \rho^0&=\max\{\rho(t^0),\rho_{i^0}(s^0)\},\\ \rho(t^0)&=\max_{t\in T_{sup}}\rho(t),\\ \rho_{i^0}(s^0)&=\max_{i\in I_n(s),\,s\in S}\rho_i(s). \end{aligned} \tag{24}$$

If $\rho^0=0$, we have the optimal control

$$u^0(t)=\begin{cases} (-1)^k\gamma_j, & t\in T_k(j),\ k=0,\dots,k(j),\ j\in J,\\ \omega(t), & t\in T_{sup}. \end{cases}$$

Otherwise we calculate a quasioutput signal $\tilde{\zeta}_i(s)$, $i\in I(s)$, $s\in S$, check inequality (19). If it holds, $\tilde{\omega}(\cdot)$ is a 0δ-solution to problem (8), otherwise we make the change of the support $K_{sup}\longrightarrow\bar{K}_{sup}$ so that the suboptimality estimate of a support [16] decreases.

A general scheme of an algorithm of constructing a new support $\bar{K}_{sup}$ consists in the following:

1. A variation $\Delta\nu$ of the function of the Lagrange multipliers is calculated;
2. A rate of the change of the dual performance index is calculated;
3. A short step σ along the direction $\Delta\nu$ providing a new zero of the varied cocontrol or zero value for a support component of the Lagrange vector is calculated. This step causes a positive jump of the dual performance index;

4. The information stored in the computer memory is transformed;
5. Repeat operations 2—4 until the value of the rate of the change of the dual performance index reaches a nonnegative value. The corresponding value σ^* of the step is called a long step;
6. The support T_{sup} is changed according to the realized combination of ρ^0 and σ^* from the four possible combinations.

The main idea of the proposed realization concerns with operations 3, 4 of the above scheme. Below we describe how to calculate the required data fast and how to transform corresponding stored information.

Investigate each of the two possible situations: 1) $\rho^0 = \rho(t^0)$; 2) $\rho^0 = \rho_{i^0}(s^0)$.

(1) $\rho^0 = \rho(t^0)$. This means that instant t^0 will be eliminated from the set T_{sup}. Let $t^0 \in T(j^0)$.

We begin constructing a new support $\bar{K}_{sup}$ from the forming of the variation $\Delta\nu(s)$, $s \in S$, of the function of the Lagrange multipliers: $\Delta\nu_n(s) = 0$, $s \in S$; $\Delta\nu_{sup}$ is calculated as a solution to the equation

$$-D'_{sup}\Delta\nu_{sup} = \Delta\delta_{sup}$$

where $\Delta\delta_{sup} = (\Delta\delta(t),\ t \in T_{sup})$, $\Delta\delta(t^0) = \mathrm{sign}\bar{u}(t^0)$, $\Delta\delta(t) = 0$, $t \in T_{sup}\backslash t^0$.

We introduce the following functions:

$$\nu_{sup}(s,\sigma) = \nu_{sup}(s) + \sigma\Delta\nu_{sup}(s), \ s \in S_{sup}, \ \sigma \geq 0,$$

(the varied function of the Lagrange multipliers);

$$\Delta\delta(t) = -\sum_{s\in S_{sup}: s>t} \Delta\nu'_{sup} d(s,t), \ t \in T_n \qquad (25)$$

(variation of the cocontrol)

$$\Delta(t,\sigma) = \Delta(t) + \sigma\Delta\delta(t), \ t \in T_h, \ \sigma \geq 0 \ \text{(varied cocontrol).} \qquad (26)$$

Further we assume that

$$\Delta\delta(t_{*k}(j))\Delta\delta(t^*_{k-1}(j)) > 0 \text{ if } t_{*k}(j) \in T_{n0}(j), \ k = 1, \ldots, k(j), \ j \in J.$$

According to [16], the rate of the change of the performance index of the problem dual to (8) is a piecewise constant function of σ. In a small right-side neighborhood of the point $\sigma = 0$ this rate equals $\alpha^1 = -|\omega(t^0) - \bar{u}(t^0)| < 0$. This value remains when increasing σ from $\sigma = 0$ to $\sigma = \sigma^1 > 0$, until either a new zero of the varied cocontrol appears: $\Delta(t(\sigma^1), \sigma^1) = 0$, or a component of the varied function of the Lagrange multipliers becomes zero: $\nu_{i(\sigma^1)}(s(\sigma^1), \sigma^1) = 0$. When $\sigma = \sigma^1$, the rate of the change of the dual performance index makes a positive jump. For changing a support, a sequence of short steps σ^l, $l = 1, \ldots, l^*$, when the rate makes jumps, is formed. We calculate a long step $\sigma^* = \sigma^{l^*}$, after which the rate of the change of the dual performance index becomes nonnegative. With the direct realization of the adaptive method [16] for detecting σ^l, $l = 1, \ldots, l^*$, it is necessary to calculate steps $\sigma(t)$ for each $t \in T_n$. This requires enormous computational resources. In the proposed realization, it is taken into account that with increasing σ a new zero of the varied cocontrol can appear only in one of the following ways:

1. At the point t^0+h or t^0-h as a result of movement of $t^0(\sigma)$, $\sigma \geq 0$: $\Delta(t^0(\sigma),\sigma)=0$, $t^0(0)=t^0$.
2. At a point t or $t-h$ as a result of movement of a point $t(\sigma)$, $\sigma \geq 0$, $t \in T_{n0}$.
3. At the point $s=s_{j-1} \in S\backslash t^* \cup t_*$ as a result of arising at the left end of the interval T_j, $j \in J$.
4. At the point $s=s_j \in S$ as a result of arising at the right end of the interval T_j, $j \in J$.
5. Inside any interval T_j from a stationary point of the function $\tilde{\Delta}(t)=(\psi_c'(t)-\sum_{s\in S_{sup}:s>t}\nu'(s)G(s,t))b(t)$, $t \in T$.

Thus, if we omit from consideration the case 5, which is studied according to the below scheme only not for the function $\tilde{\Delta}(t)$, $t \in T$, but for its derivative, it is sufficient to investigate the behaviour of the function $\Delta(t,\sigma)$ not on the whole set T_n but only for cases 1–4.

Form the following information

$$\sigma(t^0),\ \tau(t^0),\ k^0;\quad \sigma(t),\ \tau(t),\ t\in T_{n0};\quad \sigma(s+0),\ s\in S\cup t_*\backslash t^*; \qquad (27)$$
$$\sigma(s-0),\ s\in S;\ \ \sigma_i(s),\ i\in I_{sup}(s),\ s\in S,$$

where the steps $\sigma(t)$, $\sigma(s\pm 0)$ correspond to cases 1–4, and the steps $\sigma_i(s)$, $i\in I_{sup}(s)$, $s\in S$, characterize arising zero of the varied function of the Lagrange multipliers: $\nu_i(s,\sigma_i(s))=0$. The numbers $\tau(t)$ show the direction of movement of the zero of function (26) when σ increases; k^0 is an index of instant t^0 in $T_{sn}(j^0)$.

The elements (27) are calculated in the following way:

$$\begin{array}{lll}
\sigma(t^0)=-\Delta(t^0-h)/\Delta\delta(t^0-h), & \tau(t^0)=-1, & \text{if } (-1)^{k^0}\gamma_{j^0}\Delta\delta(t^0)>0;\\
\sigma(t^0)=-\Delta(t^0+h)/\Delta\delta(t^0+h), & \tau(t^0)=1, & \text{if } (-1)^{k^0}\gamma_{j^0}\Delta\delta(t^0)<0;\\
\sigma(t)=-\Delta(t-h)/\Delta\delta(t-h), & \tau(t)=-1, & \text{if } \Delta(t-h)\Delta\delta(t-h)<0;\\
\sigma(t)=-\Delta(t)/\Delta\delta(t), & \tau(t)=1, & \text{if } \Delta(t)\Delta\delta(t)<0,\ t\in T_{n0},
\end{array}$$

$$\sigma(s+0)=\begin{cases}-\Delta(s)/\Delta\delta(s) & \text{if } \Delta(s)\Delta\delta(s)<0;\\ +\infty & \text{if } \Delta(s)\Delta\delta(s)\geq 0,\ s\in S\cup t_*\backslash t^*;\end{cases}$$

$$\sigma(s-0)=\begin{cases}-\Delta(s-h)/\Delta\delta(s-h) & \text{if } \Delta(s-h)\Delta\delta(s-h)<0;\\ +\infty & \text{if } \Delta(s-h)\Delta\delta(s-h)\geq 0,\ s\in S;\end{cases}$$

$$\sigma_i(s)=\begin{cases}-\nu_i(s)/\Delta\nu_i(s) & \text{if } \nu_i(s)\Delta\nu_i(s)<0;\\ +\infty & \text{if } \nu_i(s)\Delta\nu_i(s)\geq 0;\quad i\in I_{sup}(s)\ s\in S_{sup}.\end{cases}$$

The steps $\sigma(t)$, $\sigma(s\pm 0)$ are easily calculated with the use of the information stored in computer memory: $G(s,t)$, $t\in T_{sn}$, $s\in S$; $s>t$; $\psi_c(t)$, $t\in T_{sn}$. To calculate each value σ (27) it is sufficient to integrate systems (10), (12) on intervals of the length no more than $3h$.

Before calculating short steps we make a "zero" step for transforming the stored information which is caused by deleting the element t^0 from the set T_{sup}.

The "zero" step

1. $t^0>s_{j^0-1}$, $\tau(t^0)=-1$. Set $T_{n0}^1=T_{n0}\cup t^0$, $p^1(s)=p(s)+(-1)^{k^0}\gamma_{j^0}d(s,t^0)$, $s\in S$, $s\geq s_{j^0}$.

2. $t^0 < s_{j^0} - h$, $\tau(t^0) = 1$. Set $T^1_{n0} = T_{n0} \cup t^0 + h$, $p^1(s) = p(s) - (-1)^{k^0}\gamma_{j^0}d(s,t^0)$, $s \in S$, $s \geq s_{j^0}$, store $G(s,t^0+h)$, $\psi_c(t^0+h)$, $s \in S$, $s \geq s_{j^0}$; if $t^0 > s_{j^0-1}$, delete the values $G(s,t^0)$, $\psi_c(t^0)$, $s \in S$, $s \geq s_{j^0}$, from computer memory. If $t^0 = s_{j^0-1}$, we set $\gamma^1_{j^0} = -\gamma_{j^0}$, $k^1(j^0) = k(j^0)+1$, and renumerate elements of the set $T_{sn}(j^0)$.

3. $t^0 = s_{j^0} - h$, $\tau(t^0) = 1$. This situation is interpreted as disappearance of a cocontrol zero through the right end of the interval T_{j^0}. Set $T^1_{n0} = T_{n0}$, $p^1(s) = p(s) - (-1)^{k^0}\gamma_{j^0}d(s,t^0)$, $s \in S$, $s \geq s_{j^0}$, $k^1(j^0) = k(j^0) - 1$.

4. $t^0 = s_{j^0-1}$, $\tau(t^0) = -1$ (disappearance of a cocontrol zero through the left end of the interval T_{j^0}). Set $T^1_{n0} = T_{n0}$, $p^1(s) = p(s) + \gamma_{j^0}d(s,s_{j^0-1})$, $s \in S$.

Suppose that $l-1$ short steps have been performed ($\sigma^0 = 0$) and for the lth step the following information has been prepared: 1) sets $T^l_{n0} = \bigcup_{j=1}^{j^*} T^l_{n0}(j)$; $T^l = \bigcup_{j=1}^{j^*} T^l(j)$, $T^l(j) = T^l_{n0}(j)\bigcup\{s_{j-1}+0, s_j - 0\}$; 2) numbers $\sigma^l(t)$, $t \in T^l$; $\sigma^l_i(s)$, $i \in I_{sup}(s)$, $s \in S_{sup}$; 3) numbers $\gamma^l_j, j \in J$; 4) vector p^l; 5) values $G(s,t)$, $\psi_c(t)$, $t \in T^l_{n0}(j)\bigcup s_{j-1}$, $j \in J$; 6) the rate α^l of the change of the dual performance index.

To simplify notations we assume that all the numbers $\sigma^l(t)$, $t \in T^l$; $\sigma^l_i(s)$, $i \in I_{sup}(s)$, $s \in S_{sup}$, are different except, possibly, a pair of numbers $\sigma^l(t), \sigma^l(t+h)$ for certain t, $t+h \in T(j^l)$.

The lth step

Calculate the lth short step

$$\sigma^l = \min\{\sigma(t^l), \sigma_{i^l}(s^l)\}. \tag{28}$$

While $\sigma = \sigma^l$, the rate of the change of the dual performance index gets a positive increment [16]

$$\Delta\alpha^l = \begin{cases} 2|\Delta\delta(t^l)| & \text{if } \sigma^l = \sigma(t^l); \\ (g^*_{i^l}(s^l) - g_{*i^l}(s^l))|\Delta\nu_{i^l}(s^l)| & \text{if } \sigma^l = \sigma_{i^l}(s^l), \end{cases}$$

and in the right-side of the point $\sigma = \sigma^l$ it becomes equal to $\alpha^{l+1} = \alpha^l + \Delta\alpha^l$.

If the inequality

$$\alpha^{l+1} \geq 0 \tag{29}$$

holds, we pass to the final step. Otherwise, the information 1)–6) is changed for the next ($(l+1)$th) short step.

We distinguish two possibilities: a) $\sigma^l = \sigma(t^l)$; b) $\sigma^l = \sigma_{i^l}(s^l)$.

(a) Let $t^l \in T(j^l)$; k^l be an index of instant t^l in $T^l_{sn}(j^l)$. To simplify the notations we omit the cases $s_{j-1}, s_j - h \in T_{sup}\backslash t^0$, $j \in J$, for which the described below scheme of jump of a mobile zero over a support zero is used.

The information stored in computer memory is transformed depending on the following situations.

A. Step $\sigma^l = \sigma(t^l)$ is reached for the only point $t^l \in T^l_{sn}(j^l)$.

A.1. $t^l + h \in T_{sup}(j^l)\backslash t^0$, $\tau(t^l) = 1$ or $t^l - 2h \in T_{sup}(j^l)\backslash t^0$, $\tau(t^l) = -1$.

As for a new support control $\{\bar{u}(\cdot), \bar{K}_{sup}\}$, the assumption (20) remains so while increasing $\sigma > \sigma^l$ the jump of the mobile zero over the support zero will take place in the course of the iteration.

A.1.1. $t^l + h \in T_{sup}(j^l)\backslash t^0$, $\tau(t^l) = 1$. Put $T_{n0}^{l+1} = (T_{n0}^l \backslash t^l) \cup t^l + 2h$, $p^{l+1}(s) = p^l(s) - 2(-1)^{k^l}\gamma_{j^l}^l d(s,t^l)$, $s \in S$, $s \geq s_{j^l}$, instead of $G(s,t^l)$, $\psi_c(t^l)$ we store $G(s,t^l+2h)$, $\psi_c(t^l+2h)$, $s \in S$, $s \geq s_{j^l}$. Instead of the step $\sigma(t^l)$ we calculate $\sigma(t^l+2h) = -\Delta(t^l+2h)/\Delta\delta(t^l+2h)$.

A.1.2. $t^l - 2h \in T_{sup}(j^l)\backslash t^0$, $\tau(t^l) = -1$. Set $T_{n0}^{l+1} = (T_{n0}^l \backslash t^l) \cup t^l - 2h$, $p^{l+1}(s) = p^l(s) + 2(-1)^{k^l}\gamma_{j^l}^l d(s,t^l)$, $s \in S$, $s \geq s_{j^l}$, delete the values $G(s,t^l)$, $\psi_c(t^l)$, $s \in S$, $s \geq s_{j^l}$; renumerate the points of the set $T_{sn}(j^l)$. Instead of the step $\sigma(t^l)$ we calculate $\sigma(t^l - 2h) = -\Delta(t^l - 3h)/\Delta\delta(t^l - 3h)$.

If situation A.1 does not realize, we analyze the location of the point t^l.

A.2. $t^l \in T_{n0}^l(j^l)$, $\tau(t^l) = 1$. Put $T_{n0}^{l+1} = (T_{n0}^l \backslash t^l) \cup t^l + h$, $p^{l+1}(s) = p^l(s) - 2(-1)^{k^l}\gamma_{j^l}^l d(s,t^l)$, $s \in S$, $s \geq s_{j^l}$, instead of $G(s,t^l)$, $\psi_c(t^l)$ we store $G(s,t^l+h)$, $\psi_c(t^l+h)$, $s \in S$, $s \geq s_{j^l}$; renumerate the points of the set $T_{sn}(j^l)$. Instead of the step $\sigma(t^l)$ we calculate $\sigma(t^l+h) = -\Delta(t^l+h)/\Delta\delta(t^l+h)$.

A.3. $t^l \in T_{n0}^l(j^l)$, $\tau(t^l) = -1$. Put $T_{n0}^{l+1} = (T_{n0}^l \backslash t^l) \cup t^l - h$, $p^{l+1}(s) = p^l(s) + 2(-1)^{k^l}\gamma_{j^l}^l d(s,t^l - h)$, $s \in S$, $s \geq s_{j^l}$, instead of $G(s,t^l)$, $\psi_c(t^l)$ we store $G(s,t^l - h)$, $\psi_c(t^l - h)$, $s \in S$, $s \geq s_{j^l}$. Instead of the step $\sigma(t^l)$ we calculate $\sigma(t^l - h) = -\Delta(t^l - 2h)/\Delta\delta(t^l - 2h)$.

A.4. $t^l = s_{j^l-1} + 0$ (arising a new zero of function (26) at the left end of interval T_{j^l}) . Put $T_{n0}^{l+1} = T_{n0}^l \cup s_{j^l-1} + h$, $p^{l+1}(s) = p^l(s) - 2\gamma_{j^l}^l d(s,s_{j^l-1})$, $s \in S$, $s \geq s_{j^l}$; $\gamma_{j^l}^{l+1} = -\gamma_{j^l}^l$; $\tau(t^l) = 1$; $k^{l+1}(j^l) = k^l(j^l) + 1$; renumerate the points of the set $T_{sn}(j^l)$. Calculate and store $G(s,s_{j^l-1}+h)$, $\psi_c(s_{j^l-1}+h)$, $s \in S$, $s \geq s_{j^l}$, calculate a new step $\sigma(t^l) = -\Delta(s_{j^l-1}+h)/\Delta\delta(s_{j^l-1}+h)$.

A.5. $t^l = s_{j^l} - 0$ (arising a new zero of function (26) at the right end of interval T_{j^l}). Put $T_{n0}^{l+1} = T_{n0}^l \cup s_{j^l} - h$, $p^{l+1}(s) = p^l(s) - 2(-1)^{k^l(j^l)}\gamma_{j^l}^l d(s,s_{j^l} - h)$, $s \in S$, $s \geq s_{j^l}$; $\tau(t^l) = -1$; $k^{l+1}(j^l) = k^l(j^l) + 1$; renumerate the points of the set $T_{sn}^{l+1}(j^l)$. Calculate and store $G(s,s_{j^l} - h)$, $\psi_c(s_{j^l} - h)$, $s \in S$, $s \geq s_{j^l}$, calculate a new step $\sigma(t^l - h) = -\Delta(t^l - 2h)/\Delta\delta(t^l - 2h)$.

B. Let the step σ^l in formula (28) is reached for the two points t^l, $t^l + h \in T_{n0}(j^l)$: $\sigma(t^l) = \sigma(t^l + h)$.

B.1. $t^l = s_{j^l-1} + 0$ (disappearance of the zero of function (26) through the left end of interval T_{j^l}). Put $T_{n0}^{l+1} = T_{n0}^l \backslash s_{j^l-1} + h$, $p^{l+1}(s) = p^l(s) - 2(-1)^{k^l(j^l)}\gamma_{j^l}^l d(s,s_{j^l-1})$, $s \in S$, $s \geq s_{j^l}$; $\gamma_{j^l}^{l+1} = -\gamma_{j^l}^l$; $k^{l+1}(j^l) = k^l(j^l) - 1$; renumerate the points of the set $T_{sn}(j^l)$. Delete $G(s,s_{j^l-1}+h)$, $\psi_c(s_{j^l-1}+h)$, $s \in S$, $s \geq s_{j^l}$, from the computer memory.

B.2. $t^l + h = s_{j^l} - 0$ (disappearance of the zero of function (26) through the right end of interval T_{j^l}). Put $T_{n0}^{l+1} = T_{n0}^l \backslash s_{j^l} - h$, $p^{l+1}(s) = p^l(s) - 2(-1)^{k^l(j^l)}\gamma_{j^l}^l d(s,s_{j^l} - h)$, $s \in S$, $s \geq s_{j^l}$; $k^{l+1}(j^l) = k^l(j^l) - 1$; renumerate the points of the set $T_{sn}(j^l)$. Delete $G(s,s_{j^l} - h)$, $\psi_c(s_{j^l} - h)$, $s \in S$, $s \geq s_{j^l}$, from the computer memory.

B.3. t^l, $t^l + h \in T_{n0}^l$ (sticking together and disappearance of two zeros of function (26) at point t^l). Put $T_{n0}^{l+1} = T_{n0}^l \backslash \{t^l, t^l + h\}$; $p^{l+1}(s) = p^l(s) - 2(-1)^{k^l(j^l)}\gamma_{j^l}^l d(s,t^l)$, $s \in S$, $s \geq s_{j^l}$; $k^{l+1}(j^l) = k^l(j^l) - 2$; renumerate the points of the set $T_{sn}(j^l)$. Delete $G(s,t^l)$, $\psi_c(t^l)$, $G(s,t^l+h)$, $\psi_c(t^l+h)$, $s \in S$, $s \geq s_{j^l}$, from the computer memory.

The rest of the information for the $(l+1)$th step is not changed.

(b) $\sigma^l = \sigma_{i^l}(s^l)$. This means that when $\sigma > \sigma^l$ the sign of $\nu_{i^l}(s^l,\sigma)$ will be changed. Put $\sigma_{i^l}(s^l) = \infty$ and pass to the $(l+1)$th step without any changes in the stored information.

The final step

The transition to the final step is performed after the lth step when the inequalities

$$\alpha^l < 0, \quad \alpha^{l+1} \geq 0 \tag{30}$$

are realized. The final step is performed to form the information for the next iteration.

In the case (a) $\sigma^l = \sigma(t^l)$ by virtue of assumption (20) the conditions of the transition to the final step may occur in situations A.2–A.5, B.1, B.2.

A.2, B.2. Set $\bar{T}_{sup} = T_{sup} \backslash t^0 \cup t^l$, $\bar{T}_{n0} = T^l_{n0} \backslash t^l$, $\bar{p}(s) = p^l(s) - (-1)^{k^l} \gamma^l_{j^l} d(s,t^l)$, $s \in S$, $s \geq s_{j^l}$.

A.3. Set $\bar{T}_{sup} = T_{sup} \backslash t^0 \cup t^l - h$, $\bar{T}_{n0} = T^l_{n0} \backslash t^l$, $\bar{p}(s) = p^l(s) + (-1)^{k^l} \gamma^l_{j^l} d(s,t^l - h)$, $s \in S$, $s \geq s_{j^l}$, instead of $G(s,t^l)$, $\psi_c(t^l)$ we store $G(s,t^l-h)$, $\psi_c(t^l-h)$, $s \in S$, $s \geq s_{j^l}$.

A.4. Set $\bar{T}_{sup} = T_{sup} \backslash t^0 \cup s_{j^l-1}$, $\bar{T}_{n0} = T^l_{n0}$, $\bar{p}(s) = p^l(s) - \gamma^l_{j^l} d(s, s_{j^l-1})$, $s \in S$, $s \geq s_{j^l}$.

A.5. Set $\bar{T}_{sup} = T_{sup} \backslash t^0 \cup s_{j^l} - h$, $\bar{T}_{n0} = T^l_{n0}$, $\bar{p}(s) = p^l(s) - (-1)^{k^l(j^l)} \gamma^l_{j^l} d(s, s_{j^l} - h)$, $s \in S$, $s \geq s_{j^l}$; $\bar{k}(j^l) = k^l(j^l) + 1$, store $G(s, s_{j^l} - h)$, $\psi_c(s_{j^l} - h)$, $s \in S$, $s \geq s_{j^l}$.

B.1. Set $\bar{T}_{sup} = T_{sup} \backslash t^0 \cup s_{j^l-1}$, $\bar{T}_{n0} = T^l_{n0} \backslash s_{j^l-1} + h$, $\bar{p}(s) = p^l(s) - \gamma^l_{j^l} d(s, s_{j^l-1})$, $s \in S$, $s \geq s_{j^l}$; $\bar{\gamma}_{j^l} = \gamma^l_{j^l}$; $\bar{k}(j^l) = k^l(j^l)$; delete the values $G(s, s_{j^l-1} + h)$, $\psi_c(s_{j^l-1} + h)$, $s \in S$, $s \geq s_{j^l}$.

A new support has the form $\bar{K}_{sup} = \{I_{sup}, \bar{T}_{sup}\}$. The matrix $\bar{D}_{|sup|}$ is obtained by the change of the column $d(t^0)$ to $d(t^l - h)$ in the matrix $D_{|sup|}$ in the cases A.3, A.5 and by the change of the column $d(t^0)$ to $d(t^l)$ in the other cases.

In the case (b) $\sigma^l = \sigma_{i^l}(s^l)$ we form the new support $\bar{K}_{sup} = \{\bar{I}_{sup}, \bar{T}_{sup}\}$, $\bar{I}_{sup}(s) = I_{sup}(s)$, $s \in S_{sup} \backslash s^l$, $\bar{I}_{sup}(s^l) = I_{sup}(s^l) \backslash i^l$; $\bar{S}_{sup} = S_{sup}$, if $\bar{I}_{sup}(s^l) \neq \emptyset$, $\bar{S}_{sup} = S_{sup} \backslash s^l$, if $\bar{I}_{sup}(s^l) = \emptyset$, $\bar{T}_{sup} = T_{sup} \backslash t^0$. The matrix $\bar{D}_{|sup|}$ is obtained by deleting the column $d(t^0)$ from $D_{|sup|}$. The rest of information for the new iteration remains unchanged from the previous step.

(2) Consider the situation $\rho^0 = \rho_{i^0}(s^0)$. This means that the element i^0 will be included into the set $I_{sup}(s^0)$.

To construct the variation $\Delta\nu(s)$, $s \in S$, of the function of the Lagrange multipliers we put

$$\Delta\nu_{i^0}(s^0) = \begin{cases} 1, & \text{if } \zeta_{i^0}(s^0) > g^*_{i^0}(s^0), \\ -1, & \text{if } \zeta_{i^0}(s^0) < g_{*i^0}(s^0), \end{cases} \quad \Delta\nu_n(s) = 0,\ s \in S.$$

Support values $\Delta\nu_{sup}$ are found from the equation

$$-D'_{sup}\Delta\nu_{sup} = \Big(d_{i^0}(t),\ t \in T_{sup}\Big)\Delta\nu_{i^0}(s^0).$$

The variation of the control is defined by formula (25). The initial rate of the change of the dual performance index for $\sigma > 0$ equals

$$\alpha^1 = -\rho(\zeta_{i^0}(s^0), [g_{*i^0}(s^0), g^*_{i^0}(s^0)]) < 0.$$

Operations for calculating the step σ^* are analogous to the case (1) but the "zero" step is not performed.

At the final step in the case (a) $\sigma^* = \sigma(t^l)$, to form $\bar{T}_{sup}$ the element t^l or $t^l - h$ includes in T_{sup}, as in the case (1), but the element t^0 is not deleted; $\bar{I}_{sup}(s^l) = I_{sup}(s^l) \cup i^0$. The column $d(t^l)$ or $d(t^l - h)$ is introduced in the matrix $D_{|sup|}$, as in the case (1), but the column $d(t^0)$ is not deleted. In the case (b) $\sigma^* = \sigma_{i^l}(s^l)$ we put $\bar{I}_{sup}(s^l) = I_{sup}(s^l)\backslash i^l$; $\bar{I}_{sup}(s^0) = I_{sup}(s^0) \cup i^0$; $\bar{S}_{sup} = S_{sup} \cup s^0$, if $\bar{I}_{sup}(s^l) \neq \emptyset$; $\bar{S}_{sup} = S_{sup} \cup s^0 \backslash s^l$, if $\bar{I}_{sup}(s^l) = \emptyset$; $\bar{D}_{|sup|} = D_{|sup|}$. The other operations of the final step are performed as in the case (1).

Under certain conditions of nonsingularity the described method is finite [11, part 1]. There exists a modification of the adaptive method [11, part 1] finite for any problem (9).

5. Optimization of Piecewise Linear Control Systems

The method of optimization of linear control systems with intermediate state constraints is generalized for piecewise linear control systems. Using these systems we can approximate nonlinear control systems with quite a high accuracy.

5.1. Problem Statement

Now in the class of discrete controls we study the OC problem of a piecewise linear system

$$c'x(t^*) \to \max, \quad \dot{x} = \hat{f}(x) + bu, \ x(0) = x_0, \tag{31}$$
$$Hx(t^*) = g, \quad |u(t)| \leq 1, \quad t \in T = [0, t^*].$$

Here $b \in R^n$; $g \in R^{\hat{m}}$; $H \in R^{\hat{m} \times n}$, $\text{rank}H = \hat{m} < n$, the piecewise linear function $\hat{f}(x)$ is defined on the set X as it was done in Sec. 3.

Definition 8. *A discrete control* $u(\cdot) = (u(t),\ t \in T)$ *is said to be an admissible control of problem* (31) *if it satisfies the equation* $|u(t)| \leq 1$, $t \in T$, *and the corresponding trajectory* $x(t)$, $t \in T$, *satisfies the endpoint constraint* $Hx(t^*) = g$ *and intersects the borders of the domains at discrete instants from the set* T_h. *An admissible control* $u^0(\cdot)$ *is called an optimal open-loop control of problem* (31) *if the corresponding trajectory* $x^0(t)$, $t \in T$, *provides the maximal value to the performance index of problem* (31).

To introduce the notion of an optimal feedback (a positional solution to problem (31)) we imbed problem (31) into the family of problems

$$c'x(t^*) \to \max, \quad \dot{x} = \hat{f}(x) + bu, \quad x(\tau) = z, \tag{32}$$
$$Hx(t^*) = g, \ |u(t)| \leq 1, \quad t \in T^\tau = [\tau, t^*],$$

depending on a scalar $\tau \in T_h$ and an n-vector z. Let $u^0(t|\tau, z)$, $t \in T^\tau$, be an optimal open-loop control of problem (32) for a position (τ, z); X_τ be a set of states z for which there exists a solution to problem (32). A function

$$u^0(\tau, z) = u^0(\tau|\tau, z), \quad z \in X_\tau, \ \tau \in T_h, \tag{33}$$

is said to be an optimal (discrete) feedback of problem (31).

Let us describe effective algorithms of constructing an optimal open-loop control and an optimal feedback in problem (31).

5.2. Parametrized Form of the Problem

According to the definition each admissible control $u(t)$, $t \in T$, of problem (31) generates a trajectory which successively passes through some (admissible) sequence $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ of sets from the totality $X_1, \ldots, X_p$, crossing the borders of neighboring sets at (admissible) instants $\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_k - 1} \in T_h$.

Denote by $X_1^0, \ldots, X_{j^*}^0$ and $\theta_1^0, \ldots \theta_{j^*-1}^0$ sequences of sets and instants of intersection of the borders corresponding to the optimal trajectory $x^0(t)$, $t \in T$.

Definition 9. *The sequence of sets $X^0 = \{X_1^0, \ldots, X_{j^*}^0\}$ is said to be a structure of the optimal trajectory of problem* (31).

Assume that the optimal trajectory $x^0(t)$, $t \in T$, intersects the borders of sets X_j^0, X_{j+1}^0, $j = 1, \ldots, j^* - 1$, on hyperplanes $H_j x = g_j$ where $H_j \in R^{m_j \times n}$, $g_j \in R^{m_j}$, $j = 1, \ldots, j^* - 1$.

On a set X_j^0 the function $\hat{f}(x)$ has the form $\hat{f}(x) = A_j x + a_j$ where $A_j \in R^{n \times n}$, $a_j \in R^n$; $A_j x + a_j = A_{j+1} x + a_{j+1}$ if $H_j x = g_j$, $j = 1, \ldots, j^* - 1$; $\mathrm{rank}\begin{pmatrix} A_j - A_{j+1} \\ H_j \end{pmatrix} < n$, $j = 1, \ldots, j^* - 1$. Denote $m_{j^*} = \hat{m}$; $H_{j^*} = H$, $g_{j^*} = g$; set $m = \sum_{j=1}^{j^*} m_j$.

In many problems of type (31), on the base of a priori information, the set $X_1^0, \ldots, X_{j^*}^0$ could be formed before solving problem (31). If the structure X^0 is known, problem (31) is reduced to the OC problem of a totality of linear systems

$$J(\theta, u) = c'x(\theta_{j^*}) \to \max, \tag{34}$$

$$\dot{x}(t) = A_j x(t) + a_j + bu(t),$$
$$t \in [\theta_{j-1}, \theta_j[, \ j \in J = \{1, 2, \ldots, j^*\}, \ x(\theta_0) = x_0, \tag{35}$$

$$H_j x(\theta_j) = g_j, \quad j \in J, \tag{36}$$

$$|u(t)| \leq 1, \ t \in T, \quad \theta_0 < \theta_1 < \cdots < \theta_{j^*-1} < \theta_{j^*}, \tag{37}$$

$$(x \in R^n, \quad u \in R, \quad g_j \in R^{m_j}, \ j = 1, \ldots, j^*, \quad \mathrm{rank} H_j = m_j < n,$$
$$\theta_j \in T_h, \ j = 1, \ldots, j^* - 1, \ \theta_0 = 0, \ \theta_{j^*} = t^*).$$

In problem (34)–(37) instants $\theta = (\theta_1, \theta_2, \ldots, \theta_{j^*-1})$ of transition between domains of linearity are chosen together with a control $u(t)$, $t \in T$.

Describe an algorithm of solving problem (34)–(37) for a given set $X_1, \ldots, X_{j^*}$.

Definition 10. *A vector θ and a discrete control $u(\cdot) = (u(t),\ t \in T)$ are said to be an accessible control of problem (34)–(37) if they satisfy constraints (37). An accessible control $\{\theta, u(\cdot)\}$ and the corresponding trajectory $x(t) = x(t|\theta, u(\cdot))$, $t \in T$, of system (35) are called admissible if $x(t)$, $t \in T$, satisfies constraint (36). An admissible control $\{\theta^0, u^0(\cdot)\}$ is optimal if it provides the maximal value of performance index (34).*

The solution to problem (34)–(37) consists of two procedures: 1) a linearization of problem (34)–(37) along an admissible trajectory and solution to the linearized problem; 2) a correction of solution to the linearized problem by the choice of the optimal positions of instants of transition between domains of linearity.

5.3. Optimal Open-Loop Solution to a Linearized Problem

A linearization of problem (34)–(37) consists in fixing a vector θ corresponding to an admissible trajectory. The linearized problem has form (34)–(37) but now it is assumed that the vector θ is fixed. Thus we get a linear OC problem with intermediate state constraints

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + a(t) + bu, \quad x(\theta_0) = x_0,$$
$$H(s)x(s) = g(s), \ s \in S = \{\theta_1, \ldots, \theta_{j^*}\}, \quad |u(t)| \le 1, \ t \in T. \qquad (38)$$

Here $A(t) = A_j$, $a(t) = a_j$, $t \in T_j = [\theta_{j-1}, \theta_j[$, $j = 1, \ldots, j^*$; $H(s) = H_j$, $g(s) = g_j$, $s = \theta_j$, $j = 1, \ldots, j^*$.

Let us describe how it is worthwhile to present the required information for solving problem (38) by the algorithm of Sec. 4 taking into account the structure of the matrix function $A(t)$, $t \in T$, and the necessity of further correction of instants $\theta_1, \ldots, \theta_{j^*-1}$. Instead of direct storage of matrices $G(s,t)$, $t \in T_{sn}$, $s \in S$; $s > t$, and vectors p; $\psi_c(t)$, $t \in T_{sn}$, $s \in S$; $s > t$, we suggest to use matrices $F_j(\theta_j - t)$, $t \in T_{sn}(j)$; $F_j(\theta_j - \theta_{j-1})$, $j \in J$, where $\dot{F}_j = A_j F_j$, $F_j(0) = E$, $j \in J$; and vectors $r_j = \sum_{t \in T_n(j)} \int_t^{t+h} F_j(\theta_j - \vartheta) b d\vartheta \omega(t) + \int_{\theta_{j-1}}^{\theta_j} F_j(\theta_j - \vartheta) d\vartheta a_j$, $j \in J$, in the course of iterations. In this case it is easy to calculate

$$G(\theta_k, t) = H_k \Phi_{kj} F_j(\theta_j - t), \ \ \psi_c'(t) = c' \Phi_{kj} F_j(\theta_j - t), \ k = j, \ldots, j^*, \ t \in T_{sup}(j);$$

$$p(\theta_k) = \sum_{j=1}^{k} H_k \Phi_{kj} r_j + G(\theta_k, 0) x_0, \quad p = \begin{pmatrix} p(\theta_k), \\ k \in J \end{pmatrix},$$

where

$$\Phi_{kj} = \Phi_{kj}(\theta) = \begin{cases} \prod_{r=j}^{k-1} F_{r+1}(\theta_{r+1} - \theta_r), & \text{if } k > j, \\ E, & \text{if } k = j, \\ 0, & \text{if } k < j, \end{cases} \qquad j = 0, \ldots, j^*-1, \ k = 1, \ldots, j^*.$$

Operations for tranforming $G(s,t)$, $\psi_c(t)$, $t \in T_{sn}$, $s \in S$, $s \geq t$; p in the course of iterations of the method described in Sec. 4 are naturally transferred to $F_j(\theta_j - t), t \in T_{sn}(j)$; $r_j, j \in J$.

Thus, according to the algorithm of Sec. 4, the optimal support $T^0_{sup}(\theta)$ [1] ; of problem (34)—(37) would be constructed for a fixed vector θ and information corresponding to this support is formed.

5.4. Optimization of Transition Instants

As a result of solving the linearized problem we get an optimal support $T^0_{sup}(\theta)$ and the corresponding optimal value $J^0(\theta)$ of the performance index for a fixed vector θ. The aim is to construct a new vector $\bar{\theta}$ such that $J^0(\bar{\theta}) \geq J^0(\theta)$.

Methods of constructing an optimal vector θ^0 [1] are based on calculating a gradient of the performance index of problem (31) with respect to θ. It is not difficult to show that

$$\frac{\partial J(\theta,u)}{\partial \theta_k} = c' \frac{\partial x(t^*|\theta, u^0_{sup}(\theta))}{\partial \theta_k} - \nu'(\theta) \left(H_j \frac{\partial x(\theta_j|\theta, u^0_{sup}(\theta))}{\partial \theta_k}, \atop j \in J \right), \quad k = 1, ..., j^*-1, \tag{39}$$

where $u^0_{sup}(\theta) = (u^0(t|\theta),\ t \in T^0_{sup}(\theta))$ are support values of the optimal control $u^0(t|\theta)$, $t \in T_h$; $x(t|\theta, u^0_{sup}(\theta))$, $t \in T$, is the corresponding trajectory of system (31). Then a step along the direction grad $J = (\partial J(\theta,u)/\partial\theta_k,\ k = 1, ..., j^*-1)$ is chosen [1] and a new vector θ is constructed.

For calculating $\partial x(\theta_j|\theta, u^0_{sup}(\theta))/\partial\theta_k$, $k = 1, ..., j^*-1$, $j \in J$, the information stored after solving the linearized problem is used and integrations on intervals of length h are performed in parallel. To transform the stored information for a new vector $\bar{\theta}$, it is necessary to perform parallel integrations on intervals which maximal length is $\max_{j=1,...,j^*-1}\{\bar{\theta}_j - \theta_j, \bar{\theta}_j - \theta_j - \bar{\theta}_{j-1} + \theta_{j-1}\} + h$. Thus we obtain data for solving a linearized problem for the new fixed vector $\bar{\theta}$ with the initial support $T_{sup}(\bar{\theta}) = T^0_{sup}(\theta)$. The use of the optimal support $T^0_{sup}(\theta)$ as an initial support $T_{sup}(\bar{\theta})$ allows to construct the optimal support $T^0_{sup}(\bar{\theta})$ very fast.

The transformation $T_{sup}(\theta) \longrightarrow T^0_{sup}(\theta)$, $\theta \longrightarrow \bar{\theta}$, i.e., solving the linearized problem and calculating $\bar{\theta}$ is called a large iteration of the method.

5.5. Optimal Controller

Following the approach [15, 16] to construct the optimal controller, let us analyze the use of optimal feedback (33) in a particular control process of system (31). Assume that optimal feedback (33) has been constructed and the behaviour of the closed system is described by the equation

$$\dot{x} = \hat{f}(x) + bu^0(t,x) + w, \quad x(0) = x_0, \tag{40}$$

[1] As problem (31) contains intermediate constraints-equalities so a set $T_{sup} = \{t_l \in T_h : l = 1, ..., m\}$ such that $\det D_{sup} = \det D(I, T_{sup}) \neq 0$ is considered as a support.

where $w = w(t,x)$, $x \in X$, $t \in T$, is an unknown but bounded piecewise continuous function of disturbances. The function $w(t)$, $t \in T$, reflects the action of unforeseen disturbances on the system. With the use of piecewise linear approximations for optimization of nonlinear system (1) the function $w(t), t \in T$, contains besides elements of external disturbances a deviation of piecewise linear system (31) from initial nonlinear system (1).

Let in any particular control process of system (40) a disturbance $w^*(t)$, $t \in T$, be realized. Under this disturbance and function (33) a certain trajectory $x^*(t)$, $t \in T$, of (40) is generated. In doing so the system (40) is fed by the values $u^*(t) = u^0(t, x^*(t))$, $t \in T$.

At any current instant $\tau \in T_h$, the controller knows a realized state $x^*(\tau)$ and should calculate the signal $u^*(\tau) = u^0(\tau|\tau, x^*(\tau))$.

For this purpose the controller should solve problem (32) for the position $(\tau, x^*(\tau))$. In formulating problem (32) in the parametrized form we assume that the change of the dynamics of the system may occur not only when the trajectory intersects the hyperplanes $H_j x = g_j$, $j = 1, \dots, j^*-1$, but also when it is in a certain ε-vicinity of these hyperplanes. The inaccuracy arising because of this assumption may be interpreted as an element of the disturbance $w(t)$, $t \in T$. Thus, the parametrized form of problem (32) is

$$c'x(\theta_{j^*}) \to \max,$$

$$\dot{x}(t) = A_j x(t) + a_j + bu(t),\ t \in [\theta_{j-1}, \theta_j[,$$

$$j \in J(\tau) = \{j(\tau), \dots, j^*\}, \quad x(\theta_{j(\tau)-1}) = x^*(\tau),$$

$$|H_j x(\theta_j) - g_j| \le \varepsilon,\ j \in J(\tau)\backslash j^*, \tag{41}$$

$$Hx(\theta_{j^*}) = g, \quad |u(t)| \le 1,\ t \in T^\tau,$$

$$\theta_{j(\tau)-1} < \theta_{j(\tau)} < \dots < \theta_{j^*-1} < \theta_{j^*}.$$

Assume that on the interval T^τ transition instants $\theta_{j(\tau)}, \dots, \theta_{j^*-1}$ are located, put $\theta_{j(\tau)-1} = \tau$.

The signal $u^*(\tau)$ is calculated on the base of the optimal support $K^0_{sup}(\tau|\theta^0(\tau))$ which is constructed by the proposed method. As an initial support $K_{sup}(\tau|\theta(\tau))$ the optimal support $K^0_{sup}(\tau - h|\theta^0(\tau - h))$ constructed at the previous moment $\tau - h$ is used.

Let at the initial instant $\tau = 0$ for the initial state $x(0) = x_0^*$ an optimal trajectory $x^0(t|0, x_0^*)$, $t \in T$, be calculated. Let $X^0(0) = \{X_j^0(0),\ j = 1, \dots, j^*\}$, $\theta^0(0) = (\theta_j^0(0),\ j = 1, \dots, j^*-1)$ be a structure and transition instants of $x^0(t|0, x_0^*)$, $t \in T$. If $w^*(t) \equiv 0$, $t \in T$, then at any instant $\tau \in T_h$ the structure $X^0(\tau) = \{X_j^0(\tau),\ j = j(\tau), \dots, j^*\}$, of the optimal trajectory $x^0(t|\tau, x^*(\tau))$, $t \in T^\tau$, is a part of the initial structure $X_j^0(\tau) = X_j^0(0)$, $j = j(\tau), \dots, j^*$, i.e., $X^0(\tau) \subset X^0(0)$, and the set of transition instants $\theta^0(\tau) = (\theta_j^0(\tau),\ j = j(\tau), \dots, j^*-1)$ on T^τ is a part of the initial set $\theta_j^0(\tau) = \theta_j^0(0)$, $j = j(\tau), \dots, j^*-1$, i.e. $\theta^0(\tau) \subset \theta^0(0)$. Under the disturbance $w^*(t) \not\equiv 0$, $t \in T$, the following situations are possible: 1) for $X^0(\tau) \subset X^0(0)$ transition instants change, i.e. $\theta^0(\tau) \not\subset \theta^0(0)$; 2) the structure of the optimal trajectory changes, i.e., $X^0(\tau) \not\subset X^0(0)$ for a certain $\tau \in T_h$.

In the case 1), at an instant $\tau \in T_h$ the controller constructed the solution to problem (41), i.e., found the extremal sets $\hat{\theta}^l(\tau) = (\hat{\theta}^l_j(\tau), j = j(\tau), \dots, j^*-1)$, $l = 1, \dots, l^*$, providing local maximums in problem (41), and corresponding optimal supports. Among the sets $\hat{\theta}^l(\tau)$, $l = 1, \dots, l^*$, the controller chose the optimal set $\theta^0(\tau)$ providing the global maximum. At the instant $\tau + h$, on the base of information stored at τ, the controller makes the correction of the sets $\hat{\theta}^l(\tau)$, $l = 1, \dots, l^*$, and the corresponding optimal supports. The controller constructs the extremal sets $\hat{\theta}^l(\tau + h)$, $l = 1, \dots, l^*$, and chooses the optimal set $\theta^0(\tau + h)$.

In the case 2), in addition to the case 1), the situations of changing the structure of the optimal trajectory are studied. For this purpose the behaviour of $x^0(t|\tau, x^*(\tau))$, $t \in T^\tau$, is analyzed in the inner points of T^τ for detecting a new region, which the optimal trajectory will pass through, or for detecting the increase of the number of intersected hyperplanes. The complexity of the procedures of detecting the change of the structure depends on a problem under consideration.

5.6. Example

Let us illustrate the results by an example of optimal control of a piecewise linear system without endpoint constraints. This problem for a linear model is trivial, because it is enough to integrate the adjoint system and put $u(t) = \text{sign}\psi'(t)b$, $t \in T$. For a piecewise linear model it becomes nontrivial.

Assume that an oscillating system moves along a horizontal without friction. Two springs act on the system on two intervals (Fig. 1).

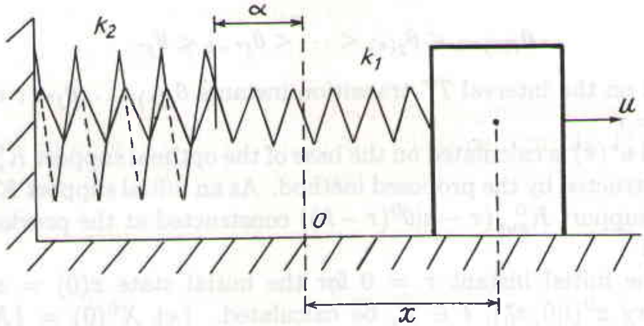


Fig. 1

We should construct a control replacing the system to the maximal distance in a fixed time.

The mathematical model of the problem is

$$x(t^*) \longrightarrow \max, \quad \ddot{x} + k_1 x = u \text{ if } x \geq -\alpha; \tag{42}$$

$$\ddot{x} + k_1 x + k_2(x + \alpha) = u, \text{ if } x < -\alpha;$$

$$x(0) = 0, \ \dot{x}(0) = 0, \quad |u(t)| \leq 1, \ t \in T = [0, t^*],$$

where $x = x(t)$ is a deviation from the equilibrium state $x = 0$ at an instant t, $u = u(t)$ is a control value, α is a distance from the equilibrium to the right end of the second spring. Using the phase coordinates $x_1 = x$, $x_2 = \dot{x}$ and fixed values of the parameters $t^* = 8$, $\alpha = 0.5$, $k_1 = 1$, $k_2 = 2$ we obtain the parametrized form of the problem

$$\begin{gathered} J(\Theta, u) = x_1(8) \longrightarrow \max, \\ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad t \in [0, \Theta_1[, \\ x_1(0) = x_2(0) = 0, \quad x_1(\Theta_1) = -0.5, \\ \dot{x}_1 = x_2, \quad \dot{x}_2 = -3x_1 - 1 + u, \quad t \in [\Theta_1, \Theta_2[, \\ x_1(\Theta_2) = -0.5, \\ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad t \in [\Theta_2, 8[, \\ |u(t)| \leq 1, t \in T = [0, 8]. \end{gathered} \tag{43}$$

The problem was solved with $h = 0.08$. The optimal values of transition instants are $\Theta^0 = (4.4, 6.16)$, the corresponding value of the performance index is $J^0(\Theta^0) = 5.2665$. As an initial value of Θ the vector $\Theta = (3.2, 4.88)$ was chosen with $J^0(\Theta) = 2.5648$. To solve the problem (32) large iterations were used. On each iteration the vector $\bar{\Theta}$ was calculated as $\bar{\Theta} = \Theta + h\Delta\Theta$ where

$$\Delta\Theta = \begin{cases} \begin{pmatrix} \text{sign}\frac{\partial J^0(\Theta)}{\partial \Theta_1} \\ 0 \end{pmatrix} & \text{if } |\partial J^0(\Theta)/\partial\Theta_1| > |\partial J^0(\Theta)/\partial\Theta_2|, \\ \begin{pmatrix} 0 \\ \text{sign}\frac{\partial J^0(\Theta)}{\partial \Theta_2} \end{pmatrix} & \text{if } |\partial J^0(\Theta)/\partial\Theta_1| < |\partial J^0(\Theta)/\partial\Theta_2|. \end{cases}$$

Fig. 2 contains current values of Θ in the course of iterations and the corresponding optimal support instants $T^0_{sup}(\Theta) = \{t^0_1(\Theta), t^0_2(\Theta)\}$. The optimal open-loop control $u^0(t)$, $t \in T$, is presented in Fig. 3.a).

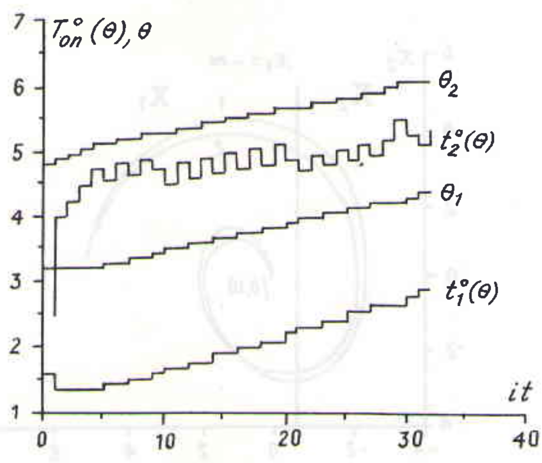


Fig. 2

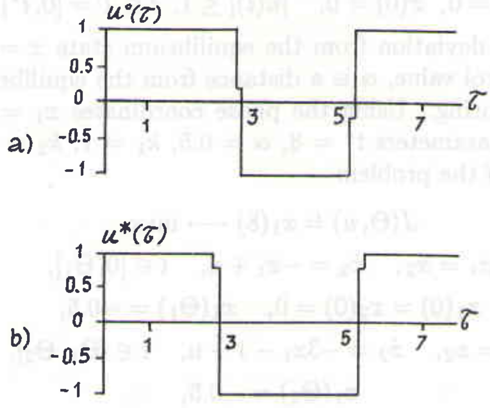


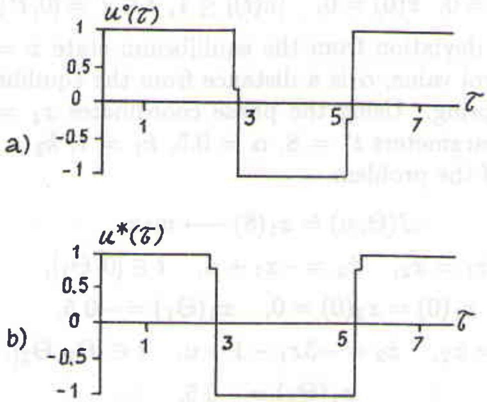
Fig. 3

In [13] a method of evaluating the effectiveness of solving OC problems is suggested. As a unit of complexity the time of integration of a primal or adjoint system on the interval T is considered. In doing so it is assumed that there are enough microprocessors at our disposal to perform the integration of several equations in parallel. Using this method of evaluation of effectiveness we obtain the complexity of construction of the optimal open-loop control in our example to be equal 4.03. This means that despite the large number of iterations, the optimization of the system is performed in time required for integration of the system on an interval of the length $8 \times 4.03 = 32.24$.

The realization of the optimal feedback was constructed under the disturbance $w^*(t) = 0.5 \sin 2t$, $t \in T$, unknown to the controller

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \begin{cases} -x_1 + u + w^*(t) & \text{if } x_1 \geq -0.5; \\ -3x_1 - 1 + u + w^*(t) & \text{if } x_1 < -0.5. \end{cases}$$



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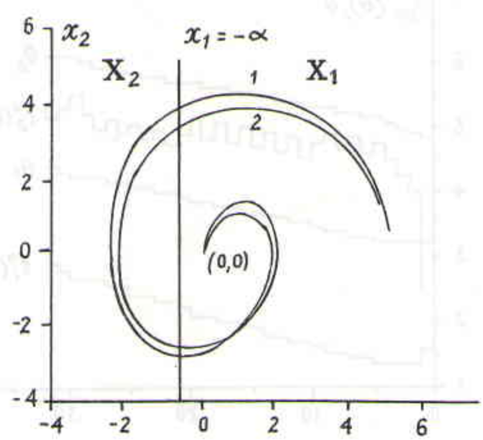

Fig. 4

The realization $u^*(t)$, $t \in T$, of the optimal feedback is presented in Fig. 3.b). Fig. 4 contains the phase trajectories of systems (43) and (44) (curve 1 is the optimal open-loop trajectory of (43), curve 2 is the trajectory of (44) generated by $u^*(t)$, $t \in T$, and $w^*(t)$, $t \in T$).

The complexity of operations of calculating current values of the optimal feedback does not exceed 0.15 at any instant $t \in T_h$. This means that for realizing the algorithm it is enough to use microprocessors able to integrate system (43) on the interval of length 8 in time τ^0 such that $0.15\tau^0 \leq h = 0.08$, i.e., in $\tau^0 = 0.53$ units of time in which problem (43) is formulated. If the time in problem (42) is measured in seconds, 0.53 second is more than enough time for modern microprocessor to integrate system (42) on the interval $[0, 8]$.

6. Asymptotic Correction of the Solution to Piecewise Linear Problem[2]

6.1. General Definitions

Imbed problem (1) into the family of problems depending on a small parameter $\mu \rightarrow 0$

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = \hat{f}(x) + \mu g(x) + bu, \ x(0) = x_0, \tag{45}$$
$$x(t^*) \in X^*, |u(t)| \leq 1, \quad t \in T,$$

where $g(x) = (f(x) - \hat{f}(x))/\delta$, $x \in X$.

For a fixed parameter μ a piecewise continuous function $u(t, \mu)$, $t \in T$, satisfying the inequality $|u(t, \mu)| \leq 1$, $t \in T$, is called an accessible control. An accessible control is called an admissible open-loop control if the corresponding trajectory $x(t, \mu)$, $t \in T$, of system (45) satisfies the constraint $x(t^*, \mu) \in X^*$. An admissible control $u^0(t, \mu)$, $t \in T$, which maximizes the performance index of problem (45), is called an optimal open-loop control. For a chosen integer s a family $u^s(t) = \{u^s_\mu(t), \ \mu \rightarrow 0\}$, $t \in T$, of accessible controls is said to be an asymptotically s-optimal open-loop control of a piecewise quasilinear problem (45) if for trajectories $x^s_\mu(t)$, $t \in T$, $\mu \longrightarrow 0$, generated by controls $u^s_\mu(t)$, $t \in T$, $\mu \rightarrow 0$, the asymptotic equalities $c'x^0(t^*, \mu) - c'x^s_\mu(t^*) = O(\mu^{s+1})$, $Hx^s_\mu(t^*) - g = O(\mu^{s+1})$ hold. Here $x^0(t, \mu)$, $t \in T$, is a trajectory of system (45) generated by the optimal control $u^0(t, \mu)$, $t \in T$.

Taking into account the form of the function $\hat{f}(x)$, $x \in X$, and assuming the structure X^0 of the optimal trajectory to be known, we write problem (45) in parametrized form as an OC problem of a stair quasilinear system

$$J(\theta, u) = c'x(\theta_{j^*}) \longrightarrow \max, \tag{46}$$

$$\dot{x} = A_j x + a_j + \mu g_j(x) + bu, \ t \in [\theta_{j-1}, \theta_j[, \ j = 1, 2, \ldots, j^*; \ \ x(\theta_0) = x_0, \tag{47}$$

[2]Results of this section have been obtained jointly with Prof. A. I. Kalinin.

$$H_j x(\theta_j) = g_j,\ j = 1, 2, \ldots, j^*; \quad (48)$$
$$|u(t)| \le 1,\ t \in T.$$

Here $g_j(x) = (f(x) - A_j x - a_j)/\delta$, $x \in X_j^0$, $j \in J$. It is obvious that functions $g_j(x)$ are infinitely differentiable in $\text{int}X_j$, $j \in J$, and $g_j(x) = g_{j+1}(x)$ if $H_j x = g_j$, $j = 1, 2, \ldots, j^* - 1$.

A vector $\theta(\mu) = (\theta_j(\mu),\ j = 1, 2, \ldots, j^* - 1)$ and a piecewise continuous function $u(t, \mu)$, $t \in T$, are said to be an accessible control if $|u(t, \mu)| \le 1$, $t \in T$. An accessible control $\{\theta(\mu); u(t, \mu), t \in T\}$ is called admissible if the corresponding continuous trajectory $x(t, \mu)$, $t \in T$, of (47) satisfies conditions (48). Admissible control providing a maximal value to performance index (46) is called optimal.

Definition 11. *A family of accessible controls $\{\theta^s_\mu; u^s_\mu(t), t \in T; \mu \to 0\}$ is said to be an asymptotically s-optimal control ($s = 0, 1, \ldots$) if for the trajectories $x^s_\mu(t)$, $t \in T$, $\mu \to 0$, of system (47) generated by controls $\{\theta^s_\mu; u^s_\mu(t), t \in T\}$, $\mu \to 0$, the following conditions hold*

$$c'x^0(\theta_{j^*}, \mu) - c'x^s_\mu(\theta_{j^*}) = O(\mu^{s+1}), \quad H_j x^s_\mu(\theta^s_{\mu j}) - g_j = O_j(\mu^{s+1}),\ j \in J.$$

Below we suggest an algorithm of constructing an asymptotically s-optimal control in problem (46)–(48). Conceptually, it is very close to the algorithm of asymptotic solution of a quasilinear OC problem proposed in [8, 19]. It is based on constructing asymptotic expansions of the vector $\theta^0(\mu)$ and switching points of OC $u^0(t, \mu)$, $t \in T$, by powers of the small parameter.

6.2. Structure and Asymptotic Properties of the Solution to Problem (46)–(48)

Let $t_1, \ldots, t_{k_1}, \theta_1, t_{k_1+1}, \ldots, t_{k_2}, \theta_2, \ldots, \theta_{j^*-1}, t_{k_{j^*-1}+1}, \ldots, t_{k^*}$ ($k_{j^*} = k^*$) be distinct points in the interval $]\theta_0, \theta_{j^*}[$ arranged in increasing order. Introduce into consideration vectors $\theta = (\theta_1, \ldots, \theta_{j^*-1})$, $T_j = (t_i,\ i = k_{j-1} + 1, \ldots, k_j)$, $j \in J$, and denote by $x(t, T_1, \ldots, T_{j^*}, \theta, \mu)$, $t \in T$, a trajectory of system (47) generated by an accessible control $\{\theta; u(t, T_1, \ldots, T_{j^*}, \theta), t \in T\}$ where

$$u(t, T_1, \ldots, T_{j^*}, \theta) = \begin{cases} \gamma_1, & t \in [\theta_0, t_1[, \\ (-1)^i \gamma_1, & t \in [t_i, t_{i+1}[,\ i = 1, 2, \ldots, k_1 - 1; \\ (-1)^{k_1} \gamma_1, & t \in [t_{k_1}, \theta_1[, \\ \ldots\ldots\ldots & \\ \gamma_{j^*}, & t \in [\theta_{j^*-1}, t_{k_{j^*-1}+1}[, \\ (-1)^{i-k_{j^*-1}} \gamma_{j^*}, & t \in [t_i, t_{i+1}[,\ i = k_{j^*-1} + 1, \ldots, k^* - 1; \\ (-1)^{k_{j^*} - k_{j^*-1}} \gamma_{j^*}, & t \in [t_{k^*}, \theta_{j^*}[. \end{cases}$$

Problem (34)–(37) formally arises from problem (46)–(48) when $\mu = 0$. Further it is called a base problem.

Let $u(t, T^0_1, \ldots, T^0_{j^*}, \theta^0) = u^0(t)$, $t \in T$, be an OC of the base problem in the class of piecewise continuous functions. Let $\nu_1, \ldots, \nu_{j^*}$ be vectors of dimensions

$m_1, \ldots, m_{j^*}$ respectively. Denote by $\psi(t, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}, \mu)$, $t \in T$, a piecewise smooth vector-function being a solution to the adjoint system

$$\dot{\psi} = -(A_j + \mu \frac{\partial g_j(x(t, T_1, \ldots, T_{j^*}, \theta, \mu))}{\partial x})'\psi, \quad t \in]\theta_{j-1}, \theta_j], \tag{50}$$
$$j = 1, 2, \ldots, j^*, \quad \psi(\theta_{j^*}) = c - H'_{j^*}\nu_{j^*},$$

with jumps at instants $\theta_1, \ldots, \theta_{j^*-1}$

$$\psi(\theta_j - 0) = \psi(\theta_j) - H'_j\nu_j, \; j = 1, 2, \ldots, j^* - 1. \tag{51}$$

One can show that under rather general conditions [8,19] in problem (46)–(48) with a quite small (in absolute value) μ there exists the unique optimal control $\{\theta(\mu);\ u^0(t, \mu),\ t \in T\}$ where

$$u^0(t, \mu) = u(t, T_1(\mu), \ldots, T_{j^*}(\mu), \theta(\mu)), \; t \in T. \tag{52}$$

Therewith the vectors $T_1(\mu), \ldots, T_{j^*}(\mu)$, $\theta(\mu)$ are infinitely differentiable functions of the small parameter, and together with $\nu_j(\mu)$, $j = 1, \ldots, j^*$, they satisfy the equations

$$H_j x(\theta_j, T_1, \ldots, T_{j^*}, \theta, \mu) - g_j = 0, \; j = 1, 2, \ldots, j^*,$$
$$\psi'(t_i, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}, \mu)b = 0, \; i = 1, 2, \ldots, k^*, \tag{53}$$
$$\nu'_j H_j(A_j x(\theta_j, T_1, \ldots, T_{j^*}, \theta, \mu) + a_j + \mu g_j(x(\theta_j, T_1, \ldots, T_{j^*}, \theta, \mu)) + (-1)^{k_j - k_{j-1}}\gamma_j b) +$$
$$(\gamma_{j+1} - (-1)^{k_j - k_{j-1}}\gamma_j)\psi'(\theta_j, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}, \mu)b = 0, \; j = 1, 2, \ldots, j^* - 1,$$

and $T_j(0) = T_j^0$, $j \in J$, $\theta(0) = \theta^0$.

6.3. Construction of Asymptotics

Let us introduce into consideration the vectors $z = (T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*})$, $z_0 = (T_1^0, \ldots, T_{j^*}^0, \theta^0, \nu_1^0, \ldots, \nu_{j^*}^0)$. Denote by $R(z, \mu)$ the vector-function composed of left sides of equations (53) and write this system in the form

$$R(z, \mu) = 0. \tag{54}$$

As the vector-functions $\theta(\mu)$, $T_j(\mu)$, $\nu_j(\mu)$, $j \in J$, are infinitely differentiable, the asymptotic expansions $\theta(\mu) = \theta^0 + \sum_{k=1}^{\infty} \mu^k \theta^k$, $T_j(\mu) = T_j^0 + \sum_{k=1}^{\infty} \mu^k T_j^k$, $\nu_j(\mu) = \nu_j^0 + \sum_{k=1}^{\infty} \mu^k \nu_j^k$, $j \in J$, take place.

Let s be a given integer. It is not difficult to prove that the family of controls $\{\theta^s(\mu);\ u^s(t, \mu),\ t \in T;\ \mu \to 0\}$ where

$$u^s(t, \mu) = u(t, T_1^s(\mu), \ldots, T_{j^*}^s(\mu), \theta^s(\mu)), \quad t \in T, \tag{55}$$

$$\theta^s(\mu) = \theta^0 + \sum_{k=1}^{s} \mu^k \theta^k, \;\; T_j^s(\mu) = T_j^0 + \sum_{k=1}^{s} \mu^k T_j^k, \;\; j = 1, 2, \ldots, j^*, \tag{56}$$

is an asymptotically s-optimal control in problem (47). To construct this control we should calculate the coefficients θ^k, $T_1^k, \ldots, T_{j^*}^k$, $k = 1, 2, \ldots, s$, of the Taylor polynomials (56). Now we describe an algorithm of their calculation. First of all, let us expand the left sides of equations (53) in powers of the small parameter.

The vector-functions $x(t, T_1, \ldots, T_{j^*}, \theta, \mu)$, $\psi(t, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}, \mu)$, $t \in T$, are infinitely differentiable with respect to μ at any point of their domains of definition and therefore admit asymptotic expansions

$$x(t, T_1, \ldots, T_{j^*}, \theta, \mu) = \sum_{k=0}^{\infty} \mu^k x_k(t, T_1, \ldots, T_{j^*}, \theta), \tag{57}$$

$$\psi(t, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}, \mu) = \sum_{k=0}^{\infty} \mu^k \psi_k(t, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}).$$

Using the Poincare formalism we set up differential equations for the functions $x_k(t) = x_k(t, T_1, \ldots, T_{j^*}, \theta)$, $\psi_k(t) = \psi_k(t, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*})$, $k = 0, 1, \ldots, s$:

$$\begin{gathered}
\dot{x}_0 = A_j x_0 + a_j + bu(t, T_1, ..., T_{j^*}, \theta), \ t \in [\theta_{j-1}, \theta_j[, \ j = 1, 2, ..., j^*, \ \ x_0(\theta_0) = x_0; \\
\dot{\psi}_0 = -A'_j \psi_0, \ t \in]\theta_{j-1}, \theta_j], \ j = 1, 2, \ldots, j^*, \ \psi_0(\theta_{j^*}) = c - H'_{j^*} \nu_{j^*}, \\
\psi_0(\theta_j - 0) = \psi_0(\theta_j) - H'_j \nu_j, \ j = 1, 2, \ldots, j^* - 1; \\
\dot{x}_1 = A_j x_1 + g_j(x_0(t)), \ t \in [\theta_{j-1}, \theta_j[, \ j = 1, 2, \ldots, j^*, \ x_1(\theta_0) = 0; \\
\dot{\psi}_1 = -A'_j \psi_1 - \frac{\partial H_j(x_0(t), \psi_0(t))}{\partial x}, \ t \in]\theta_{j-1}, \theta_j], \ j = 1, 2, \ldots, j^*, \\
\psi_1(\theta_{j^*}) = 0; \\
\dot{x}_2 = A_j x_2 + \frac{\partial g_j(x_0(t))}{\partial x} x_1(t), \ t \in [\theta_{j-1}, \theta_j[, \ j = 1, 2, \ldots, j^*, \ x_2(\theta_0) = 0; \\
\dot{\psi}_2 = -A'_j \psi_2 - \frac{\partial H_j(x_0(t), \psi_1(t))}{\partial x} - \frac{\partial^2 H_j(x_0(t), \psi_0(t))}{\partial x^2} x_1(t), \ t \in]\theta_{j-1}, \theta_j], \\
j = 1, 2, \ldots, j^*, \ \ \psi_2(\theta_{j^*}) = 0; \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{gathered} \tag{58}$$

where $H_j(x, \psi) = \psi' g_j(x)$, $j = 1, 2, \ldots, j^*$.

By virtue of (57), the asymptotic expansion

$$R(z, \mu) = \sum_{k=0}^{\infty} \mu^k R_k(z)$$

takes place, where

$$R_0(z) = \begin{bmatrix} H_j x_0(\theta_j, T_1, \ldots, T_{j^*}, \theta) - g_j, \ j = 1, 2, \ldots, j^* \\ \psi'_0(t_i, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}) b, \ i = 1, 2, \ldots, k^* \\ \nu'_j H_j(A_j x_0(\theta_j, T_1, \ldots, T_{j^*}, \theta) + a_j + (-1)^{k_j - k_{j-1}} \gamma_j b) \\ + (\gamma_{j+1} - (-1)^{k_j - k_{j-1}} \gamma_j) \psi'_0(\theta_j, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}) b, \\ j = 1, 2, \ldots, j^* - 1 \end{bmatrix},$$

$$R_1(z) = \begin{bmatrix} H_j x_1(\theta_j, T_1, \ldots, T_{j^*}, \theta), \ j = 1, 2, \ldots, j^* \\ \psi'_1(t_i, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}) b, \ i = 1, 2, \ldots, k^* \\ \nu'_j H_j(A_j x_1(\theta_j, T_1, \ldots, T_{j^*}, \theta) + g_j(x_0(\theta_j, T_1, \ldots, T_{j^*}, \theta)) \\ + (\gamma_{j+1} - (-1)^{k_j - k_{j-1}} \gamma_j) \psi'_1(\theta_j, T_1, \ldots, T_{j^*}, \theta, \nu_1, \ldots, \nu_{j^*}) b, \\ j = 1, 2, \ldots, j^* - 1 \end{bmatrix},$$

$$R_2(z) = \begin{bmatrix} H_j x_2(\theta_j, T_1, \dots, T_{j^*}, \theta),\ j = 1,2,\dots,j^* \\ \psi_2'(t_i, T_1, \dots, T_{j^*}, \theta, \nu_1, \dots, \nu_{j^*})b,\ i = 1,2,\dots,k^* \\ \nu_j' H_j(A_j x_2(\theta_j, T_1, \dots, T_{j^*}, \theta) \\ + \dfrac{\partial g_j(x_0(\theta_j, T_1, \dots, T_{j^*}, \theta))}{\partial x} x_1(\theta_j, T_1, \dots, T_{j^*}, \theta)) \\ +(\gamma_{j+1} - (-1)^{k_j - k_{j-1}}\gamma_j)\psi_2'(\theta_j, T_1, \dots, T_{j^*}, \theta, \nu_1, \dots, \nu_{j^*})b, \\ j = 1,2,\dots,j^*-1 \end{bmatrix}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

The defining elements $z_0 = (T_1^0, \dots, T_{j^*}^0, \theta^0, \nu_1^0, \dots, \nu_{j^*}^0)$ of OC $u(t, T_1^0, \dots, t_{j^*}^0, \theta^0)$, $t \in T$, of the base problem are constructed by solving the equation

$$R_0(z) = 0. \tag{59}$$

Definition 12. *The procedure of solving equation* (59) *is called a finishing one with respect to the solution to problem* (31) [11, part 2].

To solve equation (59), the Newton method is used. An approximation z_0^l of the elements z_0 is calculated as $z_0^l = z_0^{l-1} - I_0^{-1}(z_0^{l-1})R_0(z_0^{l-1})$, $l = 1, \dots, l_0$, where $I_0(z)$ is the Jacobi matrix of system (59).

We put $z_0 = z_0^{l_0}$ where $z_0^{l_0}$ is an approximation which satisfies the condition $\|R_0(z_0^{l_0})\| \le \varepsilon$ for a given $\varepsilon \ge 0$.

As an initial approximation $z_0^0 = (T_1^{0(0)}, \dots, T_{j^*}^{0(0)}, \theta^{0(0)}, \nu_1^{0(0)}, \dots, \nu_{j^*}^{0(0)})$ we take elements $T_{sup}^0(j|\theta^{0(0)}) \cup T_{n0}(j|\theta^{0(0)})$, $j \in J$, the set $\theta^{0(0)}$ and values of the Lagrange multipliers $\nu_j^{0(0)} = \nu^0(\theta_j^{0(0)})$, $j \in J$, obtained after solving problem (34)–(37) in the class of discrete controls.

On the base of the known values $F_j(\theta_j^{0(0)} - t_k^{0(0)})$, $t_k^{0(0)} \in T_{sn}(j)$; $F_j(\theta_j^{0(0)} - \theta_{j-1}^{0(0)})$, r_j, $j \in J$, we form $R_0(z_0^0)$, $I_0(z_0^0)$. On each iteration of the Newton method the integration of functions $F_j(t)$, $j \in J$, is performed on intervals of the length $|\theta_j^{0(l)} - \theta_j^{0(l-1)} + t_k^{0(l-1)} - t_k^{0(l)}|$, $k = k_{j-1}, \dots, k_j$, $|\theta_j^{0(l)} - \theta_{j-1}^{0(l)} - \theta_j^{0(l-1)} + \theta_{j-1}^{0(l-1)}|$. When these integrations are performed in parallel, the total complexity of the finishing procedure equals

$$A_{fin} = \sum_{l=1}^{l_0} \max_{j \in J} \Big\{ |\theta_j^{0(l)} - \theta_{j-1}^{0(l)} - \theta_j^{0(l-1)} + \theta_{j-1}^{0(l-1)}|, \\ \max_{k = k_{j-1}, \dots, k_j} |\theta_j^{0(l)} - \theta_j^{0(l-1)} + t_k^{0(l-1)} - t_k^{0(l)}| \Big\} / t^*.$$

Let $z_k = (T_1^k, \dots, T_{j^*}^k, \theta^k, \nu_1^k, \dots, \nu_{j^*}^k)$, $k = 1,2,\dots,s$; $z_s(\mu) = \sum_{k=0}^s \mu^k z_k$.

By using the Taylor formula we expand the vector-function $\sum_{k=0}^s \mu^k R_k(z_s(\mu))$ by powers of μ up to s and equate the coefficients of the expansion to zero. As a result, we obtain nondegenerate systems of linear equations for successive calculation of the vectors $T_1^k, \dots, T_{j^*}^k, \theta^k, \nu_1^k, \dots, \nu_{j^*}^k$, $k = 1,2,\dots,s$:

$$I_0 z_1 = -R_1(z_0),$$
$$I_0 z_2 = -\frac{\partial R_1}{\partial z}(z_0) z_1 - \frac{1}{2} z_1' \frac{\partial^2 R_0}{\partial z^2}(z_0) z_1 - R_2(z_0), \tag{60}$$
$$\cdots\cdots\cdots\cdots\cdots$$

The Jacobi matrix $I_0 = I_0(z_0)$ is calculated on the last iteration of the finishing procedure. Right sides of system (60) are formed by integrating equations (58) and differential equations for partial derivatives of the vector-functions x_k, ψ_k with respect to the components of the vectors $T_1, \dots, T_{j^*}, \theta, \nu_1, \dots, \nu_{j^*}$ (see, for example, [17]). In doing so, it is taken into account that $u(t, T_1^0, \dots, T_{j^*}^0, \theta^0) = u^0(t)$, $x_0(t, T_1^0, \dots, T_{j^*}^0, \theta^0) = x^0(t)$, $\psi(t, T_1^0, \dots, T_{j^*}^0, \theta^0, \nu_1^0, \dots, \nu_{j^*}^0) = \psi^0(t)$, $t \in T$.

We calculate coefficients z_k, $k = 1, \dots, s$, and find $t_i^s(\delta) = \sum_{k=0}^{s} \delta^k t_i^k$, $i = 1, \dots, k^*$; $\theta_j^s(\delta) = \sum_{k=0}^{s} \delta^k \theta_j^k$, $j = 1, \dots, j^*-1$. The control $u^s(t)$, $t \in T$, of the form (49) with $t_i = t_i^s(\delta)$, $i = 1, \dots, k^*$, is taken as an open-loop solution to problem (1).

As a rule, there are no strict limitations on the time of constructing the optimal open-loop control so here we are not discussing the ways of calculating required values of functions $x_k(t)$, $\psi_k(t)$, $t \in T$, $k = 1, \dots, s$, and their derivatives. Let us dwell on this question when describing a method of constructing a positional solution.

6.4. Realization of the Optimal Feedback

As shown above, in a particular control process the behaviour of a nonlinear system is described by equation (5) and the problem is to calculate a realization $u^*(\tau)$ of the optimal feedback at each current instant $\tau \in T_h$. As in the case of constructing an open-loop solution, for constructing a realization of an optimal feedback one can use either a solution to a piecewise linear problem or its asymptotic correction.

In Sec. 5, the construction of the realization of the optimal feedback in a piecewise linear problem is described. There, at each instant $\tau \in T_h$, the solution to the problem

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = \hat{f}(x) + bu, \quad x(\tau) = x^*(\tau),$$
$$x(t^*) \in X^*, \quad |u(t)| \le 1, \quad t \in T^\tau, \tag{61}$$

is constructed, the signal $u^*(\tau) = u^0(\tau|\tau, x^*(\tau))$ is calculated and required auxiliary information is formed.

Below we describe a method of constructing realizations $u^*(\tau)$, $\tau \in T_h$, with the use of asymptotic expansions.

To introduce notations concerned with the optimal feedback control we consider family (45) as an element of a more general family

$$c'x(t^*) \longrightarrow \max, \quad \dot{x} = \hat{f}(x) + \mu g(x) + bu, \quad x(\tau) = z, \tag{62}$$
$$x(t^*) \in X^*, \quad |u(t)| \le 1, \quad t \in T^\tau,$$

also depending on $\tau \in T$ and $z \in X$.

Let $u^s(t|\tau,z) = \{u^s_\mu(t|\tau,z),\ \mu \to 0\}, t \in T^\tau$, be an asymptotically s-optimal open-loop control for a position (τ, z), Ω_s be a set of positions (τ, z), for which there exist open-loop asymptotically s-optimal controls, $u^s_\mu(\tau, z) = u^s_\mu(\tau|\tau, z)$, $(\tau, z) \in \Omega_s$.

Definition 13. *A family $u^s(\tau,z) = \{u^s_\mu(\tau,z), \mu \to 0\}$, $(\tau,z) \in \Omega_s$, is said to be an s-optimal feedback control.*

Consider the behaviour of the system closed by the s-optimal feedback under unknown disturbances:

$$\dot{x} = f(x) + bu^s_\delta(t,x) + w(t), \quad x(0) = x_0. \tag{63}$$

Denote by $x^*(t)$, $t \in T$, a trajectory of equation (63) corresponding to a realizing disturbance $w^*(t)$, $t \in T$.

Definition 14. *A function $u^{s*}(t) = u^s_\delta(t, x^*(t))$, $t \in T_h$, is said to be a realization of s-optimal feedback, and any device able to calculate its values in real time is called an s-optimal controller* [9].

An s-optimal controller with a chosen value s is taken as an optimal controller solving the problem of optimization of a nonlinear system.

Let us describe an algorithm of operating a 1-optimal controller.

Before starting the controller, on the interval T we fix N points s_l, $l = 1, \dots, N$ so that N_j points s_l, $s = 1, \dots, N_j$ $(N_{j^*} = N)$, are located on the interval $]0, \theta_j[$, $j \in J$. Then we calculate and store matrices $F_j(s_l)$, $l = N_{j-1}+1, \dots, N_j$, $j \in J$.

Suppose that the controller has been constructed and operated on the interval $[0, \tau[$, and at instant $\tau \in T_h$, the system (63) is in a state $x^*(\tau)$. We solve base problem (61) for this state. The defining elements $z_0(\tau) = (T^0_{j(\tau)}(\tau), \dots, T^0_{j^*}(\tau)$, $\theta^0(\tau)$, $\nu^0_{j(\tau)}(\tau), \dots, \nu^0_{j^*}(\tau))$ are calculated from the equation

$$R_0(z(\tau)) = 0, \tag{64}$$

where

$$R_0(z(\tau)) = \begin{bmatrix} H_j x_0(\theta_j(\tau), T_{j(\tau)}(\tau), \dots, T_{j^*}(\tau), \theta(\tau)) - g_j, \ j = j(\tau), \dots, j^* \\ \psi_0'(t_i(\tau), T_{j(\tau)}(\tau), \dots, T_{j^*}(\tau), \theta(\tau), \nu_{j(\tau)}(\tau), \dots, \nu_{j^*}(\tau))b, \\ i = 1, 2, \dots, k^*(\tau) \\ \nu_j' H_j (A_j x_0(\theta_j(\tau), T_{j(\tau)}(\tau), \dots, T_{j^*}(\tau), \theta(\tau)) + \\ + a_j + (-1)^{k_j - k_{j-1}} \gamma_j b) + \\ + (\gamma_{j+1} - (-1)^{k_j - k_{j-1}} \gamma_j) \times \\ \times \psi_0'(\theta_j(\tau), T_{j(\tau)}(\tau), \dots, T_{j^*}(\tau), \theta(\tau), \nu_{j(\tau)}(\tau), \dots, \nu_{j^*}(\tau))b, \\ j = j(\tau), \dots, j^* - 1 \end{bmatrix}$$

As above we assume that on the interval T^τ the transition instants $\theta_{j(\tau)}, \dots, \theta_{j^*-1}$ are located and put $\theta_{j(\tau)-1} = \tau$.

A set $S(\tau) = \{j(\tau); k_j, \gamma_j, j = j(\tau), \dots, j^*\}$ is said to be a structure of the optimal control of the base problem. If the structure $S(\tau)$ coincides with $S(\tau - h)$, we solve equation (64) by the Newton method and take as an initial approximation $z_0^0(\tau)$ elements $z_0(\tau - h)$ constructed at a previous instant $\tau - h$. The analysis of situations of changing the structure is performed by analogy with [15].

In the course of solving the base problem we form and store information which is used further for calculation of the coefficients of the first approximation: $\int_{t_i^0(\tau)}^{t_{i+1}^0(\tau)} F_j^{-1}(\vartheta) b d\vartheta$, $F_j^{-1}(t_i^0(\tau))$, $i = k_{j-1}(\tau)+1, \dots, k_j(\tau)$, $j = j(\tau), \dots, j^*$;

$F_j(\theta_j^0(\tau))$, $\quad F_j^{-1}(\theta_{j-1}^0(\tau))$, $\quad \int_{t^0_{k_j(\tau)}(\tau)}^{\theta_j^0(\tau)} F_j^{-1}(\vartheta) b d\vartheta$, $\quad \int_{\theta_{j-1}^0(\tau)}^{t^0_{k_{j-1}(\tau)+1}(\tau)} F_j^{-1}(\vartheta) b d\vartheta$,

$j = j(\tau), \dots, j^*$; $\int_{\theta_{j-1}^0(\tau)}^{s_l} F_j^{-1}(\vartheta) a_j d\vartheta$, $l = N_{j-1}(\tau)+1, \dots, N_j(\tau)$, $j = j(\tau), \dots, j^*$, where $N_j(\tau)$ is a number of fixed in advance points s_l, $l = 1, \dots, N$, located on interval $]\tau, \theta_j^0(\tau)[$, $j = j(\tau), \dots, j^*$.

To calculate $z_1(\tau) = (T^1_{j(\tau)}(\tau), \dots, T^1_{j^*}(\tau), \theta^1(\tau), \nu^1_{j(\tau)}(\tau), \dots, \nu^1_{j^*}(\tau))$ we solve the system

$$I_0(z_0(\tau)) z_1(\tau) = -R_1(z_0(\tau)), \tag{65}$$

where $I_0(z_0(\tau))$ is the Jacobi matrix of equation (64) constructed when solving the base problem. Values $x_1(t^*)$, $\psi_1(t_i^0(\tau))$, $i = 1, \dots, k^*(\tau)$, required for forming the right side of equation (65), are calculated from the equations

$$\dot{x}_1 = A_j x_1 + g_j(x_0(t|\tau)), \; t \in [\theta_{j-1}(\tau), \theta_j(\tau)[, \; j = j(\tau), \dots, j^*; \quad x_1(\tau) = 0;$$

$$\dot{\psi}_1 = -A_j' \psi_1 - \frac{\partial H_j(x_0(t|\tau), \psi_0(t|\tau))}{\partial x}, \quad t \in]\theta_{j-1}(\tau), \theta_j(\tau)], \; j = j(\tau), \dots, j^*;$$
$$\psi_1(t^*) = 0. \tag{66}$$

The idea of fast calculation of the values $x_1(\theta_j)$, $j = 1, \dots, j^*$; $\psi_1(t_i^0(\tau))$, $i = 1, \dots, k^*(\tau)$, is based on the use of quadratures

$$x_1(\theta_j) = \sum_{r=j(\tau)}^{j} \Phi_{jr} \sum_{l=N_{r-1}(\tau)+1}^{N_r(\tau)} F_r(\theta_r) F_r^{-1}(s_l) g_r(x^0(s_l|\tau)) h_l,$$

$$\begin{aligned} \psi_1'(t_i^0(\tau)) = & \sum_{r=j+1}^{j^*} \sum_{l=N_{r-1}(\tau)+1}^{N_r(\tau)} \psi_0'(s_l|\tau) \frac{\partial g_r'}{\partial x}(x^0(s_l|\tau)) F_r(s_l) F_r^{-1}(\theta_{r-1}) \\ & \times \Phi_{r-1,j} F_j(\theta_j) F_j^{-1}(t_i(\tau)) h_l \\ & + \sum_{l:\; l \le N_j(\tau),\; s_l \ge t_i^0(\tau)} \psi_0'(s_l|\tau) \frac{\partial g_j'}{\partial x}(x^0(s_l|\tau)) F_j(s_l) F_j^{-1}(t_i(\tau)) h_l, \end{aligned}$$

where h_l, $l = 1, \dots, N$, are coefficients defined by the chosen quadrature (in the simplest case $h_l = \text{const}$, $l = 1, \dots, N$).

Required values $x^0(s_l|\tau)$, $\psi_0(s_l|\tau)$, $l = 1, \dots, N(\tau)$, are calculated with the use of information formed before starting the controller and after solving the base problem.

From equation (65) we find $z_1(\tau)$ and calculate $t_i^1(\delta|\tau) = t_i^0(\tau) + \delta t_i^1(\tau)$, $i = 1, \dots, k^*(\tau)$; $\theta_j^1(\delta|\tau) = \theta_j^0(\tau) + \delta\theta_j^1(\tau)$, $j = j(\tau), \dots, j^*-1$. If

$$\begin{aligned} \tau < t_1^1(\delta|\tau) < \cdots < t_{k_{j(\tau)}}^1(\delta|\tau) < \theta_{j(\tau)}^1(\delta|\tau) < \dots \\ < \theta_{j^*-1}^1(\delta|\tau) < t_{k_{j^*-1}+1}^1(\delta|\tau) < \cdots < t_{k^*(\tau)}^1(\delta|\tau) < t^*, \end{aligned} \tag{67}$$

the structure of the control $u^1(t, \delta|\tau, x^*(\tau))$, $t \in T^\tau$, coincides with the structure of the optimal control $u^0(t|\tau, x^*(\tau))$, $t \in T^\tau$, of the base problem. In this case, the input of system (4) is fed by the signal $u^{1*}(\tau) = \gamma_{j(\tau)}(\tau)$. The violation of condition (67) means that the number δ is greater than such a number $\mu_0 > 0$ [8] that for any μ, $|\mu| < \mu_0$ the structure of OC $u^0(t, \mu|\tau, x^*(\tau))$, $t \in T^\tau$, coincides with the structure of the control $u^0(t|\tau, x^*(\tau))$, $t \in T^\tau$. In this case, we put

$$u^{1*}(\tau) = \begin{cases} \gamma_{j(\tau)}(\tau) & \text{if } t_1^1(\delta|\tau) > \tau, \\ -\gamma_{j(\tau)}(\tau) & \text{if } t_1^1(\delta|\tau) < \tau. \end{cases}$$

In doing so, we assume that the length of intervals, where the structures of controls $u^1(t, \delta|\tau, x^*(\tau))$, $t \in T^\tau$, and $u^0(t|\tau, x^*(\tau))$, $t \in T^\tau$, are distinguished, is insignificant in comparison with the duration t^* of the process.

6.5. Example

The proposed approach to optimize nonlinear systems is illustrated by the following problem of optimal damping of a mathematical pendulum

$$\begin{aligned} &\int_0^{10} u(t)dt \longrightarrow \min, \\ &\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 + u, \quad x_1(0) = 1.5, \quad x_2(0) = 0, \\ &x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5, \quad t \in T = [0, 10], \end{aligned} \tag{68}$$

in the domain $X = \{(x_1, x_2) : \ |x_1| < \pi/2\}$.

We use two approximations of the nonlinear element $-\sin x_1$: 1) linear approximation $-x_1$, $x \in X$; 2) piecewise linear approximation $(1-4/\pi)x_1 + 1 - \pi/2$, $x \in X_1 = \{(x_1, x_2) : \ \pi/4 < x_1 < \pi/2\}$; $-x_1$, $x \in X_2 = \{(x_1, x_2) : \ |x_1| < \pi/4\}$. From the linear approximation we obtain $\delta_1 = 0.570796$, and by using the nonlinear approximation one gets $\delta_2 = 0.110721$.

Thus, in the first case, problem (68) is imbedded into the family

$$\begin{aligned} &\int_0^{10} u(t)dt \longrightarrow \min, \\ &\dot{x}_1 = x_2, \ \dot{x}_2 = -x_1 + \frac{\mu(x_1 - \sin x_1)}{\delta_1} + u, \ x_1(0) = 1.5, \ x_2(0) = 0, \\ &x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5, \ t \in T, \end{aligned} \tag{69}$$

in the second case, we use the family

$$\int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2,$$
$$\dot{x}_2 = \begin{cases} \left(1-\frac{4}{\pi}\right)x_1 + 1 - \frac{\pi}{2} + \frac{\mu((4/\pi-1)x_1-1+\pi/2-\sin x_1)}{\delta_2} + u, & x \in X_1, \\ -x_1 + \frac{\mu(x_1-\sin x_1)}{\delta_2} + u, & x \in X_2, \end{cases} \tag{70}$$
$$x_1(0) = 1.5,\ x_2(0) = 0,\ x_1(10) = x_2(10) = 0,\ 0 \le u(t) \le 0.5,\ t \in T.$$

The parametrized form of problem (70) is

$$\begin{aligned} &\int_0^{10} u(t)dt \longrightarrow \min, \\ &\dot{x}_1 = x_2, \\ &\dot{x}_2 = \left(1-\frac{4}{\pi}\right)x_1 + 1 - \frac{\pi}{2} + \frac{\mu((4/\pi-1)x_1 - 1 + \pi/2 - \sin x_1)}{\delta_2} + u,\ t \in [0, \theta_1[, \\ &x_1(0) = 1.5,\ x_2(0) = 0,\ x_1(\theta_1) = \frac{\pi}{4}, \\ &\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + \frac{\mu(x_1 - \sin x_1)}{\delta_2} + u,\ t \in [\theta_1, 10], \\ &x_1(10) = x_2(10) = 0,\ 0 \le u(t) \le 0.5,\ t \in T. \end{aligned} \tag{71}$$

The base problem for problem (71) is as follows

$$\begin{aligned} &\int_0^{10} u(t)dt \longrightarrow \min, \\ &\dot{x}_1 = x_2,\ \dot{x}_2 = (1-4/\pi)x_1 + 1 - \pi/2 + u,\ t \in [0, \theta_1[, \\ &x_1(0) = 1.5,\ x_2(0) = 0, \quad x_1(\theta_1) = \pi/4, \\ &\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ t \in [\theta_1, 10], \\ &x_1(10) = x_2(10) = 0, \quad 0 \le u(t) \le 0.5,\ t \in T. \end{aligned} \tag{72}$$

Table 1 contains results of open-loop solution to problem (68). Trajectories of system (68) have been constructed by the following controls: 1) $u_1^0(t)$, $t \in T$, is the optimal control of the linear base problem

$$\int_0^{10} u(t)dt \longrightarrow \min, \quad \dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ x_1(0) = 1.5,\ x_2(0) = 0,$$
$$x_1(10) = x_2(10) = 0,\ 0 \le u(t) \le 0.5,\ t \in T;$$

2) $u_1^1(t)$, $t \in T$, is the realization of asymptotically 1-optimal open-loop control of problem (69) for the fixed value $\mu = \delta_1$; 3) $u_2^0(t)$, $t \in T$, is the optimal control of the piecewise linear base problem (72); 4) $u_2^1(t)$, $t \in T$, is the realization of asymptotically 1-optimal open-loop control of problem (71) for the fixed value $\mu = \delta_2$; 5) $u^0(t, \delta)$, $t \in T$, is the optimal open-loop control of problem (68). The control $u^0(t, \delta)$, $t \in T$, has been constructed by iterative procedure [11, part 5] based on the Pontryagin maximum principle. In doing so the integration of the nonlinear system on interval T is performed on each iteration so this procedure is valuable comparing results in illustrative examples but not for practical use.

In each case the control has the form

$$u(t) = \begin{cases} 0, & t \in [0, t_1[\cup [t_2, t_3[\cup [t_4, 10[, \\ 0.5, & t \in [t_1, t_2[\cup [t_3, t_4[. \end{cases} \tag{73}$$

Table 1 contains switching points of these controls, the values of the transition instant θ_1 when the piecewise linear approximation is used, the values of the performance index and the endpoint states of system (68). The curve 1 in Fig. 5 presents the phase trajectory of system (68) constructed by the control $u_2^1(t)$, $t \in T$. Let us construct the positional solution to problem (68). At first, we analyze the behaviour of system (68) without disturbances. We control the system by the realization $u^{1*}(t)$, $t \in T$, constructed by the 1-optimal controller ($h = 0.01$). Necessary values of the functions $x_1(t)$, $\psi_1(t)$, $t \in T^\tau$, $\tau \in T_h$, have been constructed by the Fehlberg fourth-fifth order Runge–Kutta method [6]. The control $u^{1*}(t)$, $t \in T$, has the form (73) with switching points 1.06, 2.58, 7.567630, 9.039940 and the transition instant $\theta_1^* = 1.21$. At the instant $t^* = 10$ the trajectory of system (68) reaches the state $(-0.000129, -0.000455)$, the value of the performance index is 1.496155.

Table 1

Control	Switching points	Instant θ_1	Performance index	Endpoint state
$u_1^0(t)$, $t \in T$	0.722734 2.418858 7.005920 8.702044	—	1.696124	0.017095 -0.516580
$u_1^1(t)$, $t \in T$	1.007916 2.517443 7.547231 9.017574	—	1.489935	-0.012199 -0.055845
$u_2^0(t)$, $t \in T$	1.078442 2.593439 7.371674 8.876624	1.233553	1.509974	-0.013850 -0.111839
$u_2^1(t)$, $t \in T$	1.064668 2.573226 7.553250 9.036841	1.218658	1.496074	-0.001177 -0.008707
$u^0(t, \delta)$, $t \in T$	1.064547 2.573684 7.566068 9.049386	—	1.496228	10^{-8} 10^{-8}

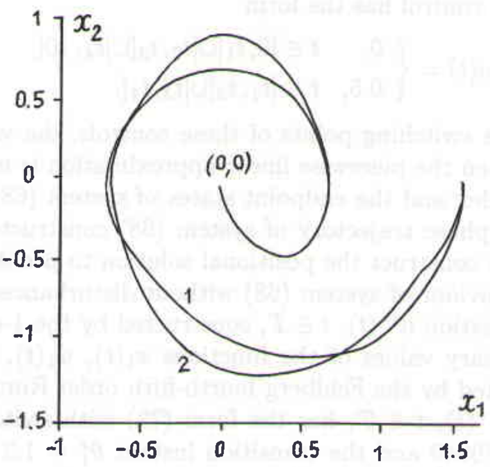

Fig. 5

Table 2 contains switching points of the realizations, the corresponding values of the performance index and the endpoint states of system (74). The curve 2 in Fig. 5 presents the phase trajectory of system (74) under control $u^{1*}(t)$, $t \in T$.

Table 2

Control	Switching points	Performance index	Endpoint state
$u^{1*}(t)$, $t \in T$	1.3 2.7 7.789350 9.287743	1.449196	0.000016 -0.001192
$u^{1*}_{10}(t)$, $t \in T$	1.3 2.7 7.717981 9.257927	1.469973	-0.003723 -0.002823
$u^{1*}_{50}(t)$, $t \in T$	1.3 2.7 7.778486 9.278203	1.449858	-0.000074 -0.001229
$u^{1*}_{100}(t)$, $t \in T$	1.3 2.7 7.778420 9.278330	1.449955	10^{-6} -0.001202
$u^{1*}_{300}(t)$, $t \in T$	1.3 2.7 7.778403 9.278366	1.449982	0.000014 -0.001133

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