Vietnam Journal of Mathematics **30**:4 (2002) 395–411

Vietnam Journal
of
MATHEMATICS

Refinable Functions from Their Values at Integers

Charles A. Micchelli[1] **and Qiyu Sun**[2]

[1]*Department of Mathematics and Statistics, State University of New York, The University at Albany, Albany, N. Y. 12222, USA*

[2]*Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Singapore*

Dedicated to Mike Powell with friendship and esteem on the occasion of his 65th birthday

Received October 5, 2001

Abstract. Any positive sequence of finite numbers such that the sum of the even indexed and odd indexed terms are one determines a refinable function which is positive in its support and whose integer translates form a partition of unity. Using this fact we introduce a recursive method to generate new refinable functions from their values at integers. This recursion is analyzed and its convergence is established.

1. Introduction

A real-valued continuous function ϕ supported on the interval $[0, N+1]$, where N is a positive integer, is said to be *refinable* provided that

$$\phi = \sum_{j\in\mathbb{Z}} a_j \phi(2\cdot - j) \tag{1.1}$$

for some sequence of real numbers $\{a_j : j \in \mathbb{Z}\}$ such that $a_j = 0$ for $j \notin \mathbb{Z}_{N+2} := \{0, 1, \ldots, N+1\}$. A necessary condition for the existence of ϕ is that the polynomial

The first author is supported in part by the U. S. National Science Foundation under grant DMS 9973427.

$$a(z) := \sum_{j\in\mathbb{Z}} a_j z^j, \quad z \in \mathbb{C} \tag{1.2}$$

has the property that

$$a(-1) = 0, \quad a(1) = 2 \tag{1.3}$$

and in this case

$$\sum_{j\in\mathbb{Z}} \phi(\cdot - j) = 1, \tag{1.4}$$

see [1]. If, in addition to (1.3),

$$a_j > 0, \quad j \in \mathbb{Z}_{N+2}, \tag{1.5}$$

then there exists a unique continuous ϕ satisfying the refinement equation (1.1) and the partition of unity condition (1.4). Moreover, in this case,

$$\phi(x) = \begin{cases} > 0, & x \in (0, N+1) \\ 0, & \text{otherwise}, \end{cases} \tag{1.6}$$

see [9] and also [4, 8]. In fact, ϕ is Hölder continuous, see [8].

Following discussion in [7] and also recently in [3], we use the values of ϕ on $\mathbb{Z}_{N+1}$ to generate another polynomial of degree $N+1$ satisfying (1.3) with positive coefficients on $\mathbb{Z}_{N+2}$. For example, the polynomial

$$a^1(z) := \frac{(1+z)^2}{2} \sum_{j\in\mathbb{Z}_N} \phi(j+1) z^j, \ z \in \mathbb{C} \tag{1.7}$$

determines a refinable function ϕ^1 satisfying all the same conditions as ϕ itself. We continue this process and form polynomials $a^n, n \in \mathbb{Z}_+$ (each of degree $N+1$) and corresponding refinable functions $\phi^n, n \in \mathbb{Z}_+$ and ask about their limit as $n \to \infty$.

This iteration is in fact a special case of the one studied in this paper. Before getting to the specifics of the iteration, let us briefly mention in passing an illustrative example of the above iteration. The cases $N = 1$ and $N = 2$, converge in at most *two* steps. For $N = 3$, we start with a symmetric quartic polynomial

$$a^0(z) = (1+z)\Big[\frac{1}{2} - t + tz + tz^2 + \Big(\frac{1}{2} - t\Big)z^3\Big], \ z \in C,$$

where $t \in (0, 1/2)$ and note for $n \in \mathbb{Z}_+$ that

$$a^n(z) = (1+z)\Big[\frac{1}{2} - t_n + t_n z + t_n z^2 + \Big(\frac{1}{2} - t_n\Big)z^3\Big], \ z \in C,$$

where

$$\begin{aligned} t_{n+1} :=& \frac{1 - t_n}{3 - 4t_n}, \ n \in \mathbb{Z}_+, \\ t_0 :=& t. \end{aligned}$$

The sequence $\{t_n : n \in \mathbb{Z}_+\}$ is strictly increasing, lies in $(0, 1/2)$ and converges to $1/2$ as $n \to \infty$. In particular,

$$t_n = \frac{t + (1-2t)n}{1 + 2(1-2t)n}, \quad n \in \mathbb{Z}_+.$$

Hence, we conclude that

$$\lim_{n\to\infty} a^n = a^\infty,$$

where

$$a^\infty(z) := \frac{z(1+z)^2}{2}, \quad z \in \mathbb{C},$$

which implies that

$$\lim_{n\to\infty} \phi^n = \phi^\infty,$$

where for $x \in \mathbb{R}$,

$$\phi^\infty(x) := \begin{cases} 0, & x \notin (1,3), \\ 1 - |2-x|, & x \in (1,3). \end{cases}$$

The convergence is uniform on compact subsets of $\mathbb{C}, \mathbb{R}$ respectively. The function ϕ^∞ is uniquely determined by the refinement equation

$$\phi^\infty = \sum_{j\in\mathbb{Z}} a_j^\infty \phi^\infty(2 \cdot -j)$$

and the condition that

$$\sum_{j\in\mathbb{Z}} \phi^\infty(\cdot - j) = 1.$$

This example prepares us for the general case which we now describe.

2. The Iteration

For any continuous function ϕ^0 on $\mathbb{R}$, positive on $(0, N+1)$ and zero otherwise, which satisfies the equation

$$\sum_{j\in\mathbb{Z}} \phi^0(\cdot - j) = 1 \tag{2.1}$$

and $\theta \in (0,1)$, we define the polynomial of degree $N+1$ by the equation

$$a^0(z) := \sum_{j\in\mathbb{Z}} L_j(\phi^0) z^j, \ z \in \mathbb{C},$$

where

$$L_j(\phi^0) := \theta\phi^0(j-1) + \phi^0(j) + (1-\theta)\phi^0(j+1), \quad j \in \mathbb{Z}.$$

This polynomial begins the iteration which we describe next.

Proposition 2.1. *For every $\theta \in (0,1)$ and function ϕ^0 described above there exists a sequence of functions $\{\phi^n : n \in \mathbb{Z}_+\}$ such that*

(i) $\phi^n(x) = \begin{cases} > 0, & x \in (0, N+1) \\ 0, & \text{otherwise}, \end{cases}$

(ii) $\sum_{j\in\mathbb{Z}} \phi^n(\cdot - j) = 1$,

(iii) $\phi^{n+1} = \sum_{j\in\mathbb{Z}} L_j(\phi^n)\phi^{n+1}(2\cdot -j)$.

Proof. According to our definition of the polynomial a^0 and our remarks in Sec. 1, there is a unique refinable function ϕ^1 such that

$$\phi^1 = \sum_{j\in\mathbb{Z}} L_j(\phi^0)\phi^1(2\cdot -j)$$

and (i) and (ii) of Proposition 2.1 hold with $n = 1$. The polynomial of degree $N + 1$

$$a^1(z) := \sum_{j\in\mathbb{Z}} L_j(\phi^1)z^j, \quad z \in \mathbb{C}$$

has positive coefficients and satisfies (1.3). This means that it likewise determines a refinable function ϕ^2 which satisfies (i)–(iii) of the proposition. In this way, we define inductively for $n \in \mathbb{Z}_+$ refinable functions ϕ^n and polynomials

$$a^n(z) := \sum_{j\in\mathbb{Z}} L_j(\phi^n)z^j,\ z \in \mathbb{C}. \tag{2.2}$$

■

Note that in our definition of ϕ^n and $a^n, n \in \mathbb{Z}_+$ we only require the values $\phi^0(j+1), j \in \mathbb{Z}_N$, the fact that they are positive and sum to one. The case $\theta = 1/2$ of the iteration corresponds to the case described in Sec. 1. For any θ, when $N = 1$ all the iterates are the same and so henceforth we assume $N \geq 2$.

We now consider two special refinable functions central in our analysis of convergence of our iteration. The first is defined by the refinement equation

$$\psi = t\psi(2\cdot) + \psi(2\cdot -1) + (1-t)\psi(2\cdot -2) \tag{2.3}$$

and the requirement that

$$\sum_{j\in\mathbb{Z}} \psi(\cdot - j) = 1,$$

where $t \in (0,1)$. This function has support $[0,2]$ and is positive in $(0,2)$. Moreover, as a special case of general results from [9, 4] it was proved in [8] that ψ is strictly increasing on $[0,1]$. We display its dependency on t by the notation ψ_t and observe for $t = 1/2$ that $\psi_t(x) = \max(0, 1 - |1-x|),\ x \in \mathbb{R}$.

The next function we require is defined by the equation

$$\Gamma_t(x) := \int_{x-1}^{x} \psi_t(\sigma)d\sigma, \ x \in \mathbb{R} \tag{2.4}$$

or alternatively by the requirement that

$$\Gamma_t = \frac{1}{2}\big[(1-t)\Gamma_t(2\cdot) + (2-t)\Gamma_t(2\cdot -1) + (1+t)\Gamma_t(2\cdot -2) + t\Gamma_t(2\cdot -3)\big] \tag{2.5}$$

and

$$\sum_{j\in\mathbb{Z}} \Gamma_t(\cdot - j) = 1. \tag{2.6}$$

We are now prepared to define the limit of the iteration described above. To this end, we introduce an integer $m \in \mathbb{Z}_{N-1}$ and $\tau \in [0,1)$ defined by the equation

$$m + \tau = \sum_{j\in\mathbb{Z}} j\phi^0(j+1). \tag{2.7}$$

These definitions lead us to the function

$$\phi^\infty(\cdot|\theta) := \begin{cases} \psi_\theta(\cdot - N + 1), & \text{if} \ \ \theta \in (0,1/2), \\ \Gamma_\tau(\cdot - m), & \text{if} \ \ \theta = 1/2, \\ \psi_\theta, & \text{if} \ \ \theta \in (1/2,1). \end{cases}$$

Theorem 2.2. *If $\theta \in (0,1)$, then there are constants $q \in (0,+\infty)$ and $r \in (0,1)$ such that for all $n \in \mathbb{Z}_+$ and $x \in \mathbb{R}$, either*
(i) *$\theta \neq 1/2$ and $|\phi^n(x) - \phi^\infty(x|\theta)| \le qr^n$, or*
(ii) *$\theta = 1/2$, $\tau \neq 0$ and $|\phi^n(x) - \phi^\infty(x|\theta)| \le qr^n$, or*
(iii) *$\theta = 1/2$ and $\tau = 0$ and $|\phi^n(x) - \phi^\infty(x|\theta)| \le q(n+1)^{-1}$.*

We remark that the estimate above in (iii) of Theorem 2.2 cannot be improved, since for the refinable function ϕ^n and ϕ in Section 1 and for any $t \in (0,1/2)$ there is a positive constant c such that for all $n \in \mathbb{Z}_+$ we have that

$$|\phi^n(2) - \phi^\infty(2|\theta)| = |4t_{n+1} - 2| \ge c(1+n)^{-1}.$$

3. Estimates

We find it convenient to introduce the sequence of polynomial of degree $N-1$

$$b^n(z) := \sum_{j\in\mathbb{Z}} \phi^n(j+1)z^j, \quad z \in \mathbb{C}, \ \ n \in \mathbb{Z}_+ \tag{3.1}$$

and also make use of the quadratic polynomial

$$C_\theta(z) := \theta + z + (1-\theta)z^2, \quad z \in \mathbb{C}. \tag{3.2}$$

From these definitions there follows the formula

$$a^n = C_\theta b^n, \quad n \in \mathbb{Z}_+. \tag{3.3}$$

Our first observation concerns an equivalent way to state the iteration described in Sec. 2 in terms of these polynomials.

Lemma 3.1. *For any $n \in \mathbb{Z}_+$ there holds*

$$b^n(1) = 1 \tag{3.4}$$

while for $z \in \mathbb{C}$

$$a^n(z)b^{n+1}(z) - a^n(-z)b^{n+1}(-z) = 2zb^{n+1}(z^2). \tag{3.5}$$

Proof. The first claim follows immediately from (ii) of Proposition 2.1. For the second claim we specialize (iii) of Proposition 2.1 to obtain for $i, j \in \mathbb{Z}, n \in \mathbb{Z}_+$ that $d_{2i+1} = \phi^{n+1}(i+1)$, where

$$d_i := \sum_{j\in\mathbb{Z}} L_j(\phi^n)\phi^{n+1}(i+1-j).$$

Recalling definition (2.2) and (3.1) we obtain from the above equation that $d = a^n b^{n+1}$ and hence for all $z \in \mathbb{C}$,

$$\begin{aligned} 2zb^{n+1}(z^2) &= 2\sum_{l\in\mathbb{Z}} d_{2l+1}z^{2l+1} \\ &= a^n(z)b^{n+1}(z) - a^n(-z)b^{n+1}(-z). \end{aligned}$$ ■

Recall for $n \in \mathbb{Z}_+$ that all the coefficients of b^n are positive. In view of the first fact stated in Lemma 3.1 these polynomials are bounded on compact subsets of $\mathbb{C}$. In our next lemma we identify all possible cluster points of this sequence of polynomials.

Lemma 3.2. *If $\theta \in (0,1)$ and b is a Laurent polynomial with nonnegative coefficients such that for all $z \in \mathbb{C} \setminus \{0\}$*

$$C_\theta(z)(b(z))^2 - C_\theta(-z)(b(-z))^2 = 2zb(z^2) \tag{3.6}$$

and

$$b(1) = 1, \tag{3.7}$$

then either $\theta = 1/2$ and there exist $\alpha \in [0,1)$ and integer $k \in \mathbb{Z}$ such that

(i) $b(z) = [\alpha + (1-\alpha)z]z^k,\ z \in \mathbb{C}$,

or $\theta \neq 1/2$ and there exists $k \in \mathbb{Z}$ such that

(ii) $b(z) = z^k,\ z \in \mathbb{C}$.

Moreover, any polynomial of either of these forms satisfies (3.6) *and* (3.7).

Proof. Certainly (3.6) and (3.7) represent the equations necessarily satisfied by any cluster point of the sequence of $\{b^n : n \in \mathbb{Z}_+\}$.

Let us begin the proof of the lemma by confirming that any polynomial of the type described in (i) and (ii) provide a solution to (3.6) and (3.7). To this end, we observe whenever b satisfies (3.6) and (3.7), then the Laurent polynomial $\tilde{b}$ defined by the equation $\tilde{b}(z) := z^k b(z), z \in \mathbb{C} \setminus \{0\}$ for any $k \in \mathbb{Z}$ also satisfies these equations. Likewise, recalling the definition of the quadratic polynomial C_θ the Laurent polynomial defined by the equation $\hat{b}(z) := b(z^{-1}), z \in \mathbb{C} \setminus \{0\}$ satisfies (3.6) and (3.7), with θ replaced by $1-\theta$ when b itself is a solution to these equations. Consequently, since the polynomial $b = 1$ clearly satisfies (3.6) and (3.7), we see the polynomials in (ii) is also a solution. In a similar manner, when $\theta = 1/2$, a direct computation confirms the polynomial in (i) satisfies (3.6) and (3.7).

For the converse, we let b be any solution to (3.6) and (3.7) with nonnegative coefficients. By our previous remarks, we can assume b is an algebraic polynomial with $b(0) > 0$ and $\theta \in (0, 1/2]$. We make use of this additional information by differentiating both sides of (3.6) and setting $z = 0$ to obtain the formula

$$b(0) + 2\theta b'(0) = 1.$$

Moreover, since

$$1 = b(1) = \sum_{j \in \mathbb{Z}_+} \frac{b^{(j)}(0)}{j!} \geq b(0) + b'(0) \geq b(0) + 2\theta b'(0) = 1,$$

we conclude that $b^{(j)}(0) = 0$ for all $j \geq 2$. If $\theta \in (0, 1/2)$ we also have that $b'(0) = 0$. This proves the result. ■

Our next lemma provides useful inequalities for the coefficients of the polynomial b^n,

$$b^n(z) = \sum_{j \in \mathbb{Z}_N} b_j^n z^j, \quad z \in \mathbb{C},$$

assumed to satisfy (3.6) and (3.7), and to have positive coefficients. For the proof of the next result we find it convenient to say $a(z) \geq b(z)$ whenever $a_j \geq b_j$, $j \in \mathbb{Z}$ for any two Laurent polynomials a and b.

Lemma 3.3. *If $n \in \mathbb{Z}_+$, $\theta \in (0,1)$ and $l \in \mathbb{Z}_{N-1}$. Then*

$$b_{l+1}^{n+1} \geq \frac{(1-\theta) b_l^{n+1}}{1 - (1-\theta) b_l^n} b_{l+1}^n \tag{3.8}$$

and

$$b_l^{n+1} \leq [1 - \rho(1 - b_l^{n+1})] b_l^n + 2(1-\theta)^{-1} \sum_{j \in \mathbf{Z}_l} (b_j^n + b_j^{n+1}), \tag{3.9}$$

where $\rho := \dfrac{1-2\theta}{1-\theta}$.

Proof. We start with the inequality (3.8). Since the coefficient of b^n are nonnegative we have that

$$b^{n+1}(z) \geq b_l^{n+1} z^l + b_{l+1}^{n+1} z^{l+1}.$$

Therefore we conclude that

$$C_\theta(z)b^n(z)b^{n+1}(z) \geq (1-\theta)(b_l^{n+1}b_{l+1}^{n+1} + b_{l+1}^{n+1}b_l^n)z^{2l+3}.$$

Combining this inequality with the equation (3.5) proves (3.8).

For the second inequality we set

$$e^n(z) := \sum_{j\in\mathbb{Z}_l} b_j^n z^j, \ \ z\in\mathbb{C}$$

and use the convention that whenever r^n appears below it represents some polynomial which does *not* contain a factor z^{2l+1}. Our first step is to write

$$b^{n+1}b^n = e^{n+1}b^n + e^n b^{n+1} + (b^{n+1}-e^{n+1})(b^n - e^n) - e^{n+1}e^n$$

and conclude that

$$\begin{aligned} C_\theta(z)b^{n+1}(z)b^n(z) \ &\leq C_\theta(z)e^{n+1}(z)b^n(z) + C_\theta(z)b^{n+1}(z)e^n(z) \\ &\quad + C_\theta(z)(b^{n+1}(z) - e^{n+1}(z))(b^n(z) - e^n(z)). \end{aligned}$$

For the last term we note that each factor has terms starting with z^l. Therefore, we obtain that it is

$$\leq \left[b_l^{n+1}b_l^n + \theta(b_{l+1}^{n+1}b_l^n + b_l^{n+1}b_{l+1}^n)\right]z^{2l+1} + r^n(z).$$

Our estimate for the first term is

$$\begin{aligned} &C_\theta(z)e^{n+1}(z)b^n(z) \\ &\leq C_\theta(z)e^{n+1}(z)\sum_{j\in\mathbb{Z}_{N-1}} z^j + r^n(z) \\ &\leq (\theta e^{n+1}(1) + e^{n+1}(1) + (1-\theta)e^{n+1}(1))z^{2l+1} + r^n(z) \\ &= 2e^{n+1}(1)z^{2l+1} + r^n(z). \end{aligned}$$

A similar estimate holds for the second term, and in total we get

$$\begin{aligned} &C_\theta(z)b^{n+1}(z)b^n(z) \\ &\leq \Big[b_l^{n+1}b_l^n + \theta(b_{l+1}^{n+1}b_l^n + b_l^{n+1}b_{l+1}^n) + 2\sum_{j\in\mathbb{Z}_l}(b_j^n + b_j^{n+1})\Big]z^{2l+1} + r^n(z). \end{aligned}$$

Referring back to the equation (3.5) we obtain,

$$b_l^{n+1} \leq b_l^{n+1}b_l^n + \theta(b_{l+1}^{n+1}b_l^n + b_l^{n+1}b_{l+1}^n) + 2\sum_{j\in\mathbb{Z}_l}(b_j^n + b_j^{n+1}). \tag{3.10}$$

We bound the second factor on the right hand side by using the fact that $b_{l+1}^n \leq 1 - b_l^n$ and upon simplification obtain the desired result. ■

In the next lemma, we use the inequality above to establish that

$$\lim_{n\to\infty} b_j^n = 0, \quad j \in \mathbb{Z}_{N-1},$$

whenever $\theta \in (0, 1/2)$ and $N > 1$. In fact, we shall demonstrate that the convergence is *geometrically* fast.

Lemma 3.4. *If $\theta \in (0, 1/2)$ and $j \in \mathbb{Z}_{N-1}$, then there are a positive constant q and $\mu \in (0, 1)$ such that for all $n \in \mathbb{Z}_+$*

$$b_j^n \le q\mu^n. \tag{3.11}$$

Proof. We first prove the result for $j = 0$. To this end, we specialize (3.9) to this case, and obtain for $n \in \mathbb{Z}_+$

$$b_0^{n+1} \le \left[1 - \rho(1 - b_0^{n+1})\right] b_0^n \le b_0^n,$$

that is, $\{b_0^n : n \in \mathbb{Z}_+\}$ is a nonincreasing sequence. Therefore, we conclude from the first inequality that

$$b_0^{n+1} \le \left[1 - \rho(1 - b_0^0)\right] b_0^n,$$

and so, indeed (3.11) holds for this case.

The proof of the general case proceeds by induction on $j \in \mathbb{Z}_{N-1}$. So, we assume that (3.11) holds for all $j \in \mathbb{Z}_k$ for some $k \in \mathbb{Z}_{N-1}$. If $k = N - 1$, we are finished, otherwise, we have that $1 \le k \le N - 2$. Again, we make use of (3.9) and the induction hypothesis to obtain for $n \in \mathbb{Z}_+$

$$b_k^{n+1} \le [1 - \rho(1 - b_k^{n+1})] b_k^n + s\mu^n, \tag{3.12}$$

where $s := 2k(1-\theta)^{-1}(\mu + 1)$ from which we derive that the sequence $\{b_k^n + s(1-\mu)^{-1}\mu^n : n \in \mathbb{Z}_+\}$ is nonincreasing. Hence there is a $r \in (0, 1]$ such that

$$\lim_{n\to\infty} b_k^n = r.$$

Let us first demonstrate that $r \ne 1$. If, to the contrary $r = 1$, then for all $j \in \mathbb{Z}_{N-1} \setminus \{k\}$ we obtain that

$$\lim_{n\to\infty} b_j^n = 0.$$

However, since $k < N - 1$ we can use (3.8) to conclude that, since $\theta \in (0, 1/2)$, that

$$b_{k+1}^{n+1} \ge \frac{(1-\theta) b_k^{n+1}}{1 - (1-\theta) b_k^n} b_{k+1}^n \ge \frac{1}{2\theta} b_{k+1}^n$$

for sufficiently large n. In other words,

$$b_{k+1}^n \ge c(2\theta)^{-n}$$

for some positive constant c, which is a clear contradiction as the lower bound tends to ∞ as $n \to \infty$.

We now know that $r \neq 1$ and therefore from (3.12) for all $n \in \mathbb{Z}_+$

$$b_k^{n+1} \leq \epsilon b_k^n + s\rho^n, \tag{3.13}$$

where $\epsilon := 1 - \rho(1-\beta)$ and $\beta \in (0,1)$ is chosen so that for all $n \in \mathbb{Z}_+$, we have that $b_k^n \leq \beta$. Using inequality (3.13) repeatedly in n gives us for $n \in \mathbb{Z}_+$ the estimate

$$\begin{aligned} b_k^{n+1} &\leq \max(s,1)(\rho^n + \rho^{n-1}\epsilon + \cdots + \epsilon^n) \\ &\leq \max(s,1)(n+1)[\max(\rho,\epsilon)]^n \end{aligned} \tag{3.14}$$

which advances the inductive hypothesis and proves the lemma. ∎

This lemma has sufficient information to prove Theorem 2.2 for $\theta \neq 1/2$. In the next lemma, we use the inequalities in Lemma 3.3 to establish

$$\lim_{j\to\infty} b_j^n = 0, \ j \in \mathbb{Z}_m$$

when $\theta = 1/2$ and $\tau \in (0,1)$.

Lemma 3.5. *If $\theta = 1/2$, m, τ are chosen as in* (2.7) *with $\tau \in (0,1)$, then there are constants $q > 0$ and $\mu \in (0,1)$ such that for all $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_m$,*

$$b_j^n \leq q\mu^n.$$

Proof. Since $\theta = 1/2$, we see that the constant ρ appearing in the bound (3.9) is one. This is a barrier to obtaining our goal to establish exponential decay of the first m coefficients of the polynomial b^n. To circumvent this problem we return to use the proof of (3.9) and see that an exponential bound would be available provided that we can find a $\eta \in (0,1)$ such that for all $l \in \mathbb{Z}_m$ and $n \in \mathbb{Z}_+$

$$b_{l+1}^n + b_l^n \leq \eta. \tag{3.15}$$

Indeed, if this were the case then inequality (3.10) in the proof of Lemma 3.3 yields for all $l \in \mathbb{Z}_m$ and $n \in \mathbb{Z}_+$ the inequality

$$b_l^{n+1} \leq \Big(\frac{\eta}{2-\eta}\Big) b_l^n + \frac{4}{3-2\eta} \sum_{j \in \mathbb{Z}_l} (b_j^n + b_j^{n+1}). \tag{3.16}$$

We now follow the argument in Lemma 3.4 and use (3.16) to prove the lemma. Hence, there remains the proof of (3.15). The first step is to observe that for all $n \in \mathbb{Z}_+$

$$\sum_{j \in \mathbb{Z}} j b_j^n = \sum_{j \in \mathbb{Z}} j b_j^0. \tag{3.17}$$

To this end, we differentiate both sides of (3.5) and evaluate the resulting expression at $z = 1$ upon simplification there follows equation (3.17). Hence, for any $l \in \mathbb{Z}_m$ we conclude that

$$\begin{aligned}\sum_{j=l+2}^{N-1} b_j^n &\geq \frac{1}{N-1}\sum_{j=l+2}^{N-1} jb_j^n \\ &\geq \frac{1}{N-1}\Big(\sum_{j=0}^{N-1} jb_j^0 - (l+1)\sum_{j=0}^{l+1} b_j^n\Big) \geq \frac{1}{N-1}(m+\tau-l-1).\end{aligned} \tag{3.18}$$

Consequently, recalling the fact that $l \leq m-1$ and $m \leq N-2$, we obtain the desired inequality

$$b_{l+1}^n + b_l^n \leq 1 - \sum_{j=l+2}^{N-1} b_j^n \leq 1 - \frac{m+\tau-l-1}{N-1} \leq \frac{N-1-\tau}{N-1} < 1$$

and hence completes the proof. ∎

This lemma has sufficient information to prove Theorem 2.2, for $\theta = 1/2$ and $\tau \in (0,1)$. In the next lemma, we use the inequalities in Lemma 3.3 to study the remaining case $\theta = 1/2$ and $\tau = 0$.

Lemma 3.6. *If $\theta = 1/2$, m, τ are chosen as in (2.7) with $\tau = 0$, then there are a positive constant q and a $\mu \in (0,1)$ such that for all $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_{m-1}$,*

$$b_j^n \leq q\mu^n,$$

and $\lim_{n\to\infty} b_{m-1}^n$ exists.

Proof. For $l \in \mathbb{Z}_{m-1}$ and $m \leq N-2$, we obtain from (3.18) that

$$b_{l+1}^n + b_l^n \leq 1 - \sum_{j=l+2}^{N-1} b_j^n \leq 1 - \frac{m-l-1}{N-1} \leq \frac{N-2}{N-1} < 1.$$

This leads to (3.15) for all $l \in \mathbb{Z}_{m-1}$. We now follow the argument in Lemma 3.4 to obtain the first claim.

For $l = m-1$, we have from (3.9) that

$$b_{m-1}^{n+1} \leq b_{m-1}^n + 4\sum_{j\in\mathbb{Z}_{m-1}} (b_j^n + b_j^{n+1}).$$

Using this fact and the estimate of b_j^n for $j \in \mathbb{Z}_{m-1}$ we proceed again as in the proof of Lemma 3.3 and conclude that $\lim_{n\to\infty} b_{m-1}^n$ exists. ∎

4. Perturbation of Refinable Functions

The previous results concentrate on the asymptotic behavior of the coefficient polynomials $\{b^n : n \in \mathbb{Z}_+\}$ as they generate and reflect properties of the refinable functions $\{\phi^n : n \in \mathbb{Z}_+\}$. We now need to make an observation on the dependency of the refinable function ϕ in (1.1) on the polynomial a. For this purpose, we amplify our notation for ϕ to ϕ_a as a means to display its dependency on a. Our goal is to estimate the difference $\phi_a - \phi_{\tilde{a}}$. The bound we present comes from delving into the proof of the existence of ϕ_a (and $\phi_{\tilde{a}}$) as presented in [8], see also [1], and making appropriate additional observation. The polynomial a is our "target" and $\tilde{a}$ is meant to be "close" to a. With this distinction in mind we prepare for the next fact.

For any biinfinite sequence $c = \{c_j : j \in \mathbb{Z}\}$ we let $\|c\|_\infty = \sup\{|c_j| : \ j \in \mathbb{Z}\}$. Likewise we use $\|\phi\|_\infty$ for the suprenorm of any function ϕ defined on $\mathbf{R}$.

Proposition 4.1. *If $\theta \in (0,1)$, $a = C_\theta b$ and $\tilde{a} = C_\theta \tilde{b}$, where b and $\tilde{b}$ are polynomials of degree at most $N-1$, have nonnegative coefficients, and satisfy $b(1) = \tilde{b}(1) = 1$, $\phi_a, \phi_{\tilde{a}}$ are the corresponding refinable functions satisfying* (1.1) *and* (1.4), *then*

$$\|\phi_a - \phi_{\tilde{a}}\|_\infty \le 2N \frac{\|b - \tilde{b}\|_\infty}{\min(\theta, 1-\theta)}. \tag{4.1}$$

Proof. Central to our proof of the inequality (4.1) is the subdivision operator S_a defined as a bounded map on $\ell^\infty(\mathbb{Z})$ by the formula

$$(S_a c)_i = \sum_{j \in \mathbb{Z}} a_{i-2j} c_j, \quad i \in \mathbb{Z},$$

where $c := \{c_j : \ j \in \mathbb{Z}\}$. Clearly, $a, \tilde{a}$ are polynomials of degree at most $N+1$ having nonnegative coefficients and satisfying (1.3). Hence, we obtain that

$$\|S_a\|_\infty \le 1, \tag{4.2}$$

where

$$\|S_a\|_\infty := \sup\{\|S_a c\|_\infty : \ \|c\|_\infty \le 1\}.$$

Also, we have that $\|S_a\|_\infty \le (N+1)\|a\|_\infty$.

Let $d(z) := (\theta + (1-\theta)z) b(z) := \sum_{j \in \mathbb{Z}_N} d_j z^j, \ \ z \in \mathbb{C}$, and ∇ be the difference operator defined by

$$(\nabla c)_j = c_j - c_{j+1}, \quad j \in \mathbb{Z},$$

where $c = \{c_j : j \in \mathbb{Z}\}$. By direct computation, we obtain that

$$\|S_d\|_\infty \le \max(\theta, 1-\theta) \tag{4.3}$$

and

$$\nabla S_a = S_d \nabla. \tag{4.4}$$

So, for all $k \in \mathbb{Z}_+$, we conclude that

$$
\begin{aligned}
\|S_a^k - S_{\tilde{a}}^k\|_\infty &\le \sum_{l=0}^{k-1} \|S_{\tilde{a}}^l (S_a - S_{\tilde{a}}) S_a^{k-1-l}\|_\infty \\
&\le \sum_{l=0}^{k-1} \|(S_a - S_{\tilde{a}}) S_a^{k-1-l}\|_\infty \\
&\le N\|b - \tilde{b}\|_\infty \sum_{l=0}^{k-1} \|\nabla S_a^{k-1-l}\|_\infty \\
&\le N\|b - \tilde{b}\|_\infty \sum_{l=0}^{k-1} \|S_d^{k-1-l} \nabla\|_\infty \\
&\le 2N\|b - \tilde{b}\|_\infty \sum_{l=0}^{k-1} (\max(\theta, 1-\theta))^{k-l-1} \\
&\le \frac{2N\|b - \tilde{b}\|_\infty}{\min(\theta, 1-\theta)}.
\end{aligned}
\tag{4.5}
$$

Let

$$
M(x) := \max\{0, 1 - |x|\}, \quad x \in \mathbb{R},
$$

and recall that the sequence of functions

$$
\Omega_a^k := \sum_{j \in \mathbb{Z}} (S_a^k \delta)_j M(2^k \cdot - j),
$$

where $k \in \mathbb{Z}_+$ and $\delta := (\delta_j : j \in \mathbb{Z})$ is defined as

$$
\delta_j := \begin{cases} 0, & j \ne 0, \\ 1, & j = 0. \end{cases}
$$

Furthermore we set $a_0(z) := (1+z)^2/2,\ z \in \mathbb{C}$, and conclude that

$$
\begin{aligned}
&\|\Omega_a^{k+1} - \Omega_a^k\|_\infty \le \|S_a^{k+1}\delta - S_{a^0} S_a^k \delta\|_\infty \\
&\le 2N\|\nabla S_a^k \delta\|_\infty \le 2N(\max(\theta, 1-\theta))^k.
\end{aligned}
\tag{4.6}
$$

This shows that Ω_a^k converges uniformly to ϕ_a as $k \to \infty$ and satisfies the estimate

$$
\|\Omega_a^k - \phi_a\|_\infty \le 2N \frac{(\max(\theta, 1-\theta))^k}{\min(\theta, 1-\theta)},\ k \in \mathbb{Z}_+.
$$

Therefore, for any $p \in \mathbb{Z}_+$ we have that

$$
\begin{aligned}
\|\phi_a - \phi_{\tilde{a}}\|_\infty &\le \|\phi_a - \Omega_a^p\|_\infty + \|\Omega_a^p - \Omega_{\tilde{a}}^p\|_\infty + \|\Omega_{\tilde{a}}^p - \phi_{\tilde{a}}\|_\infty \\
&\le 4N \frac{(\max(\theta, 1-\theta))^p}{\min(\theta, 1-\theta)} + 2N \frac{\|b - \tilde{b}\|_\infty}{\min(\theta, 1-\theta)}.
\end{aligned}
$$

We now let p tend to infinity in this upper bound to finish the proof of (4.1). ∎

We remark that the effect of "truncating" the coefficients in a refinement equation (1.1) on the refinable function has been studied, [2, 5]. Although the results in [2, 5] are of a general nature. Proposition 4.1 is directly pertinent to the class of refinable functions of our focus here, and is not covered by the results in [2, 5].

5. Convergence

Proof of Theorem 2.2. First, we discuss the case $\theta \in (0, 1/2)$. According to (3.4) and Lemma 3.4 there are constant $q > 0$ and $\mu \in (0, 1)$ such that for all $n \in \mathbb{Z}_+$

$$\|a^n - a^\infty(\cdot|\theta)\|_\infty \le q\mu^n, \tag{5.1}$$

where

$$a^\infty(z|\theta) := z^{N-1} C_\theta(z), \quad z \in \mathbb{C}.$$

In the inequality above, μ has the same meaning as in Lemma 3.4 while q may be different. Similarly, for $\theta \in (1/2, 1)$, (5.1) holds with

$$a^\infty(z|\theta) := C_\theta(z), \quad z \in \mathbb{C}.$$

The reason for this is that in this case the polynomial $\hat{b}^n$ defined for $n \in \mathbb{Z}_+$ by the equation

$$\hat{b}^n(z) := z^{N-1} b^n(z^{-1}), \quad z \in \mathbb{C}$$

satisfies all the conditions of Lemma 3.4 with θ replaced by $1-\theta$ and consequently

$$\lim_{n\to\infty} b^n(z) = 1, \quad z \in \mathbb{C}$$

uniformly on compact subsets of $\mathbb{C}$.

Now we make use of Lemma 3.5 to consider the case $\theta = 1/2$ and $\tau \in (0, 1)$. Here both b^n and $\hat{b}^n$ satisfy the hypothesis of the lemma. Moreover, since

$$\sum_{j\in\mathbb{Z}_N} j\hat{b}_j^0 = N - 1 - \sum_{j\in\mathbb{Z}_N} jb_j^0$$

we have that

$$\widehat{m} = N - 2 - m \quad \text{and} \quad \widehat{\tau} = 1 - \tau.$$

This means for some positive constants $q, \mu \in (0, 1)$, $n \in \mathbb{Z}_+$ and for all j such that $0 \le j \le m - 1$ or $m + 2 \le j \le N - 1$ we have that

$$b_j^n \le q\mu^n.$$

However, since for all $n \in \mathbb{Z}_+$,

$$\sum_{j\in\mathbb{Z}_N} b_j^n = 1 \tag{5.2}$$

and

$$\sum_{j\in\mathbb{Z}_N} jb_j^n = m + \tau \tag{5.3}$$

see (3.17), we conclude that there exists a positive constant $\tilde{q}$ such that for all $n \in \mathbb{Z}_+$

$$|b_m^n - (1-\tau)| \le \tilde{q}\mu^n, \quad |b_{m+1}^n - \tau| \le \tilde{q}\mu^n. \tag{5.4}$$

Consequently, we have proved that (5.1) holds for $\theta = 1/2$ and $\tau \in (0,1)$, where

$$a^\infty(z|\theta) := z^m((1-\tau) + \tau z), \quad z \in \mathbb{C}.$$

The final case $\theta = 1/2$ and $\tau = 0$ requires the use of Lemma 3.6. As in the proof above, using Lemma 3.6 we conclude that for some positive constant q and $\mu \in (0,1)$,

$$b_j^n \le q\mu^n \tag{5.5}$$

for all j such that $0 \le j \le m-2$ or $m+2 \le j \le N-1$, and moreover $\lim_{n\to\infty} b_{m-1}^n$ and $\lim_{n\to\infty} b_{m+1}^n$ exist. Therefore, so does $\lim_{n\to\infty} b_m^n$ by (5.2). According to Lemma 3.2 there are a $k \in \mathbb{Z}_+$ and an $\alpha \in [0,1]$ such that

$$\lim_{n\to\infty} b^n(z) = z^k((1-\alpha) + \alpha z), \quad z \in \mathbf{C}. \tag{5.6}$$

We combine this fact with (5.2) and (5.3) to conclude that

$$\lim_{n\to\infty} b^n(z) = z^m, \quad z \in \mathbb{C},$$

that is, $\lim_{n\to\infty} b_{m-1}^n = \lim_{n\to\infty} b_{m+1}^n = 0$ and $\lim_{n\to\infty} b_m^n = 1$. We must now estimate the rate at which these *three* sequences tend to their respective limits. The *two* equations (5.2) and (5.3) and the estimate (5.5) guarantee that it suffices to estimate the rate at which the first sequence tends to the limit. Specifically, there exists a positive constant $q_1 \ge q$ such that for all $n \in \mathbb{Z}_+$

$$|b_{m+1}^n - b_{m-1}^n| \le q_1\mu^n \tag{5.7}$$

and

$$|b_m^n + 2b_{m-1}^n - 1| \le q_1\mu^n. \tag{5.8}$$

This inequality together with the estimate (5.5), shows that

$$|b^n(z) - z^{m-1}\beta^n(z)| \le q_1\mu^n \sum_{j\in\mathbb{Z}_N} z^j,$$

where $\beta^n(z) := b_{m-1}^n + (1-2b_{m-1}^n)z + b_{m-1}^n z^2$. Substituting this inequality into (3.5) with $\theta = 1/2$, we conclude that there is a positive constant q_2 such that for all $n \in \mathbb{Z}_+$,

$$\begin{aligned}&|C_{1/2}(z)\beta^n(z)\beta^{n+1}(z) - C_{1/2}(-z)\beta^n(-z)\beta^{n+1}(-z) - 2z\beta^{n+1}(z^2)|\\ &\le 2q_2\mu^n \sum_{j=0}^{2} z^{2j}.\end{aligned}$$

Identifying the coefficients of z of the polynomials appearing in the above inequality yields, upon simplification, the estimate

$$\left|b_{m-1}^{n+1}-\frac{b_{m-1}^{n}}{1+2b_{m-1}^{n}}\right|\le 2q_2\mu^n,$$

and, in particular, we conclude that

$$0<b_{m-1}^{n+1}\le \frac{b_{m-1}^{n}}{1+2b_{m-1}^{n}}+2q_2\mu^n.$$

Let $q_3\ge 1$ be a positive constant chosen so that $q_3\ge 2q_2\tau^n(n+3)^2$ for all $n\in\mathbb{Z}_+$. We now prove that

$$b_{m-1}^{n}\le q_3(n+1)^{-1} \tag{5.9}$$

for all $n\in\mathbb{Z}_+$. Clearly (5.9) holds for $n=0$. The proof of the general case proceeds by induction on n. So we assume that (5.9) holds for some $n\in\mathbb{Z}_+$. Therefore using the monotonicity of the function $x\to x/(1+2x)$ on $(0,\infty)$ we get that

$$b_{m-1}^{n+1}=\frac{b_{m-1}^{n}}{1+2b_{m-1}^{n}}+2q_2\tau^n\le\frac{q_3}{n+1+2q_3}+\frac{q_3}{(n+3)^2}\le\frac{q_3}{n+2}.$$

This proves that the inequality (5.9) holds for all $n\in\mathbb{Z}_+$. As a consequence of (5.9), for $\theta=1/2$ and $\tau=0$ there exists a positive constant q_4 such that

$$\|b^n-a^\infty(z|\theta)\|_\infty\le\frac{q_4}{n+1},\quad n\in\mathbf{Z}_+, \tag{5.10}$$

where

$$a^\infty(z|\theta):=z^m,\quad z\in\mathbb{C}.$$

In all cases, we have by our definition, (2.8) that

$$\phi^\infty(\cdot|\theta)=\sum_{j\in\mathbb{Z}}a_j^\infty(\theta)\phi^\infty(2\cdot-j|\theta),$$

and therefore Theorem 2.2 follows from Proposition 4.1 by choosing $\tilde a:=a^n$ and $a=a^\infty(\cdot|\theta)$. ■

Acknowledgement. We wish to thank Johan De Villiers, Karin Goosen and Seng Luan Lee for helpful conversation concerning the problem studied here, especially to Seng Luan Lee in the formulation of the problem.

References

1. A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, Stationary subdivision. *Mem. Amer. Math. Soc.* **93** (1991) 1–186.
2. I. Daubechies and Y. Huang, How does truncation of the mask affect a refinable function?, *Constr. Approx.* **11** (1995) 365–380.
3. J. De Villiers, C. A. Micchelli, and T. Sauer, Building refinable functions from their values at integers, *Calcolo* **37** (2000) 139–158.

4. T. N. T. Goodman and C. A. Micchelli, On refinement equations determined by Polya frequency sequences, *SIAM J. Math. Anal.* **23** (1992) 766–784.
5. B. Han and T. A. Hogan, How is a vector pyramid scheme affected by perturbation in the mask?, In *Approximation Theory*, C. K. Chui and L. L. Schumaker (eds.), Vanderbilt University, 1998, pp. 97–104.
6. S. L. Lee, A. Sharma, and H. H. Tan, Spline interpolation and wavelet construction, *Appl. Comput. Harmon. Anal.* **5** (1998) 249–276.
7. C. A. Micchelli, Banded matrices with bounded inverse, *J. Comp. Appl. Math.* **41** (1992) 281–300.
8. C. A. Micchelli, *Mathematical Aspects of Geometric Modeling*, CBMS-NSF Regional Conference Series in Applied Mathematics, 65. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1995.
9. C. A. Micchelli and A. Pinkus, Descartes systems from corner cutting, *Constr. Approx.* **7** (1991) 161–194.