On Continuity Properties of the Solution Map in Linear Complementarity Problems

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Abstract. In an earlier paper [3] we have proved that, in a linear complementarity problem with a Q-matrix, the Lipschitzian continuity and the lower semicontinuity of the solution map are equivalent. In this paper, this fact is proved in the general case where the underlying matrix M of the problem need not have any prescribed special structure.

1. Introduction

For a given $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem corresponding to $M$ and $q$ is to find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad x^T(Mx + q) = 0. \quad (1.1)$$

The solution set of (1.1) is denoted by $S_M(q)$. Thus, for a fixed $M$, $S_M$ is a set-valued map from $\mathbb{R}^n$ into $\mathbb{R}^n$. It was known [1] that

$$\text{Dom} S_M = \bigcup_{\alpha \subseteq I} K_{\alpha}, \quad (1.2)$$

where $I = \{1, 2, \ldots, n\}$ and $K_{\alpha}$ is the complementarity cone corresponding to the index set $\alpha$ which is defined by setting

$$K_{\alpha} := \left\{ \sum_{i \in \alpha} \lambda_i (-M^i) + \sum_{j \in I \setminus \alpha} \mu_j e_j \mid \lambda_i \geq 0, i \in \alpha; \mu_j \geq 0, j \in I \setminus \alpha \right\}, \quad (1.3)$$

with $M^i$ standing for the $i^{th}$ column vector in $M$ and $e_j$ being the $j^{th}$ unit vector in $\mathbb{R}^n$. 
In Sec. 3 we shall prove that, for any $M \in \mathbb{R}^{n \times n}$ the solution map $S_M$ is Lipschitz on its effective domain if and only if it is lower semicontinuous on the set. To this end, we first show that if $S_M$ is lower semicontinuous on $\text{Dom}S_M$ then $M$ is nondegenerate, and then, by utilizing results in [2, 4] we deduce that in this case $S_M$ is also Lipschitz continuous on $\text{Dom}S_M$.

From now on, let $M$ be an $n \times n$-matrix with elements $a_{ij} \in \mathbb{R}$, $1 \leq i, j \leq n$. For $\alpha \subseteq \{1, 2, \ldots, n\}$, let $M_\alpha$ denote the submatrix of $M$ with the elements $a_{ij}$, $i, j \in \alpha$. The determinants of these matrices are called the principal minors of $M$. A matrix is said to be nondegenerate if all of the principal minors are nonzero. If at least one of the principal minors is zero then $M$ is a degenerate matrix. For abbreviation, we write $M_k$ instead of $M_{\{1,2,\ldots,k\}}$.

Recall that, a set-valued map $F$ from $\mathbb{R}^n$ into $\mathbb{R}^n$ is said to be Lipschitz on a subset $U \subseteq \mathbb{R}^n$ if there exists a constant number $L$ such that

$$H(F(p), F(q)) \leq L\|p - q\|; \quad \forall p, q \in U,$$

(1.4)

where $H(\cdot, \cdot)$ denotes the Hausdorff distance. $F$ is called lower semicontinuous (l.s.c. for short) at $q \in \text{Dom}F$ if for any $\bar{x} \in F(q)$ and $\epsilon > 0$ there exists $\delta > 0$ such that $F(q) \cap B(\bar{x}, \epsilon) \neq \emptyset$ for all $q \in B(\bar{q}, \delta) \cap \text{Dom}F$. Or, equivalently, for any $\bar{x} \in F(q)$ and any sequence $(q^n) \subseteq \text{Dom}F$ converging to $q$ there exists a sequence $(\bar{x}^m)$ such that $\bar{x}^m \in F(q^n)$ for each $m \in \mathbb{N}$ and $\bar{x}^m \to \bar{x}$. Finally, $F$ is said to be l.s.c. if it is l.s.c. at every point of $\text{Dom}F$.

2. Lower Semicontinuity of $S_M$ Implies Nondegeneracy of $M$

Theorem 2.1 below is one of the two main results of this paper. For the proof of that theorem we shall need the following lemma.

**Lemma 2.1.** Let $M \in \mathbb{R}^{n \times n}$. For every $n \geq k \geq 2$ and $k \geq l \geq 1$ there exists a vector $v = (v_1, v_2, \ldots, v_k)^T \in \mathbb{R}^k$ such that

$$v_l = \det (M_{\{1,\ldots,k\} \backslash \{l\}})$$

and

$$v^T M_k = \det(M_k) e_l^T.$$

(2.1) (2.2)

**Proof.** For each $i = 1, \ldots, k$ we define $v_i$ as the cofactor of $a_{il}$ in the matrix $M_k$. By $M_k$ we denote the $j$-th column vector of $M_k$. From the theory of determinants it follows that

$$v^T M_k = \begin{cases} 0 & \text{if } j \neq l, \\ \det(M_k) & \text{if } j = l. \end{cases}$$

Or, $v^T M_k = \det(M_k) e_l^T$.Besides, $v_l = \det (M_{\{1,\ldots,k\} \backslash \{l\}})$ by definition. The proof is complete. ■

**Theorem 2.1.** For any $M \in \mathbb{R}^{n \times n}$, if $S_M(\cdot)$ is l.s.c. then $M$ is nondegenerate.

**Proof.** We first consider the case $n = 1$. If $M$ is degenerate then $M = (0)$ and
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\[ S_M(q) = \begin{cases} \mathbb{R}_+ & \text{if } q = 0, \\ 0 & \text{if } q > 0, \\ \emptyset & \text{if } q < 0. \end{cases} \]

So \( S_M \) is not l.s.c. at \( q = 0 \in \text{Dom} S_M \).

Now, for the case \( n \geq 2 \), we suppose, by contrary, that \( S_M \) is l.s.c. and \( M \) is degenerate. Denote by \( M_\alpha \) the singular submatrix of \( M \) having the property that all its proper principal minors are nonzero. Without loss of generality, we can assume that \( \alpha = \{1, 2, \ldots, k\}, \ k \leq n. \) So, \( \det(M_k) = 0 \) and, if \( k > 1 \), \( \det(M_{\{1, \ldots, k\}\setminus\{l\}}) \neq 0 \) for all \( l \in \{1, \ldots, k\} \).

If \( k = 1 \) then \( a_{11} = 0 \). Choose \( \bar{x} := (1, 0, \ldots, 0)^T \in \mathbb{R}^n \) and, for each \( m \in \mathbb{N} \), set

\[ q := \left( \begin{array}{c} 0 \\ \vdots \\ 0 \\ r \end{array} \right) \in \mathbb{R}^n, \quad q^m := \left( \begin{array}{c} 1 \\ \vdots \\ m \\ r \end{array} \right) \in \mathbb{R}^n, \quad (2.3) \]

where \( r := \max\{|a_{21}|, |a_{31}|, \ldots, |a_{n1}|\} + 1 \geq 1. \) It is not difficult to verify that \( x \in S_M(q), \ 0 \in S_M(q^m) \) for every \( m \in \mathbb{N} \). So \( q \in \text{Dom} S_M \) and \( q^m \in \text{Dom} S_M \) for every \( m \in \mathbb{N} \). Furthermore, \( q^m \to q. \) By the lower semicontinuity of \( S_M(.) \) there exists a sequence \( (x^m) \) satisfying \( x^m \in S_M(q^m) \) for all \( m \in \mathbb{N} \) and

\[ \lim_{m \to \infty} x^m = \bar{x} = (1, 0, \ldots, 0)^T. \quad (2.4) \]

We have

\[ \lim_{m \to \infty} (Mx^m + q^m) = M\bar{x} + q = \left( \begin{array}{c} 0 \\ a_{21} + r \\ \vdots \\ a_{n1} + r \end{array} \right) \geq \left( \begin{array}{c} 0 \\ 1 \\ \vdots \\ 1 \end{array} \right). \quad (2.5) \]

It follows from (2.4) and (2.5) that for some \( m_0 \) large enough we have

\[ \{ (Mx^{m_0} + q^{m_0})_j > 0; \ \forall j \geq 2, \ x^{m_0}_1 > 0. \quad (2.6) \]

Since \( x^{m_0} \in S_M(q^{m_0}) \), from (2.6) we obtain

\[ \left\{ \begin{array}{l} x^{m_0}_j = 0; \ \forall j \geq 2, \\ (Mx^{m_0} + q^{m_0})_1 = 0. \end{array} \right. \quad (2.7) \]

Using the first property in (2.7) and the assumption \( a_{11} = 0 \), one has

\[ (Mx^{m_0} + q^{m_0})_1 = \sum_{j=1}^{n} a_{1j} x^{m_0}_j + q^{m_0}_1 = \frac{1}{m_0} > 0. \]

This contradicts the second property in (2.7).

Now assume that \( k > 1. \) Since \( M_k \) is singular, \( k \) column vectors of \( M_k \) are linearly dependent. By Lemma 2.1 in [2] we can find \( \lambda_1, \ldots, \lambda_k \geq 0 \) such that at least one of them equals zero and
Since all the columns of $M_k$ have the same role in the sense that $M_{\{1, \ldots, k\}\backslash \{l\}}$ is nonsingular for all $l \in \{1, \ldots, k\}$, without loss of generality we can assume that $\lambda_k = 0$ and (2.8) can be rewritten as follows

$$
\sum_{j=1}^{k} \lambda_j M_k^j = \sum_{j=1}^{k} \lambda_j M_k^j.
$$

(2.8)

Now let $\bar{x}, x', \bar{q}, q^m (m \in \mathbb{N})$ be the vectors in $\mathbb{R}^n$ defined by

$$
\bar{x} := \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} (k^{th}) \end{pmatrix}, \quad x' := \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix},
$$

(2.10)

$$
\bar{q} := \begin{pmatrix} -\sum_{j=1}^{k} a_{1j} \\ \vdots \\ -\sum_{j=1}^{k} a_{kj} \end{pmatrix} \begin{pmatrix} r \end{pmatrix}, \quad q^m := \bar{q} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} (k^{th}) \end{pmatrix},
$$

(2.11)

where

$$
r := \max\left\{ \sum_{j=1}^{k} a_{k+1,j}, \ldots, \sum_{j=1}^{k} a_{nj}, \sum_{j=1}^{k-1} \lambda_j a_{k+1,j}, \ldots, \sum_{j=1}^{k-1} \lambda_j a_{nj} \right\} + 1.
$$

Then

$$
M \bar{x} + \bar{q} = \begin{pmatrix} \sum_{j=1}^{k} a_{1j} \\ \sum_{j=1}^{k} a_{2j} \\ \vdots \\ \sum_{j=1}^{k} a_{nj} \end{pmatrix} + \bar{q} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} (k^{th}) \end{pmatrix}.
$$

(2.12)

Using (2.9)-(2.11), we obtain
Combining (2.12) and (2.13) with (2.10) it implies that \( \bar{x} \in S_M(\bar{q}) \) and \( x' \in S_M(q^m) \); \( \forall m \in \mathbb{N} \). Furthermore, \( q^m \rightarrow \bar{q} \) as \( m \rightarrow \infty \). By the lower semicontinuity of \( S_M \), there exists a sequence \( (x^m) \) converging to \( \bar{x} \) and \( x^m \in S_M(q^m), \ m \in \mathbb{N} \). Since \( x^m \rightarrow \bar{x} \) and \( Mx^m + q^m \rightarrow M\bar{x} + \bar{q} \), from (2.10) and (2.12) it follows that there exists \( m_0 \) large enough such that

\[
\begin{align*}
\left\{ \begin{array}{l}
x^m_i > 0, & \forall i = 1, \ldots, k, \\
(Mx^m + q^m)_j > 0; & \forall j = k + 1, \ldots, n.
\end{array} \right.
\tag{2.14}
\end{align*}
\]

Since \( x^{m_0} \in S_M(q^{m_0}) \), (2.14) implies

\[
\begin{align*}
\left\{ \begin{array}{l}
(Mx^{m_0} + q^{m_0})_i = 0; & \forall i = 1, \ldots, k, \\
x^{m_0}_j = 0; & \forall j = k + 1, \ldots, n.
\end{array} \right.
\tag{2.15}
\end{align*}
\]

Thus, by setting \( z := Mx^{m_0} + q^{m_0} \) one gets

\[
\left(\begin{array}{c}
0 \\
\vdots \\
0 \\
z_{k+1} \\
\vdots \\
z_n
\end{array}\right) = z = Mx^{m_0} + q^{m_0}
\]

\[
= M^{j^2} \left(\begin{array}{c}
x^{m_0}_1 \\
\vdots \\
0
\end{array}\right) + \left(\begin{array}{c}
-\sum_{j=1}^{k} a_{1j} \\
\vdots \\
0
\end{array}\right) + \left(\begin{array}{c}
0 \\
\vdots \\
1/m_0 \ (k^{th})
\end{array}\right). \tag{2.16}
\]

Noting that \( M^j_{k} \) is the \( j^{th} \) column vector of \( M_k \), one derives from (2.16) that
By virtue of Lemma 2.1 we can find \( v = (v_1, \ldots, v_k)^T \in \mathbb{R}^k \) satisfying
\[
\begin{align*}
\sum_{j=1}^k x_j^{m_0} M_j^i - \sum_{j=1}^k M_j^i + \left(\begin{array}{c}
0 \\
\vdots \\
0 \\
\frac{1}{m_0} \\
0
\end{array}\right) &= \left(\begin{array}{c}
0 \\
\vdots \\
0 \\
0
\end{array}\right) \\
&\in \mathbb{R}^k.
\end{align*}
\] (2.17)

By virtue of Lemma 2.1 we can find \( v = (v_1, \ldots, v_k)^T \in \mathbb{R}^k \) satisfying
\[
\begin{align*}
v^T M_k &= (0, \ldots, 0)^T \in \mathbb{R}^k \\
\text{and} \\
v_k &= \text{det}(M_{k-1}).
\end{align*}
\] (2.18) (2.19)

Taking the scalar product of both sides of the equality in (2.17) with \( v \) we have
\[
\sum_{j=1}^k (x_j^{m_0} - 1)v^T M_j^i + \frac{1}{m_0} v_k = 0.
\]
This together with (2.18) gives \( \text{det}(M_{k-1}) = v_k = 0 \), a contradiction with the definition of \( M_k \). So \( M \) is nondegenerate and the proof is complete.

3. Equivalence Between the Two Continuity Properties

The next theorem is the second main result of this paper.

Theorem 3.1. Let \( M \in \mathbb{R}^{n \times n} \). Then \( S_M \) is Lipschitz on \( \text{Dom} S_M \) if and only if it is lower semicontinuous.

Proof. Obviously, we need only verify the sufficient condition. Assume that \( S_M \) is l.s.c. on \( \text{Dom} S_M \). By Theorem 2.1 \( M \) is nondegenerate, and hence, by [1] \( S_M(q) \) is a finite set for every \( q \in \text{Dom} S_M \). Besides, by virtue of [4, Proposition 1], \( S_M \) is uniformly locally upper Lipschitz on \( \text{Dom} S_M \). That is, with a certain positive number \( \lambda > 0 \), for all \( q \in \text{Dom} S_M \) there exists \( \delta(q) > 0 \) such that
\[
S_M(q) \subset S_M(q) + \lambda \|q - \bar{q}\| B(0, 1); \quad \forall q \in B(\bar{q}, \delta(q)).
\] (3.1)

The proof of the theorem now can be divided into three lemmas.

Lemma 3.1. For any \( \bar{q} \in \text{Dom} S_M \) there exists \( \eta > 0 \) such that
\[
H(S_M(q), S_M(\bar{q})) \leq \lambda \|q - \bar{q}\|; \quad \forall q \in B(\bar{q}, \eta) \cap \text{Dom} S_M.
\] (3.2)

Proof. Take any \( \bar{q} \in \text{Dom} S_M \) and assume that \( S_M(\bar{q}) = \{x^1, \ldots, x^k\} \). We set
\[
\epsilon := \min \{\|x^i - x^j\|, 1 \leq i < j \leq k\} > 0.
\] (3.3)

Since \( S_M \) is l.s.c. at \( \bar{q} \) and \( S_M(\bar{q}) \) is finite, there exists \( \delta_1 > 0 \) such that
\[
S_M(q) \cap B\left(x^i, \frac{\epsilon}{2}\right) \neq \emptyset; \quad \forall q \in B(\bar{q}, \delta_1) \cap \text{Dom} S_M, \forall i = 1, 2, \ldots, k.
\] (3.4)
We now choose $\eta := \min\{\delta_1, \delta(q), \epsilon/2\lambda\}$. Then for all $q \in B(q,\eta) \cap \text{Dom}S_M$ both (3.1) and (3.4) hold. For every $x^i \in S_M(q)$, by (3.4) there exists $v$ such that

$$v \in S_M(q) \quad \text{and} \quad \|v - x^i\| < \frac{\epsilon}{2}. \quad (3.5)$$

By the definition of $\epsilon$ it follows that

$$v \notin x^j + \frac{\epsilon}{2} B(0,1), \quad \forall j \neq i. \quad (3.6)$$

hence, noting that $\lambda\|q - \bar{q}\| < \lambda\eta \leq \epsilon/2$ we have

$$v \notin x^j + \lambda\|q - \bar{q}\| B(0,1), \quad \forall j \neq i. \quad (3.7)$$

On the other hand, from (3.1) it follows that

$$v \in x^i + \lambda\|q - \bar{q}\| B(0,1), \quad k$$

which together with (3.7) yields (3.2).

Lemma 3.2. For all $p, q \in \text{Dom}S_M$ such that $[p, q] \subset \text{Dom}S_M$ we have

$$H(S_M(p), S_M(q)) \leq \lambda\|p - q\|, \quad (3.9)$$

where $[p, q]$ denotes the segment $co\{p, q\}$.

Proof. This lemma can be derived from Lemma 3.1 and the compactness of the segment $[p, q]$.

Lemma 3.3. There exists $L \geq 0$ such that

$$H(S_M(p), S_M(q)) \leq L\|p - q\|, \quad \forall p, q \in \text{Dom}S_M.$$ 

From this lemma the theorem follows.

Proof. Applying [2, Corollary 2.1] for the class of polyhedral convex cones $\{K_\alpha, \alpha \subset I\}$ there exists $\gamma > 0$ such that for all $p \in K_\alpha$, $q \in K_\beta$ with $\alpha \subset I$, $\beta \subset I$, there exists $u \in K_\alpha \cap K_\beta$ satisfying

$$\|p - q\| \geq \gamma(\|p - u\| + \|q - u\|). \quad (3.10)$$

Now we set $L := \lambda/\gamma$. For all $p, q \in \text{Dom}S_M$ there are $\alpha \subset I$ and $\beta \subset I$ such that $p \in K_\alpha$ and $q \in K_\beta$. Denoting $u \in K_\alpha \cap K_\beta$ the vector satisfying (3.10) we have
From (3.9) one gets
\[ H(S_M(p), S_M(u)) \leq \lambda \|u - p\|, \]
\[ H(S_M(u), S_M(q)) \leq \lambda \|u - q\|. \]
Combining these two inequalities we obtain
\[ H(S_M(p), S_M(q)) \leq H(S_M(p), S_M(u)) + H(S_M(u), S_M(q)) \]
\[ \leq \lambda (\|u - p\| + \|u - q\|) \]
\[ \leq \frac{\lambda}{\gamma} \|p - q\| = L\|p - q\|. \]
The proof is complete. \[ \square \]

References