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Some Bounds for the Higher Schur Multiplier of a Finite Group and Its Number of Generators

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Abstract. This paper is devoted to give some inequalities for the higher Schur multiplier of finite groups with respect to the variety of nilpotent groups of class at most $c (c \geq 1)$, which generalize the works of M. R. Jones in 1972 and 1973, intensively. Also we give an upper bound for the minimal number of generators of higher Schur multiplier of a finite p-group of class at most n.

1. Introduction

Let $1 \to R \to F \to G \to 1$ be a free presentation of a group G, where F is a free group. Then the higher Schur multiplier of the group G, denoted by $M^{(c)}(G)$, is defined to be $\dfrac{R \cap \gamma_{c+1}(F)}{[R,_c F]}, c \geq 1$ (see also [8]), where $\gamma_{c+1}(F) = [\underbrace{F, \ldots, F}_{(c+1)-times}]$ is the $(c+1)^{st}$-term of the lower central series and $[R,_c F]$ denotes $[R, \underbrace{F, \ldots, F}_{c-times}]$. In particular, if $c = 1$, we obtain the Schur multiplier $\dfrac{R \cap F'}{[R, F]}$, which is denoted by $M(G)$.

One may easily check that the higher Schur multiplier of the group G is always abelian and independent of the choice of the free presentation of G (see [6]).

Let G be a finite group, then the positive integer $r(G)$ is defined to be the *rank* of G, if every subgroup of G may be generated by $r(G)$ elements and there is at least one subgroup of G, which can not be generated by fewer than $r(G)$

elements. Also $d(G)$ and $e(G)$ denote the minimal number of generators and exponent of G, respectively.

In the next section, we present some inequalities for the higher Schur multiplier of groups. Also we give some lower and upper bounds for the order of higher Schur multiplier of a finite p-group, when $c = 2$. Our results somehow are similar to the works of Jones in 1972 and 1973 (see [4, 5]).

In the final section, if G is a finite p-group of class at most n with d generators then a bound for the minimal number of generators of $M^{(c)}(G)$ is obtained.

2. Some Inequalities for the Higher Schur Multiplier of Finite Groups

The following definition from [7] is vital in our investigation.

Definition 1. *Let A and G be two groups, then A is said to be a G-crossed module, when there exist a homomorphism $\sigma : A \to G$ and an action of G on A such that*
(i) $\sigma(a^g) = g^{-1}\sigma(a)g$ *and*
(ii) $a_2{}^{\sigma(a_1)} = a_1{}^{-1}a_2a_1$.

Now, let $1 \to R \to F \to G \to 1$ be a free presentation of a group G and N a normal subgroup of G such that $N \subseteq Z_c(G) \cap \gamma_{c+1}(G)$. Put $N \cong \frac{S}{R}$ and $Q = \frac{G}{N} \cong \frac{F}{S}$ and consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 1 & \longrightarrow & R & \longrightarrow & S & \longrightarrow & N \longrightarrow 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & = & F & & \\
 & & & & \downarrow \pi & & \downarrow \pi' & & \\
1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & = & \dfrac{G}{N} \longrightarrow 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

in which all the rows and columns are exact. Clearly $[S,_c F]$ is a normal subgroup of F and hence F acts on it by conjugation. By the assumption

$$\frac{S}{R} \subseteq Z_c(\frac{F}{R}) \cap \frac{\gamma_{c+1}(F)R}{R}.$$

Thus $S \subseteq \gamma_{c+1}(F)R$ and $[S,_c F] \subseteq R$, which imply that

$$\begin{aligned}
[S, [S,_c F]] &\subseteq [\gamma_{c+1}(F)R, [S,_c F]] \\
&= [\gamma_{c+1}(F), [S,_c F]][R, [S,_c F]] \\
&\subseteq [\gamma_{c+1}(F), R][R, [S,_c F]] \\
&\subseteq [R, \gamma_{c+1}(F)] \subseteq [R,_c F].
\end{aligned}$$

Therefore the above action induces an action of $Q = F/S$ on $\frac{[S,_cF]}{[R,_cF]} = T_c$, say.

Now trivial homomorphism $\sigma : \frac{[S,_cF]}{[R,_cF]} \to Q$ implies that T_c is a Q-crossed module and hence, using [7, Corollary 2], one may obtain the following lemma.

Lemma 2.1. *There exists an epimorphism* $\otimes^c(N,Q) = N \otimes \underbrace{Q \otimes \ldots \otimes Q}_{c-times} \to T_c$ *of* Q*-crossed modules.*

The following theorem gives the upper bound for higher Schur multiplier of a finite group, which generalizes the work of Jones [5], intensively (see also [9]).

Theorem 2.2. *Let* N *be a normal subgroup of a finite group* G *and*

$$1 \to N \to G \to Q = \frac{G}{N} \to 1$$

be an exact sequence such that $N \subseteq Z_c(G) \cap \gamma_{c+1}(G)$*. Then*
(i) $|M^{(c)}(G)|\,|N|$ *divides* $|M^{(c)}(Q)||\otimes^c(N,Q)|$;
(ii) $d(M^{(c)}(G)) \le d(M^{(c)}(Q)) + d(\otimes^c(N,Q))$;
(iii) $e(M^{(c)}(G))$ *divides* $e(M^{(c)}(Q))e(\otimes^c(N,Q))$.

Proof. By [7, Theorem 4], the following sequence

$$\otimes^c(N, G/\gamma_{c+1}(G)) \to M^{(c)}(G) \to$$
$$M^{(c)}(Q) \xrightarrow{\zeta} N \to G/\gamma_{c+1}(G) \to Q/\gamma_{c+1}(Q) \to 1$$

is exact. As $N \subset \gamma_{c+1}(G)$, ζ is surjective, hence

$$|M^{(c)}(G)||N| \text{ divides } |M^{(c)}(Q)||\otimes^c(N, G/\gamma_{c+1}(G))|,$$
$$d(M^{(c)}(G)) \le d(M^{(c)}(Q)) + d(\otimes^c(N, G/\gamma_{c+1}(G))),$$

and

$$e(M^{(c)}(G)) \text{ divides } e(M^{(c)}(Q))e(\otimes^c(N, G/\gamma_{c+1}(G))).$$

The result follows by noting that $\otimes^c(N, G/\gamma_{c+1}(G))$ is a homomorphic image of $\otimes^c(N,Q)$. ■

Definition 2. *An exact sequence* $1 \to A \to G^* \to G \to 1$ *of* G *is called a c-stem cover if* $A \subseteq Z_c(G^*) \cap \gamma_{c+1}(G^*)$ *and* $A \cong M^{(c)}(G)$*. In this case,* G^* *is said to be a c-covering group of* G*.*

The following interesting corollary is a wide generalization of Iwahori and Matsumoto [3], which gives a criterion for triviality of higher Schur multiplier of groups.

Corollary 2.3. *Let* G^* *be a c-covering group of a finite perfect group* G (i.e. $G' = G$), $c \ge 1$. *Then* $M^{(c)}(G^*)$ *is trivial.*

In the remaining of this section, we present some upper and lower bounds for the 2-nilpotent multiplier of a finite p-group. We start with the following useful lemmas.

Lemma 2.4.
(i) *Let Z_p^r be the direct product of r copies of cyclic groups. Then*

$$|M^{(2)}(Z_p^r)| = p^{\frac{r(r^2-1)}{3}}.$$

(ii) *Let G be a group such that $G/Z_2(G)$ is of order p^r, then $\gamma_3(G)$ is of order at most $p^{\frac{r(r^2-1)}{3}}$.*

Proof.
(i) Apply Theorem 3.1 of [1].
(ii) The proof follows by using Moghaddam [10, Theorem 3.1] and (i). ∎

Lemma 2.5. *Let N be a normal subgroup of a finite group G and $F/R \cong G$ be a free presentation of G. If S is a normal subgroup of F such that SR/R corresponds to N, then the following sequence is exact:*

$$1 \to \frac{R \cap [S,_c F]}{[R,_c F] \cap [S,_c F]} \to M^{(c)}(G) \to M^{(c)}(G/N) \to \frac{N \cap \gamma_{c+1}(G)}{[N,_c G]} \to 1. \quad (1)$$

Proof. The kernel of the map $N/[N,_c G] \to G/\gamma_{c+1}(G)$ is $N \cap \gamma_{c+1}(G)/[N,_c G]$. So, combining [1, Theorem 2.2] and [1, Theorem 2.6 and its proof], we get the exact sequence (1). ∎

The following corollary is an immediate consequence of the above lemma.

Corollary 2.6. *Let N be a normal subgroup of a finite group G. If the higher Schur multiplier $M^{(c)}(G)$ $(c > 1)$ is trivial, then $M^{(c)}(G/N) \cong \frac{N \cap \gamma_{c+1}(G)}{[N,_c G]}$.*

Now we are ready to prove the following theorem.

Theorem 2.7. *Let G be a finite p-group with $|G| = p^n$ and $d(G) = d$. Then*
a) $p^{\frac{d(d^2-1)}{3}} \leq |M^{(2)}(G)||\gamma_3(G)|$;
b) $|M^{(2)}(G)||\gamma_3(G)| \leq p^{\frac{n(n^2-1)}{3}}$;
c) $|M^{(2)}(G)||\gamma_3(G)| < |N \otimes Q \otimes Q|\, p^{\frac{\alpha(\alpha^2-1)}{3}}$, *where N is a normal subgroup of G, $N \subset Z_2(G) \cap \gamma_3(G)$, $Q = G/N$ and $|G/N| = p^\alpha$.*

Proof. (a) The exact sequence (1) of Lemma 2.5 implies that

$$|M^{(c)}(G/N)| \leq |M^{(c)}(G)||N \cap \gamma_{c+1}(G)/[N,_c G]| \leq |M^{(c)}(G)||\gamma_{c+1}(G)|.$$

Now the result follows by setting $c = 2$ and $N = \Phi(G)$, and using Lemma 2.4(i).
(b) By [10, Corollary 3.12], we obtain $|M^{(2)}(G)||\gamma_3(G)| \leq |M^{(2)}(Z_p^n)|$. Hence Lemma 2.4 (i) gives the inequality (b).
(c) Again, assume that $G \cong F/R$ is a free presentation of the p-group G, where F is a free group. Set $N \cong T/R$ for a suitable normal subgroup T of F and $Q = G/N \cong F/T$. Then

$$\gamma_3(G) \cong \frac{\frac{\gamma_3(F)}{[R,F,F]}}{\frac{R\cap\gamma_3(F)}{[R,F,F]}},$$

$$M^{(2)}(G) = \frac{R\cap\gamma_3(F)}{[R,F,F]} \text{ and } M^{(2)}(Q) = \frac{T\cap\gamma_3(F)}{[T,F,F]}.$$

So $|M^{(2)}(G)|\,|\gamma_3(G)|$ is the product of $\left|\frac{[T,F,F]}{[R,F,F]}\right|$ by $\left|\frac{\gamma_3(F)}{[T,F,F]}\right|$. By Lemma 2.1, the first factor divides $|N\otimes Q\otimes Q|$. Since $\left|\frac{F}{[T,F,F]}/Z_2(\frac{F}{[T,F,F]})\right| \le |Q|$, using Lemma 2.4, the second factor divides $p^{\frac{\alpha(\alpha^2-1)}{3}}$, and hence (c) is proved. ∎

Now the above theorem has the following corollary, which is similar to the result of Green [2] for the Schur multiplier of G.

Corollary 2.8. *Let G be a finite p-group of order p^n, then*

$$|M^{(2)}(G)| \le p^{\frac{n(n^2-1)}{3}}.$$

Remark. Let G be a finite p-group with $d(G) = 3$ and $d(\gamma_3(G)) \le 7$. Then [1, Theorem 3.1] and the inequality

$$d(M^{(2)}(Q)) \le d(\frac{T\cap\gamma_3(F)}{[R,F,F]}) \le d(M^{(2)}(G)) + d(\gamma_3(G)\cap N)$$

imply that $M^{(2)}(G)$ is non-trivial. In particular, if G is a finite p-group of order at most p^{10} and $d(G) = 3$, then $M^{(2)}(G)$ is non-trivial.

3. Upper Bound for the Number of Generators of Higher Schur Multiplier

Given a natural number n, we denote by $\mu(n)$ the Mobius function defined as follows: $\mu(1) = 1$; for $n = p_1^{e_1}p_2^{e_2}\dots p_s^{e_s}$, where $p_1,\dots,p_s$ are distinct primes, $\mu(n) = 0$ if $e_i > 1$ for some $i \in \{1,\dots,s\}$, and $\mu(p_1p_2\dots p_s) = (-1)^s$. Let F denote the free group of rank d. Clearly the factor group $\frac{\gamma_q(F)}{\gamma_{q+1}(F)}$ is a free abelian group generated by the basic commutators of weight q on d generators, and its rank $m_d(q)$, say, is given by Witt's formula

$$m_d(q) = \frac{1}{q}\sum_{s|q}\mu(s)d^{\frac{q}{s}}.$$

In this section, we give an upper bound for the $(c-)$ higher Schur multiplier of finite p-groups $(c \ge 1)$. In fact, we prove the following.

Theorem 3.1. *Let G be a finite p-group of class at most n and generated by d elements. Then $M^{(c)}(G)$ is generated by $\sum_{k=1}^{n} m_d(k+c)$ elements.*

We start with the following useful lemma.

Lemma 3.2. *Let G be a finite nilpotent group of class at most $n \geq 2$ with a free presentation $F/R \cong G$, where F is a free group of finite rank. Then the following sequence is exact:*

$$1 \to \frac{\gamma_{n+c}(F)}{[R,_c F] \cap \gamma_{n+c}(F)} \to M^{(c)}(G) \to M^{(c)}(\frac{G}{\gamma_n(G)}) \to \gamma_n(G) \to 1.$$

Proof. We use the sequence (1) of Lemma 2.5, for $N = \gamma_n(G)$. Clearly $\gamma_n(G) = \frac{\gamma_n(F)R}{R}$, and hence $S = \gamma_n(F)$. This implies that $[S,_c F] = \gamma_{n+c}(F)$. Using the nilpotency class of G, one obtains $\gamma_{n+c}(F) \subseteq \gamma_{n+1}(F) \subseteq R$ and so $R \cap [S,_c F] = \gamma_{n+c}(F)$. Now, if $c+1 > n$, then $\gamma_n(G) \cap \gamma_{c+1}(G) = 1$, and, if $c+1 \leq n$, then $\gamma_n(G) \cap \gamma_{c+1}(G) = \gamma_n(G)$. Thus, in the first case, we have $\frac{N \cap \gamma_{c+1}(G)}{[N,_c G]} = 1$, and in the latter case, $\frac{N \cap \gamma_{c+1}(G)}{[N,_c G]} = \gamma_n(G)$, which completes the proof. ∎

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We use induction on n. Clearly, by the assumption, for a suitable normal subgroup R of F, $G \cong F/R$. If $n = 1$, then G (or F/R) is an abelian group and hence $\gamma_{c+1}(F) \subseteq F' \subseteq R$. It follows that $M^{(c)}(G) = \frac{\gamma_{c+1}(F)}{[R,_c F]}$.

One notes that $\gamma_{c+2}(F) \subseteq [R,_c F]$ and $\frac{\gamma_{c+1}(F)}{\gamma_{c+2}(F)}$ is generated by $m_d(c+1)$ elements, which gives the result.

Now assume that $n > 1$, then by induction hypothesis, $M^{(c)}(\frac{G}{\gamma_n(G)})$ is generated by t elements, where

$$t = \sum_{k=1}^{n-1} m_d(k+c).$$

By Lemma 3.2, the following sequence is exact:

$$1 \to \frac{\gamma_{n+c}(F)}{[R,_c F] \cap \gamma_{n+c}(F)} \xrightarrow{\theta} M^{(c)}(G) \xrightarrow{\psi} M^{(c)}\Big(\frac{G}{\gamma_n(G)}\Big) \to \gamma_n(G) \to 1.$$

Put $N = \operatorname{Im}\theta = \ker \psi$, we see that $\frac{M^{(c)}(G)}{N}$ is isomorphic to a subgroup of $M^{(c)}(\frac{G}{\gamma_n(G)})$. Hence $\frac{M^{(c)}(G)}{N}$ is generated by t elements. On the other hand, $R \supseteq \gamma_{n+1}(F)$, and so $[R,_c F] \cap \gamma_{n+c}(F) \supseteq \gamma_{n+c+1}(F)$.

Thus N is a homomorphic image of $\frac{\gamma_{n+c}(F)}{\gamma_{n+c+1}(F)}$. Consequently, N is generated by $m_d(n+c)$ elements. This implies that $M^{(c)}(G)$ is generated by $t + m_d(n+c)$, which completes the proof. ∎

The above theorem has an interesting corollary as follows:

Corollary 3.3. *Let G be a finite p-group of class at most n and generated by d elements. Then*

$$d(M^{(c)}(G)) \leq \sum_{k=1}^{n} m_d(c+k).$$

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