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Milnor Number of Positive Polynomials

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Abstract. We prove that if a positive polyomial is "good" at infinity, then its total Milnor number must be odd.

Let $f \in \mathbb{C}[z_1, z_2, \dots, z_n]$ be a polynomial function. Let $\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_n}\right)$ be the ideal generated by the partial derivations of f. The *total Milnor number* of f, denoted by $\mu_{\text{total}}(f)$, is defined by

$$\mu_{\text{total}}(f) := \dim_{\mathbb{C}} \frac{\mathbb{C}[z_1, z_2, \dots, z_n]}{\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_n}\right)}.$$

This number is finite if and only if all critical points of f are isolated. In this case, it is the sum of local Milnor numbers at all singular points. The total Milnor number can be interpreted as the degree of the map

$$\mathbb{S}_r^{2n-1} \to \mathbb{S}_1^{2n-1}, \quad z \mapsto \frac{\text{grad} f(z)}{\|\text{grad} f(z)\|},$$

where $\mathbb{S}_r^{2n-1} = \partial B_r$ for a sufficient large ball B_r such that all the singular points of f are contained in B_r.

In this note, we prove the following

Theorem. *Let $f \in \mathbb{C}[z_1, z_2, \dots, z_n]$. Suppose that*

(i) *All the coefficients of f are real numbers;*
(ii) *The restriction of f on $\mathbb{R}^n$ is bounded from below.*
(iii) *There is $r_0 \gg 1, \delta > 0$ such that*

$$\|\text{grad} f(z)\| \geq \delta$$

for $z \in \mathbb{C}^n \setminus B_{r_0}$. Then

$$\mu_{\text{total}}(f) = 1 \bmod 2.$$

Proof. (1) We show that the conditions (ii) and (iii) imply that the restriction of f on $\mathbb{R}^n$ is proper.

Put

$$V = \{x \in \mathbb{R}^n \mid f(x) = \min_{\|y\|=\|x\|, y\in\mathbb{R}^n} f(y)\}.$$

It is not hard to see that V is an unbounded semi-algebraic set. A version at infinity of the Curve Selection Lemma [4] gives the existence of a meromorphic real curve

$$\theta : (0, \epsilon] \to \mathbb{R}^n$$

such that $\theta(t) \in V$ for $t > 0$ and $\|\theta(0)\| = \infty$.

It is easy to see that $\mathrm{grad} f(\theta(t)) = \lambda(t)\theta(t)$ for some $\lambda(t) \in \mathbb{R}$.

Let

$$f(\theta(t)) = at^\alpha + \text{terms of higher degrees,}$$
$$\theta(t) = bt^\beta + \text{terms of higher degrees.}$$

One has

$$\frac{d}{dt} f(\theta(t)) = \Big\langle \frac{d\theta}{dt}, \mathrm{grad} f(\theta(t))\Big\rangle = \lambda(t)\Big\langle \frac{d\theta}{dt}, \theta(t)\Big\rangle.$$

Hence

$$2\frac{d}{dt} f(\theta(t)) = \lambda(t)\frac{d}{dt}\|\theta(t)\|^2.$$

This implies

$$\Big|2\frac{d}{dt} f(\theta(t))\Big| = \frac{\|\mathrm{grad} f(\theta(t))\|}{\|\theta(t)\|}\frac{d}{dt}[\|\theta(t)\|^2];$$

or

$$2|at^{\alpha-1} + \cdots| \geq \delta\|b\|t^{\beta-1} + \cdots .$$

Thus

$$\alpha < \beta < 0.$$

So $|f(\theta(t))| \to \infty$ as $t \to 0$. Since the restriction of f on $\mathbb{R}^n$ is bounded from below, $f(\theta(t)) \to +\infty$ as $t \to 0$. From this we see that if $\{a_k\} \subset \mathbb{R}^n$, $\|a_k\| \to \infty$, then $f(a_k) \to +\infty$; i.e., $f|_{\mathbb{R}^n}$ is proper.

(2) By the index of f we means the index of the restriction of f on $\mathbb{R}^n$; i.e., the index of the gradient field of f on $\mathbb{R}^n$ [1]. We denote it by $i(f)$. We will show that if the restriction of f on $\mathbb{R}^n$ is proper and bounded from below, then $i(f) = 1$.

Put

$$D_1 = \Big\{x \in \mathbb{R}^n \mid \mathrm{rank}\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \leq 1\Big\},$$
$$D_2 =: \{x \in \mathbb{R}^n \mid \langle x, \mathrm{grad} f(x)\rangle < 0\}.$$

Then the set

$$D = D_1 \cap D_2 = \{x \in \mathbb{R}^n \mid \exists\lambda < 0 \text{ such that } \mathrm{grad} f(x) = \lambda x\}$$

is semi-algebraic.

We will show that D is bounded. Assume that it is not the case. Let $\eta(t)$ be a meromorphic curve, $\eta(t) \in D$ for $t \in (0, \epsilon]$ and $\|\eta(0)\| = \infty$.

Let
$$\|\eta(t)\| = ct^{\kappa} + \cdots ,$$
where $\kappa < 0$ and $c > 0$.

As in the proof of (1),
$$2\frac{d}{dt}f(\eta(t)) = \lambda(t)\frac{d}{dt}[\|\eta(t)\|^2] = 2\kappa c\lambda(t)(t^{2\kappa-1} + \cdots) > 0$$
for $t > 0$, t close to 0. Thus, $f(\eta(t))$ is monotonouslly deacreasing as $t \to 0^+$. Since f is proper, $f(\eta(t)) \to -\infty$. This contradicts the fact that the restriction of f on $\mathbb{R}^n$ is bounded from below.

Using the boundedness of D, it is easy to show that the map $\frac{\text{grad}f}{\|\text{grad}f\|}$ is homotopic to the indentify map on the sphere $\mathbb{S}_r \subset \mathbb{R}^n$ with sufficiently large r. Hence $i(f) = 1$.

(3) Let r_0 and δ be as in the condition (iii) of the theorem. Let $a = (a_1, a_2, \dots ,$
$a_n) \in \mathbb{R}^n$ be a real regular value of the map grad$f : \mathbb{C}^n \to \mathbb{C}^n$. All the singular points of the polynomial $f_a(z) = f(z) + \sum_{i=1}^n a_i z_i$ are non-degenerated. We can assume that $\|a\| < \delta/2$. It is easy to check that the maps $\frac{\text{grad}f}{\|\text{grad}f\|}$ and $\frac{\text{grad}f_a}{\|\text{grad}f_a\|}$ are homotopic, here these maps are considered both in $\mathbb{C}^n$ and $\mathbb{R}^n$. Thus
$$\mu_{\text{total}}(f) = \mu_{\text{total}}(f_a), \qquad i(f) = i(f_a) = 1.$$
Since the coefficients of f are all real, the number of nonreal singular points of f_a is even. Thus
$$\mu_{\text{total}}(f_a) = k \bmod 2,$$
where k is the number of real singular points of f_a. Let them be $A_1, A_2, \dots , A_k \in \mathbb{R}^n$ and let $i_{A_i}(f)$ be the index of f at A_i. (Here f is considered as a map from $\mathbb{R}^n$ to $\mathbb{R}$.) Since $A_i, i = 1, 2, \dots , k$ is nondegenerated, $i_{A_i}(f_a) = +1$ or -1.

We have
$$1 = i(f_a) = \sum_{i=1}^{k}(\pm 1).$$
It follows that $k = 1 \bmod 2$. Therefore
$$\mu_{\text{total}}(f) = \mu_{\text{total}}(f_a) = 1 \bmod 2.$$

The theorem is proved. ∎

Remark. The condition (iii) says that, in some sense f has no singularities at infinity. The following example shows that this condition cannot be omitted.

Let
$$f_1(x, y) = (x^2y - x - 1)^2 + (x^2 - 1)^2, \qquad ([2])$$
$$f_2(x, y) = (x^2y + x + 1)^2 + x^2 + 2.$$
We have $\mu_{\text{total}}(f_1) = 2$ and $\mu_{\text{total}}(f_2) = 0$. Using Theorem B of [3], it is easy to see that f_1 has 0 and f_2 has 2 as critical values, corresponding to the singularities at infinity.

References

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