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Stochastic Models of Games Which Become Fairer with Stopping Time*

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Abstract. In this paper we consider two stochastic models of games which include martingales in the limit and games fairer with time. It turns out that there is a close relationship between them. This remark together with a stopping time technique allows us to obtain some limit theorems of Doob's type for these models.

1. Notations and Statements of the Results

Throughout this note we are dealing with a complete probability space $(\Omega, \mathcal{A}, P)$ and a stochastic basis $(\mathcal{A}_n)$, i.e., an increasing sequence of sub σ-algebras of $\mathcal{A}$ with $\mathcal{A}_n \uparrow \mathcal{A}$. By T we denote the set of all bounded stopping times w.r.t. $(\mathcal{A}_n)$. Then equipped with the usual order "$\leq$", given by $\sigma \leq \tau$ iff $\sigma(\omega) \leq \tau(\omega)$, a.s., T becomes a directed set.

Thus one can regard the set N of all positive integers as a cofinal subset of T. From now on instead of the usual set N, we shall look at the set T of all bounded stopping times and consider only the sequences (τ_n) of T for which each τ_n satisfies:

$$n \leq \tau_n \leq \min\{k \in N,\ P(\{\tau_{n+1} = k\}) > 0\}.$$

To avoid any confusion we will denote by $\{\tau_n\}$ the set of all elements of (τ_n) and by U we always mean a cofinal subset of N. Thus for any $p \in N$ and $\tau \in T$ with $p \leq \tau$ we can use the following notations:

$$U(p) = \{k \in U,\ k \geq p\}$$

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and

$$U(p, \tau) = \{k \in U,\ p \leq k \leq \tau\}.$$

Now let $L^1(\mathcal{A})$ denote the Banach space of all (equivalence classes of) $\mathcal{A}$-measurable integrable random variables (r.v.'s) $X : \Omega \to \mathbb{R}$ with

$$E(|X|) = \int_\Omega |X| dP < \infty.$$

Unless otherwise stated, we shall consider in the sequel only the sequences (X_n) in $L^1(\mathcal{A})$ which are assumed to be adapted to $(\mathcal{A}_n)$, i.e., each X_n is $\mathcal{A}_n$-measurable. Such a sequence (X_n) would be regarded sometimes as a (stochastic) game.

For other notations we refer to [2,4]. Here the following stochastic models of games are the first important starting point of our consideration.

Definition 1.1. *A sequence (X_n) is said to be:*
a) *a martingale in the limit, if*

$$\lim_n \sup_{m \geq n} |E^n(X_m) - X_n| = 0, \ a.s.,$$

where given $\tau \in T$ and $X \in L^1(\mathcal{A})$, $E^\tau(X)$ denotes the $\mathcal{A}_\tau$-conditional expectation of X (cf. [8]).
b) *a mil, if for every $\varepsilon > 0$ there exists $p \in N$ such that for all $n \in N(p)$ we have*

$$P\big(\sup_{q \in N(p,n)} |E^q(X_n) - X_q| > \varepsilon\big) < \varepsilon.$$

c) *a game which becomes fairer with time (or briefly, a game fairer with time), if for every $\varepsilon > 0$ there exists $p \in N$ such that for any $n \in N(p)$ we have*

$$\sup_{q \in N(p,n)} P\big(|E^q(X_n) - X_q| > \varepsilon\big) < \varepsilon.$$

Martingales in the limit were first introduced by Mucci [7] who proved that *every L^1-bounded real-valued martingale in the limit converges a.s.* Then an immediate natural question arises if Mucci's result still holds for L^1-bounded martingales in the limit, taking values in a Banach space with the Radon–Nykodym property? During about the next ten years, one had found a series of positive answers but for only particular cases. Finally, Talagrand [9] positively solved not only this problem but also for the next class of mils, which was also pointed out by him to be essentially more general than that involving the question. However the limit problem is still open for the third class of games fairer with time, earlier introduced by Blake [1] and then considered by Mucci [6] since it seems to be too large. An attempt to improving the Blake's result was later made by Luu [3], who brought out for the first time a close relationship between mils and games fairer with time which can be formulated that *every game fairer with time contains a subsequence that is a mil.* This simple remark together with the Talagrand's result [9, Theorem 4] allowed us to conclude in [3] that *every L^1-bounded real-valued game fairer with time converges in probability.* But in many situations, a game would converge a.s. or in probability, resp., although it is not a mil or a game fairer with time, resp. This led us to consider in [2, 4] the following two reasonable stochastic models in which stopping times are taken into account.

Definition 1.2. *A sequence (X_n) is said to be:*
a) *a $\{\tau_n\}$-mil, if for every $\varepsilon > 0$ there exists $p \in N$ such that for all $n \in N(p)$ we have*

$$P\Big(\sup_{q \in N(p,\tau_n)} |E^q(X_{\tau_n}) - X_q| > \varepsilon\Big) < \varepsilon.$$

b) *a game which becomes fairer with a sequence (τ_n) of stopping times (or briefly, a $\{\tau_n\}$-game), if for every $\varepsilon > 0$ there exists $p \in N$ such that for any $n \in N(p)$ we have*

$$\sup_{q \in N(p,\tau_n)} P\Big(\big|E^q(X_{\tau_n}) - X_q\big| > \varepsilon\Big) < \varepsilon. \tag{1}$$

It is clear that in the usual case, when each $\tau_n = n$, these models are reduced to the last two classes of games given in the previous definition.

In reality, the notions of $\{\tau_n\}$-mils and $\{\tau_n\}$-games have been recently introduced in ([2], Definition 0.2) and ([4], Definition 2.3 and Remark 3.2), resp., where the author has proved that *every real-valued $\{\tau_n\}$-mil (X_n) with*

$$\liminf_n E(|X_{\tau_n}|) < \infty \tag{2}$$

converges a.s. However for a $\{\tau_n\}$-game (X_n) together with (2), the author had to require another additional assumption to get the convergence in probability of (X_n). The reason is one could not apply a procedure of proof like it was done in [4] for games fairer with time. The main aim of this note is to continue [2, 4] overcoming the difficulties we have met before to obtain the convergence in probability of a $\{\tau_n\}$-game satisfying only (2).

2. Main Results

First, we present the following important property of $\{\tau_n\}$-games which is of particular interest.

Lemma 2.1. *Let (X_n) be a $\{\tau_n\}$-game satisfying (2). Suppose that it contains a subsequence which converges to zero in probability. Then so does (X_n).*

Proof. To prove the lemma, let (X_n) be a $\{\tau_n\}$-game satisfying (2) which contains a subsequence, say $(X_u,\ u \in U)$ for some cofinal subset U of N which converges to zero in probability. Suppose on the contrary that (X_n) does not converge to zero in probability. Then there exists some positive real number $a > 0$ such that

$$\limsup_n P\Big(|X_n| > \frac{6a}{5}\Big) > a. \tag{3}$$

We conclude that

For every $0 < \varepsilon < a/4$ and $n_1 \in N$ there exists $n_2 \in N(n_1+1)$ such that for each $A \in \mathcal{A}_{\tau_{n_1}}$ with $P(A) < a/4$ and $n \in N(n_2)$ one can find $B \in \mathcal{A}_{\tau_{n_2}}$ with $B \cap A = \phi$, $P(B) < \varepsilon$ and such that

$$\int_B |X_{\tau_n}| dP \geq \frac{a^2}{8}. \tag{4}$$

To see the conclusion, let $0 < \varepsilon < a/4$ and $n_1 \in N$ be given. Then by (1) and (3) there exists $p \geq \tau_{n_1}$ so large that for every $n \geq p$ we have

$$P\Big(|E^p(X_{\tau_n}) - X_p| > \frac{a}{5}\Big) < \frac{\varepsilon}{2} \tag{5}$$

and

$$P(C) > a \quad \text{with} \quad C = \Big\{|X_p| > \frac{6a}{5}\Big\}$$

since (X_n) is a $\{\tau_n\}$-game and (X_n) does not converge to zero in probability. Thus if we put

$$C_1^1 = \Big\{X_p > \frac{6a}{5}\Big\} \quad \text{and} \quad C_2^1 = \Big\{-X_p > \frac{6a}{5}\Big\} \setminus C_1^1$$

then it is evident that

$$C = C_1^1 \cup C_2^1 \text{ with } C_1^1 \cap C_2^1 = \phi.$$

On the other hand, as the subsequence $(X_u,\ u \in U)$ converges to zero in probability there exists $u_1 \in U(p)$ such that

$$P(D) < \frac{\varepsilon}{2} \quad \text{with} \quad D = \Big\{|X_{u_1}| > \frac{3a}{5}\Big\}. \tag{6}$$

Now let define $n_2 = u_1 + 1$. We shall show that n_2 satisfies all the requirements of the conclusion. For the purpose let $A \in \mathcal{A}_{\tau_{n_1}}$ with $P(A) < a/4$ and $n \in N(n_2)$ be given. By (5) we get

$$P(G) < \frac{\varepsilon}{2} \quad \text{with} \quad G = \Big\{|E^p(X_{\tau_n}) - X_p| > \frac{a}{5}\Big\}. \tag{7}$$

Further, let define

$$C_1^2 = C_1^1 \setminus (G \cup A); \quad C_2^2 = C_2^1 \setminus (G \cup A)$$

and $C^2 = C_1^2 \cup C_2^2$. It is easily seen that

$$C_1^2 \cap C_2^2 = \phi \quad \text{and} \quad C^2 = C \setminus (G \cup A).$$

Then by a simple calculation we have

$$P(C^2) > \frac{5a}{8}. \tag{8}$$

Similarly, by (5) we also have

$$P(H) < \frac{\varepsilon}{2} \quad \text{with} \quad H = \Big\{|E^{u_1}(X_{\tau_n}) - X_{u_1}| > \frac{a}{5}\Big\}. \tag{9}$$

Finally, take

$$D^1 = D \cup H; \quad B_1 = C_1^2 \cap D^1; \quad B_2 = C_2^2 \cap D^1$$

and

$$B = B_1 \cup B_2.$$

It is easily checked that $B \in \mathcal{A}_{u_1}$ with

$$P(B) \leq P(D^1) \leq \varepsilon.$$

We shall show that constructed in such a way the set B satisfies (4), proving the conclusion.

Indeed, since C^2 is contained in the outside of G, by (7) on C^2 we have

$$|E^p(X_{\tau_n}) - X_p| \leq \frac{a}{5}.$$

Equivalently,

$$X_p - \frac{a}{5} \leq E^p(X_{\tau_n}) \leq X_p + \frac{a}{5}.$$

Thus by the definition of C_1^2 and C_2^2 we get

$$\int_{C_1^2} X_{\tau_n} dP = \int_{C_1^2} E^p(X_{\tau_n}) dP \geq aP(C_1^2), \tag{10}$$

since on C_1^2 we have

$$E^p(X_{\tau_n}) \geq X_p - \frac{a}{5} > \frac{6a}{5} - \frac{a}{5} = a$$

and

$$-\int_{C_2^2} X_{\tau_n} dP = -\int_{C_2^2} E^p(X_{\tau_n}) dP \geq aP(C_2^2), \tag{11}$$

since on C_2^2 the following inequality holds

$$-E^p(X_{\tau_n}) \geq -X_p - \frac{a}{5} > \frac{6a}{5} - \frac{a}{5} = a.$$

Similarly, let define

$$D_1^2 = C_1^2 \setminus D^1 = C_1^2 \setminus (D \cup H)$$

and

$$D_2^2 = C_2^2 \setminus D^1 = C_2^2 \setminus (D \cup H).$$

By (6) and (9) on $D_1^2 \cup D_2^2$ we get the following inverse inequalities

$$|X_{u_1}| \leq \frac{3a}{5} \quad \text{and} \quad X_{u_1} - \frac{a}{5} \leq E^{u_1}(X_{\tau_n}) \leq X_{u_1} + \frac{a}{5}.$$

But note that $D_1^2, D_2^2 \in \mathcal{A}_{u_1}$. It follows that

$$\int_{D_1^2} X_{\tau_n} dP = \int_{D_1^2} E^{u_1}(X_{\tau_n}) dP \leq \frac{4a}{5} P(D_1^2) \leq \frac{4a}{5} P(C_1^2), \tag{12}$$

since on D_1^2 we have

$$E^{u_1}(X_{\tau_n}) \leq X_{u_1} + \frac{a}{5} < \frac{3a}{5} + \frac{a}{5} = \frac{4a}{5}$$

and

$$-\int_{D_2^2} X_{\tau_n} dP = -\int_{D_2^2} E^{u_1}(X_{\tau_n}) dP \le \frac{4a}{5} P(D_2^2) \le \frac{4a}{5} P(C_2^2), \tag{13}$$

since on D_2^2 one sees

$$-E^{u_1}(X_{\tau_n}) \le -X_{u_1} + \frac{a}{5} \le \frac{3a}{5} + \frac{a}{5} = \frac{4a}{5}.$$

But note that $B_1 \cap D_1^2 = \phi$ and $C_1^2 = B_1 \cup D_1^2$ then by (10) and (12), it follows that

$$\int_{B_1} |X_{\tau_n}| dP \ge \int_{B_1} X_{\tau_n} dP = \int_{C_1^2} X_{\tau_n} dP - \int_{D_1^2} X_{\tau_n} dP$$

$$\ge aP(C_1^2) - \frac{4aP(C_1^2)}{5} = \frac{aP(C_1^2)}{5}. \tag{14}$$

Similarly, as $B_2 \cap D_2^2 = \phi$ and $C_2^2 = B_2 \cup D_2^2$, it follows from (11) and (13) that

$$\int_{B_2} |X_{\tau_n}| dP \ge -\int_{B_2} X_{\tau_n} dP = -\Big(\int_{C_2^2} X_{\tau_n} dP - \int_{D_2^2} X_{\tau_n} dP\Big)$$

$$= \int_{D_2^2} X_{\tau_n} dP - \int_{C_2^2} X_{\tau_n} dP$$

$$\ge -\frac{4aP(C_2^2)}{5} + aP(C_2^2) = \frac{aP(C_2^2)}{5}.$$

This with (14) gives

$$\int_B |X_{\tau_n}| dP = \int_{B_1} |X_{\tau_n}| dP + \int_{B_2} |X_{\tau_n}| dP \ge \frac{a\big(P(C_1^2) + P(C_2^2)\big)}{5} = \frac{aP(C^2)}{5}.$$

Therefore substituting (8) in the right hand side we obtain

$$\int_B |X_{\tau_n}| dP \ge \frac{a^2}{8}.$$

It proves (4) and the conclusion.

Returning to the proof of the lemma, one can apply the conclusion to construct by induction a strictly increasing subsequence (n_p) of N with the following property:

Whenever $A \in \mathcal{A}_{\tau_{k_p}}$ with $P(A) < a/4$ and $n \in N(k_{p+1})$ there exists $B \in \mathcal{A}_{\tau_{k_{p+1}}}$ with $B \cap A = \phi$, $P(B) < a.2^{-(p+1)}$ and such that

$$\int_B |X_{\tau_n}| dP \ge \frac{a^2}{8}.$$

Thus given an arbitrarily big constant $M > 0$, there exists $p \in N$ such that

$$\frac{(p-1)a^2}{8} \ge M.$$

We shall construct next by finite induction with $j \leq p$ disjoint sets $B_j \in \mathcal{A}_{\tau_{k_j}}$ with $B_1 = \phi$, $P(B_j) < a.2^{-(j+1)}$ and

$$\int_{B_j} |X_{\tau_n}| dP \geq \frac{a^2}{8}.$$

Therefore, by taking $B = \bigcup_{j \leq p} B_j$ we have

$$\int_B |X_{\tau_n}| dP \geq \frac{(p-1)a^2}{8} \geq M,$$

which implies that

$$\lim_n E(|X_{\tau_n}|) = \infty.$$

It contradicts the assumption (2), proving the lemma. ∎

Further, as the main result of this note we shall apply the previous lemma together with Theorem 2.5 [5] to obtain the following statement.

Theorem 2.2. *Let (X_n) be a $\{\tau_n\}$-game which satisfies (2). Then (X_n) converges in probability to some $X \in L^1(\mathcal{A})$.*

Proof. Let (X_n) be a $\{\tau_n\}$-game satisfying (2). Then by passing to a subsequence of (τ_n), if necessary one can suppose directly that the sequence (X_{τ_n}) is L^1-bounded. We shall show first that there exists a cofinal subset U of N such that (X_n) becomes *a $\{\tau_n\}$-mil relative to U taken in the sense of Definition* 1.3 [5], i.e., for every $\varepsilon > 0$ there exists $p \in U$ such that for any $n \geq p$ we have

$$P\Big(\sup_{q \in U(p,\tau_n)} |E^q(X_{\tau_n}) - X_q| > \varepsilon\Big) < \varepsilon. \tag{15}$$

Indeed, since (X_n) is a $\{\tau_n\}$-game, by (1) one can find a strictly increasing sequence (k_n) of N such that for any $m, n \in N$ with $k_m \leq n$ we have

$$P\Big(|E^{k_m}(X_{\tau_n}) - X_{k_m}| > 2^{-m}\Big) < 2^{-m}. \tag{16}$$

Further, let define $U = \{k_n,\ n \in N\}$. We claim that constructed in such a way (X_n) becomes a $\{\tau_n\}$-mil relative to U. To see this let $\varepsilon > 0$ be given. Clearly there exists $m_0 \in N$ such that $2^{-m_0+1} < \varepsilon$. This with (16) yields that for every $n \geq k_{m_0}$ by setting $p = k_{m_0} \in U$ and

$$q(n) = \max\{m \in N,\ k_q \leq \tau_m\}$$

we have

$$P\Big(\sup_{q \in U(p,\tau_n)} |E^q(X_{\tau_n}) - X_q| > \varepsilon\Big) = P\Big(\sup_{m \in N(m_0,q(n))} |E^{k_m}(X_{\tau_n}) - X_{k_m}| > \varepsilon\Big)$$

$$\leq P\Big(\sup_{m \in N(m_0,q(n))} |E^{k_m}(X_{\tau_n}) - X_{k_m}| > 2^{-m}\Big) \leq \sum_{m=m_0}^{q(n)} 2^{-m} < 2^{-m_0+1} < \varepsilon.$$

It proves (15), hence the claim. Here is the best place to inform that such a $\{\tau_n\}$-mil (X_n) relative to U satisfies all the requirements of Theorem 2.5 [5] which guarantees that the subsequence $(X_u,\ u \in U)$ converges a.s. to some $X \in L^1(\mathcal{A})$. Thus if we define finally the sequence (P_n) by

$$P_n = X_n - E^n(X) \quad \text{or} \quad X_n = E^n(X) + P_n, \tag{17}$$

$n \in N$ then (P_n) is also a $\{\tau_n\}$-game satisfying

$$\liminf_n E(|P_{\tau_n}|) \le \liminf_n E(|X_{\tau_n}|) + E(|X|) < \infty$$

and (P_n) contains the subsequence $(P_u,\ u \in U)$ converging to zero a.s. since by Levy's Theorem the regular martingale $\big(E^n(X)\big)$, hence its subsequence $\big(E^u(X),\ u \in U\big)$, converges a.s. just to X as $\big(X_u,\ u \in U\big)$.

Therefore, by the previous lemma, (P_n) converges to zero in probability. This with the decomposition (17) and Levy's Theorem proves that the $\{\tau_n\}$-game (X_n) converges to X in probability as $\big(E^n(X)\big)$. It completes the proof. ■

It is clear that (2) follows immediately from the stronger assumption that (X_{τ_n}) is uniformly integrable. The main sense of this remark is that under the stronger hypothese we get the following converse conclusion:

Proposition 2.3. *Let (X_n) be a game with (X_{τ_n}) being uniformly integrable and (X_n) converging in probability. Suppose that either there exists some $d \in N$ such that each τ_n satisfies*

$$\text{Card}\,\{k \in N,\ P(\{\tau_n = k\}) > 0\} \le d,$$

or each τ_n is independent of the increments

$$\{(X_m - X_n),\ m \in N(n)\}.$$

Then (X_n) must be a $\{\tau_n\}$-game. Moreover the statement fails true if we remove the above two additional assumptions simultaneously.

Proof. To see the first part of the proposition, let (X_n) be a game with (X_{τ_n}) being uniformly integrable. Suppose that (X_n) converges in probability to some r.v. X. For the first case, assume in addition that there exists $d \in N$ such that for each $n \in N$ we have

$$\text{Card}\Big\{k \in N,\ P(\{\tau_n = k\}) > 0\Big\} \le d.$$

Then this with the stochastic convergence of (X_n) to X, implies that for a given $\varepsilon > 0$ there exists $p \in N$ such that

$$\sup_{n \ge p} P(|X_n - X| > \varepsilon) < \frac{\varepsilon}{d}.$$

Thus for all $n \ge p$ one obtains

$$
\begin{aligned}
P(|X_{\tau_n} - X| > \varepsilon) &\le \sum_{k \ge p} P\Big(\{\tau_n = k\} \cap \{|X_k - X| > \varepsilon\}\Big) \\
&\le \mathrm{Card}\Big\{k \ge p,\ P(\{\tau_n = k\}) > 0\Big\}.\sup_{k \ge p} P(|X_k - X| > \varepsilon) \\
&\le d.\frac{\varepsilon}{d} = \varepsilon,
\end{aligned}
$$

since $\tau_n \ge p$.

It means that the sequence (X_{τ_n}) converges also to X in probability. But (X_{τ_n}) was assumed to be uniformly integrable, it follows that (X_{τ_n}) converges to X in L^1 as well.

This with Chebyshev's inequality and Levy's Theorem guarantees that (X_n) is a $\{\tau_n\}$-game since for any $\varepsilon > 0$ and $p, n \in N$ with $p \le n$ we have

$$
\begin{aligned}
P\Big(\big|E^p(X_{\tau_n}) - X_p\big| > \varepsilon\Big) &\le P\Big(\big|E^p(X_{\tau_n}) - E^p(X)\big| > \frac{\varepsilon}{3}\Big) \\
&\quad + P\Big(\big|E^p(X) - X\big| > \frac{\varepsilon}{3}\Big) + P\Big(\big|X - X_p\big| > \frac{\varepsilon}{3}\Big).
\end{aligned}
$$

It proves the proposition for the first case.

To see the proposition in the second situation, where each τ_n is independent of the increments $\{(X_m - X_n),\ m \ge n\}$, as in the first case it is sufficient to show that (X_{τ_n}) converges also to X in probability.

To see this, let $\varepsilon > 0$, $p, n \in N$ be given with $p \le n$. Since τ_n is independent of $\{(X_m - X_n),\ m > n\}$ it follows that τ_n is independent of $(X_n - X)$ as well. Hence we have

$$
\begin{aligned}
P(|X_{\tau_n} - X| > \varepsilon) &\le \sum_{m=n}^{\infty} P(\{\tau_n = m\} \cap \{|X_m - X| > \varepsilon\}) \\
&\le \sum_{m \ge p} P(\tau_n = m).P\big(|X_m - X| > \varepsilon\big) \\
&= \sup_{m \ge p} P\big(|X_m - X| > \varepsilon\big),
\end{aligned}
$$

since $\tau_n \ge n$.

But by the common assumption, (X_n) converges to X in probability, then so does (X_{τ_n}). This with the last argument, used in the first case implies that (X_n) should be a $\{\tau_n\}$-game which completes the proof of the second situation of the proposition.

To construct a counter-example mentioned in the second part of the proposition, let $(\Omega, \mathcal{A}, P)$ be the usual Lebesgue probability space on $[0,1)$.

For each $m \in N$, let Q_m be the partition of $[0,1)$, given by

$$
Q_m = \Big\{\Big[\frac{j-1}{m}, \frac{j}{m}\Big),\ 1 \le j \le m\Big\}.
$$

Further, set $a_0 = 0$ and $a_m = a_{m-1} + m$, $m \in N$. Then for every $n \in N$, there is a unique pair $(m, j) \in N^2$ such that $1 \le j \le m$ and $n = a_{m-1} + j$.

Thus we can define .

It is clear that $0 \leq X_n \leq 1$ and (X_n) converges to zero in probability.

Now we are looking at the sequence (τ_n) of bounded stopping times given by

$$\tau_n = \sum_{j=1}^{n} (a_{n-1} + j).I_{\left[\frac{j-1}{m}, \frac{j}{m}\right)}.$$

It is clear that

$$n \leq a_{n-1} \leq \tau_n \leq a_n \leq \tau_{n+1}, \quad n \in N.$$

Moreover, each $X_{\tau_n} = 1$. Consequently, for any $q, n \in N$ with $2 \leq q \leq \tau_n$ we have

$$P\Big(|E^q(X_{\tau_n}) - X_q| > \frac{1}{2}\Big) = 1 - P(X_q \neq 0) > \frac{1}{2}.$$

Hence (X_n) cannot be a $\{\tau_n\}$-game. It completes the construction and the proof of the proposition. ∎

Finally, to convince the readers that the last two stochastic models are truely more realistic we propose the following simple remark.

Remark 1. There exists a nonnegative L^1-bounded game (X_n) which is not fairer with time. But it is a $\{\tau_n\}$-mil with each $X_{\tau_n} = 0$. Hence, it converges to zero, a.s.

Indeed, first we take $\Omega = [0, 1)$ and $a_0 = 0$. For $n \geq 1$, let define

$$a_n = \sum_{j=1}^{n} 2^{-(1+j)} \quad \text{and} \quad \mathcal{A}_n = \sigma - \big([a_{k-1}, a_k),\ k \leq n\big).$$

Let $\mathcal{A}$ be the smallest σ-algebra generated by $\bigcup_n \mathcal{A}_n$ and P the usual Lebesgue probability measure on $[0, 1)$, restricted to $\mathcal{A}$. On the probability space $\big([0, 1), \mathcal{A}, P\big)$ with its stochastic basis $(\mathcal{A}_n)$ we shall construct the desired example (X_n) as follows

$$X_n = 2^{n+1}.I_{[a_{n-1}, a_n)}, \quad n \in N,$$

where I_A denotes the characteristic function of $A \in \mathcal{A}$. It is evident that each $X_n \geq 0$ and $E(X_n) = 1$. However, note that for any $p, n \in N$ with $p < n$ we have

$$E^p(X_n) = (1 - a_p)^{-1}.I_{[a_p, 1)}.$$

Then

$$\Big\{|E^p(X_n) - X_p| > \frac{1}{2}\Big\} = [a_{p-1}, 1).$$

This implies

$$P\Big(|E^P(X_n) - X_p| > \frac{1}{2}\Big) = 1 - a_{p-1} > \frac{1}{2}.$$

It shows that (X_n) cannot be a game fairer with time.

On the other hand, if we look at the sequence (τ_n) of bounded stopping times given by

$$\tau_n(t) = \begin{cases} n, & t \in [a_n, 1) \\ n+1, & \text{elsewhere} \end{cases}$$

then it is clear that for every $n \in N$ we have

$$n \leq \tau_n \leq n+1 = \min \Big\{ k \in N,\ P\big(\{\tau_{n+1} = k\}\big) > 0 \Big\}.$$

Moreover, each $X_{\tau_n} = 0$. In this case, the statement that (X_n) is a $\{\tau_n\}$-mil, is equivalent to the fact that (X_n) converges to zero a.s. But the fact is trivial. The construction, hence the proof of the remark is complete.

For another kind of such examples, constructed on a purely nonatomic probability space, the interested reader is referred to ([2, Example 1]).

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