Vietnam Journal of Mathematics **30**:4 (2002) 343-363

Vietnam Journal
of
MATHEMATICS

On the C-Lipschitz Continuities and C-Approximations of Multivalued Mappings

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Received February 02, 2001

Revised August 8, 2001

Abstract. The notions of locally, locally upper and locally lower C-Lipschitz continuities of multivalued mappings at a given point are introduced and some properties of these mappings are provided. Further, we define locally C-approximations, upper and lower contingent derivatives of a multivalued mapping and derive some necessary and sufficient conditions for multiobjective optimization problems.

1. Introduction

In recent years, in order to study the theory of vector multiobjective optimization and equilibrium problems, several authors have considered the properties of multivalued vector mappings in locally convex Hausdorff spaces with a cone and derive necessary and sufficient conditions for efficient solutions of vector problems. For example, Luc and Malivert [10] have studied the concept of invexity to multivalued mappings and optimality conditions for multiobjecttive optimization with invex data. Aubin and Frankowska [2] have studied some other properties of multivalued mappings and various questions concerning multivalued vector mappings...

In [12 - 13] we already introduced the notions of upper, lower C-continuities, upper, lower C-convexities of multivalued mappings with values in locally convex Hausdorff spaces with a cone and showed some necessary and sufficient conditions for the upper and lower C-continuities and relations between them. We also used these properties in the study of the existence of solutions of optimization and equilibrium problems. In this paper we continue the study of the mentioned papers by investigating the C-Lipschitz continuities and contingent derivatives of multivalued mappings. The paper is organized as follows. In Sec. 2, after recalling the definitions of the upper and lower C-continuities, we introduce the concepts of locally upper and lower C-Lipschitz continuities and show some necessary and sufficient conditions for them. In Sec. 3 we introduce the notions of upper and lower C-approximations, upper and lower contingent derivatives of multivalued mappings. Further, we give some conditions for the existence of contingent derivatives and for an upper (lower) contingent derivative to be an upper (a lower) C-approximation. In Sec. 4, we consider a multiobjective optimization problem and give some necessary and sufficient conditions for the existence of solutions.

2. C-Lipschitz Continuities of Multivalued Mappings

In what follows we assume that X and Y are normed spaces with the topological duals X' and Y', respectively, and $D \subset X$, C a cone in Y. The symbol $\|\,.\,\|$ stands for the norms in X and Y. The closure and the interior of D is denoted by $\overline{D}$ and $\operatorname{int} D$, respectively. By 2^Y we denote the family of all subsets of Y and $F : D \to 2^Y$ denote a multivalued mapping from D into 2^Y with $\operatorname{dom} F = \{x \in D | F(x) \neq \emptyset\}$. We first recall the following definitions

Definition 2.1.
a) *A multivalued mapping F is said to be upper (lower) C-continuous at $x_0 \in$* $\operatorname{dom} F$ *if for any neighborhood V of the origin in Y there exists a neighborhood U of x_0 in X such that*

$$F(x) \subset F(x_0) + V + C$$

$$(F(x_0) \subset F(x) + V - C, \quad \textit{respectively})$$

holds for all $x \in D \cap U$.
b) *If F is upper and lower C-continuous at x_0 simultaneously, we say that F is C-continuous at x_0.*
c) *We say that F is upper (lower) C- continuous on D if it is upper (lower) C-continuous at any point of* $\operatorname{dom} F$.

Remark 1.

i) In the case $C = \{0\}$, if F is a singlevalued mapping, then the above definitions coincide with those in the usual sense.
ii) If $Y = \mathbb{R}$, the space of real numbers, and $\mathbb{R}_+ = \{\alpha \in \mathbb{R} | \alpha \geq 0\}$ and if F is upper (lower) $\mathbb{R}_+$-continuous at x_0 then we say that F is upper (lower) semicontinuous at x_0.
iii) The above definitions are little bit different from Definition 7.1, Chapter 1 in [8]. It is easy to verify that if $F(x_0)$ is compact then they coincide with those given in [8].

Definition 2.2.

1) *A singlevalued mapping $f : D \to \mathbb{R}$ is said to be locally upper (lower) Lipschitz at $x_0 \in$ $\operatorname{dom} f = \{x \in D | f(x) < +\infty\}$ if there exist a neighborhood U of x_0 and a constant $L > 0$ such that*

$$f(x) - f(x_0) \leq L\|x - x_0\|$$

$$(f(x_0) - f(x) \leq L\|x - x_0\|, \quad \textit{respectively})$$

holds for all $x \in U \cap D$.

2) f *is called a locally Lipschitz mapping around* x_0 *if there exist a neighborhood* U *of* x_0 *and a constant* $L > 0$ *such that*

$$|f(x) - f(x')| \leq L\|x - x'\|$$

holds for all $x, x' \in U \cap D$.

3) *A family of singlevalued mappings* $f_\alpha : D \to R, \alpha \in I$, *is said to be locally (upper, lower) equi-Lipschitz at* x_0 *if there exist a neighborhood* U *of* x_0 *and a constant* $L > 0$ *such that*

$$|f_\alpha(x) - f_\alpha(x')| \leq L\|x - x'\|$$
$$(f_\alpha(x) - f_\alpha(x_0) \leq L\|x - x_0\|, f_\alpha(x_0) - f_\alpha(x) \leq L\|x - x_0\|, \quad \textit{respectively})$$

holds for all $x, x' \in U \cap D$ *and* $\alpha \in I$.

Next, we introduce the notions of locally (upper, lower) C-Lipschitz of multivalued mappings. By $B_Y(0, r)$ we denote the closed ball with center at the origin and radius r in Y i.e.

$$B_Y(0, r) = \{y \in Y \mid \|y\| \leq r\}.$$

We also write simply $B_Y(0, 1) = B_Y$.

Definition 2.3.

a) *A multivalued mapping* $F : D \to 2^Y$ *is said to be locally upper (lower) C-Lipschitz at* $x_0 \in \operatorname{dom} F$ *if there exist a neighborhood* U *of* x_0 *and a constant* L *such that*

$$F(x) \subset F(x_0) + L\|x - x_0\|B_Y + C$$

$$(F(x_0) \subset F(x) + L\|x - x_0\|B_Y - C, \textit{respectively})$$

holds for all $x \in U \cap D$.

b) *A multivalued mapping* $F : D \to 2^Y$ *is said to be locally C-Lipschitz around* $x_0 \in \operatorname{dom} F$ *if there exist a neighborhood* U *of* x_0 *and a constant* L *such that*

$$F(x) \subset F(x') + L\|x - x'\|B_Y - C$$

holds for all $x, x' \in U \cap D$.

c) F *is said to be locally (upper, lower) C-Lipschitz in* D *if it is locally (upper, lower) C-Lipschitz around (at) any point of* D.

d) *If* $C = \{0\}$, *we say simply that* F *is locally (upper, lower) Lipschitz instead of locally (upper, lower, respectively)* $\{0\}$ *- Lipschitz.*

Remark 2. It is easy to show that if F is locally upper (lower) C-Lipschitz at $x_0 \in \operatorname{dom} F$, then it is upper (lower) C-continuous at x_0.

We recall that the set

$$C' = \{\xi \in Y' \mid \ <\xi, y> \geq 0 \quad \text{for all} \quad y \in C\}$$

is called a polar cone of the cone C. For any $\xi \in C'$ we define function $g_\xi, G\xi : D \to R$ by

$$g_\xi(x) = \inf_{y \in F(x)} <\xi, y>,$$

$$G_\xi(x) = \sup_{y \in F(x)} <\xi, y>, \quad x \in D.$$

We have

Theorem 2.4. *Let $F : D \to 2^Y$. Let x_0 be in* $\operatorname{dom} F$ *such that $F(x_0) + C$ $(F(x_0) - C)$ is convex. Then F is locally upper (lower, respectvely) C-Lipschitz at x_0 if and only if the family $\{g_\xi | \xi \in C', \|\xi\| = 1\}$ $(\{G_\xi | \xi \in C', \|\xi\| = 1\})$ is locally lower (upper, respectively) equi-Lipschitz at x_0.*

Proof. We prove the first case. The second case is similar. First, we assume that F is locally upper C-Lipschitz at x_0. It follows that there exist a neighborhood U of x_0 and a constant L such that

$$F(x) \subset F(x_0) + L\|x - x_0\| B_Y + C, \quad \text{for all} \quad x \in U \cap D.$$

Hence,

$$g_\xi(x) = \inf_{y \in F(x)} <\xi, y> \geq \inf_{y \in F(x_0)} <\xi, y> - L\|x - x_0\|$$

and then

$$g_\xi(x_0) - g_\xi(x) \leq L\|x - x_0\|, \quad \text{for all} \quad x \in U \cap D, \xi \in C', \|\xi\| = 1.$$

This shows that the family $\{g_\xi | \xi \in C', \|\xi\| = 1\}$ is locally lower equi-Lipschitz at x_0.

Now, assume that this family is locally lower equi-Lipschitz at x_0, but F is not locally upper C-Lipschitz at x_0. It follows that for any neighborhood U_α of $x_0, \alpha \in I$, where I is an index set, and for any constant $L_\alpha > 0$ one can find $x_\alpha \in U_\alpha \cap D$ with

$$F(x_\alpha) \not\subseteq F(x_0) + L_\alpha \|x_\alpha - x_0\| B_Y + C.$$

Therefore, we can take $y_\alpha \notin F(x_\alpha)$ and

$$y_\alpha \notin F(x_0) + L_\alpha \|x_\alpha - x_0\| B_Y + C.$$

Take $\delta_\alpha \in (0, L_\alpha)$ arbitrary. Since the set $cl(F(x_0) + (L_\alpha - \delta_\alpha)\|x_\alpha - x_0\| B_Y + C)$ is closed and convex, one can find $\xi_\alpha \in Y'$ with unit norm such that

$$\xi_\alpha(y_\alpha) < \gamma_\alpha < \xi_\alpha(F(x_0) + (L_\alpha - \delta_\alpha)\|x_\alpha - x_0\| B_Y + C) \tag{1}$$

for some $\gamma_\alpha \in R$. Assume in contrary that $\xi_{\alpha_0} \notin C'$ for some $\alpha_0 \in I$. Then, there exists $y_0 \in C$ with $\xi_{\alpha_0}(y_0) < 0$.

Since $\beta y_0 \in C$ for all $\beta \geq 0$, we deduce

$$\xi_{\alpha_0}(y_{\alpha_0}) < \gamma_{\alpha_0} < \xi_{\alpha_0}(F(x_0)) + (L_\alpha - \delta_\alpha)\|x_\alpha - x_0\| + \beta \xi_{\alpha_0}(y_0).$$

Letting $\beta \to +\infty$, the right hand side tends to $-\infty$ and we have a contradiction. It follows that $\xi_\alpha \in C'$ for all $\alpha \in I$. Further, it implies from (1) that

$$g_{\xi_\alpha}(x_\alpha) < \gamma_\alpha \leq \inf_{y \in F(x_0)} < \xi, y > -(L_\alpha - \delta_\alpha)\|x_\alpha - x_0\|,$$

for arbitrary $\delta_\alpha > 0$. This implies

$$g_\alpha(x_\alpha) < \gamma_\alpha \leq \inf_{y \in F(x_0)} < \xi_\alpha, y > -L_\alpha\|x_\alpha - x_0\|$$

or

$$g_{\xi_\alpha}(x_0) - g_{\xi_\alpha}(x_\alpha) > L_\alpha\|x_\alpha - x_0\|,$$

which contradicts the fact that the family $\{g_\xi | \xi \in C', \|\xi\| = 1\}$ is locally lower equi-Lipschitz at x_0. This completes the proof the theorem. ■

Analogously, we can prove

Theorem 2.5. *Let $F : D \to 2^Y$. Let x_0 be in* $\mathrm{dom}\, F$ *such that $F(x_0) - C$ ($F(x_0) + C$) is convex. Then F is locally C-Lipschitz ((-C)-Lipschitz) at x_0 if and only if the family $\{G_\xi | \xi \in C', \|\xi\| = 1\}$, ($\{g_\xi | \xi \in C', \|\xi\| = 1\}$, respectively) is locally equi-Lipschitz at x_0.*

We recall the following definition introduced in [8].

Definition 2.6. *We say that a set $A \subset Y$ is C-bounded if for any neighborhood V of the origin in Y there exists a constant $\rho > 0$ such that*

$$A \subset \rho V + C.$$

Remark 3. It is easy to verify that if $\mathrm{int}\, C \neq \emptyset$, then A is C-bounded if and only if there exists an element $c \in \mathrm{int}\, C$ such that $A \subset c - C$ ($A \subset -c + C$ respectively).

We recall that a set $B \subset Y$ generates the cone C and write $C = \mathrm{cone}\,(B)$ if $C = \{tb | b \in B, t \geq 0\}$. If in addition, B does not contain the origin and for each $c \in C, c \neq 0$, there are unique $b \in B, t > 0$ such that $c = tb$, then we say that B is a base.

Proposition 2.7. *Let C have a closed convex bounded base and* $\mathrm{int}\, C \neq \emptyset$. *Then A is a bounded set in the usual sense if and only if A is C-bounded and $-C$-bounded simultaneously.*

Proof. It is clear that if A is bounded, then it is C-bounded and $-C$-bounded. Now, we assume that A is C-bounded and $-C$-bounded simultaneously. By Remark 3, there exist $c_1, c_2 \in \mathrm{int}\, C$ such that

$$A \subset (c_1 - C) \cap (-c_2 + C).$$

Let $B_Y(0, r)$ be a given ball in Y. Using Proposition 1.8 in [8], we conclude that there exists $\delta > 0$ such that

$$(B_Y(0, \delta) + C) \cap (B_Y(0, \delta) - C) \subseteq B_Y(0, r).$$

Choosing $\rho > 0$ such that $\rho c_1, \rho c_2 \in B_Y(0,\delta)$, we obtain

$$(-\rho c_1 + C) \cap (\rho c_2 - C) \subseteq B_Y(0, r),$$

or

$$(-c_1 + C) \cap (c_2 - C) \subseteq B_Y(0, \frac{r}{\rho}).$$

Consequently,

$$A \subset (-c_1 + C) \cap (c_2 - C) \subseteq B_Y\Big(0, \frac{r}{\rho}\Big),$$

which means that A is bounded. This completes the proof of the proposition. ■

Definition 2.8. *Let D be a nonempty convex set in X and C be a cone in Y. A multivalued mapping $F : D \to 2^Y$ is said to be upper (lower) C-convex if*

$$\alpha F(x) + (1-\alpha)F(y) \subset F(\alpha x + (1-\alpha)y) + C$$
$$(F(\alpha x + (1-\alpha)y \subset \alpha F(x) + (1-\alpha)F(y) - C, \quad \textit{respectively})$$

holds for all $x, y \in D$ and $\alpha \in [0,1]$.

We have

Theorem 2.9. *Let X be a finite dimensional space and Y be a Banach space. Let $C \subset Y$ be a closed convex cone with $\operatorname{int} C \neq \emptyset$ and $C' =$ cone (conv $\{\xi_1, ..., \xi_n\}$) for some $\xi_1, ..., \xi_n \in Y'$ and $0 \notin$ conv $\{\xi_1, ..., \xi_n\}$. Let D be a nonempty closed convex subset in X and $F : D \to 2^Y$ be a lower C-convex multivalued mapping with $F(x)$ being C-bounded and $F(x) - C$ convex for all $x \in$ dom F. Then F is locally C-Lipschitz in* ri$D$.

Proof. Without loss of generality we may assume that $0 \in$ ri$D$ and $0 \in F(0)$. Let $x_1, ..., x_n$ be vertices of n-simplex S in D with $0 \in$ ri$S$. Since $F(x_1), ..., F(x_n)$ are upper C-bounded, it follows from Remark 3 that there exists $c_1, ..., c_n \in \operatorname{int} C$ with

$$F(x_i) \subset c_i - C, i = 1, ..., n.$$

Further, take a neighborhood $U \subset S$ of the origin in the space spanned by S. For arbitrary $x \in U, x = \sum_{i=1}^n \lambda_i x_i, \lambda_i > 0, \sum_{i=1}^n \lambda_i = 1$, we have

$$F(x) = F(\sum_{i=1}^n \lambda_i x_i) \subset \sum_{i=1}^n \lambda_i F(x_i) - C \subset \sum_{i=1}^n \lambda_i c_i - C.$$

Therefore, for $\xi \in C'$, we have

$$G_\xi(x) = \sup_{y \in F(x)} < \xi, y > \leq \sum_{i=1}^n \lambda_i < \xi, c_i > \leq \max_{i=1,...,n} < \xi, c_i > = \rho < +\infty. \quad (2)$$

We may also assume $\rho > 1$. Putting $U_0 = \frac{1}{\rho}(U \cap (-U))$ for $x \in U_0, -\rho x \in U$, we obtain

$$\begin{aligned}0 \in F(0) &= F\Big(\frac{1}{1+\frac{1}{\rho}}x + \frac{\frac{1}{\rho}}{1+\frac{1}{\rho}}(-\rho x)\Big)\\ &\subset \frac{1}{1+\frac{1}{\rho}}F(x) + \frac{\frac{1}{\rho}}{1+\frac{1}{\rho}}F(-\rho x) - C\\ &\subset \frac{1}{1+\frac{1}{\rho}}F(x) + \frac{\frac{1}{\rho}}{1+\frac{1}{\rho}}\sum_{i=1}^{n}\lambda_i c_i - C\\ &\subset \frac{1}{1+\frac{1}{\rho}}\Big(F(x) + \frac{1}{\rho}\sum_{i=1}^{n}\lambda_i c_i - C\Big)\end{aligned}$$

and hence

$$0 \in F(x) + \frac{1}{\rho}\sum_{i=1}^{n}\lambda_i c_i - C, \quad \text{for all} \quad x \in U_0.$$

It follows that

$$0 \leq \sup_{y\in F(x)} < \xi, y > + \sum_{i=1}^{n}\frac{\lambda_i}{\rho} < \xi, c_i > \leq G_\xi(x) + 1$$

or

$$G_\xi(x) \geq -1, \quad \text{for all} \quad x \in U_0. \tag{3}$$

A combination of (2) and (3) yields

$$|G_\xi(x)| \leq \rho \quad \text{for all} \quad x \in U_0.$$

Thus, G_ξ is bounded in U_0 uniformly in $\xi \in C'$, $\|\xi\| = 1$. Since F is lower C-convex, then G_ξ is also convex (see the proof in [1, 3]). As it was shown that G_ξ is bounded on U_0. It follows from the well-known result in [3] that G_ξ is locally Lipschitz on U. Further, for C' is a polyhedral pointed cone, $C' = \text{cone}\,(\text{conv}\,\{\xi_1, ..., \xi_n\})$ and $0 \notin \text{conv}\,\{\xi_1, ..., \xi_n\}$ for some $\xi_1, ..., \xi_n \in Y'$. For $i = 1, ..., n$, one can find a neighborhood U_i of origin in X and a constant L_i such that

$$|G_{\xi_i}(x) - G_{\xi_i}(x')| \leq L_i\|x - x'\|$$

holds for all $x, x' \in U_i$. Putting $U_0 = \cap_{i=1}^{n} U_i, L = \max_{i=1,...,n} L_i$, we conclude that the family $\{G_{\xi_i} | i = 1, ...n\}$ is locally equi-Lipschitz on U_0. Now, we claim that the family $\{G_\xi | \xi \in C', \|\xi\| = 1\}$ is also locally equi-Lipschitz on U_0. Indeed, for $\xi \in C', \|\xi\| = 1$ we can write $\xi = \beta\sum_{i=1}^{n}\lambda_i\xi_i$ for some $\beta \geq 0, \lambda_i \geq 0, \sum_{i=1}^{n}\lambda_i = 1$. Set

$$\gamma_0 = \min\big\{\big\|\sum_{i=1}^{n}\lambda_i\xi_i\big\| \,|\lambda_i \geq 0, \sum_{i=1}^{n}\lambda_i = 1\big\}.$$

It is clear that $\gamma_0 > 0$. We have

$$1 = \big\|\beta\sum_{i=1}^{n}\lambda_i\xi_i\| = \beta\|\sum_{i=1}^{n}\lambda_i\xi_i\big\|.$$

and hence

$$\beta = \frac{1}{\|\sum_{i=1}^{n}\lambda_i\xi_i\|} \leq \frac{1}{\gamma_0} = \beta_0.$$

Since

$$G_{\xi_i}(x') - L_i\|x - x'\| \le G_{\xi_i}(x) \le G_{\xi_i}(x') + L_i\|x - x'\| \tag{4}$$

holds for all $i = 1, ..., n,\ x, x' \in U_0$, multiplying both sides of (4) with $\beta\lambda_i$ and taking the sum of them we obtain

$$G_\xi(x') - \beta L\|x - x'\| \le G_\xi(x) \le G_\xi(x') + \beta L\|x - x'\| \tag{5}$$

for all $x, x' \in U_0$. This implies

$$|G_\xi(x) - G_\xi(x')| \le \beta L\|x - x'\| \le \beta_0 L\|x - x'\| \tag{6}$$

for all $\xi \in C',\ \|\xi\| = 1,\ x, x' \in U_0$. Thus, the family $\{G_\xi | \xi \in C',\ \|\xi\| = 1\}$ is locally equi-Lipschitz at the origin in X. Applying Theorem 2.5, we conclude that F is locally C-Lipschitz at the origin. This completes the proof of the theorem. ∎

Corollary 2.10. *Let X, Y, D, C and F be as in Theorem* 2.9. *Then F is upper C-continuous and lower (-C)-continuous in* ri$D$.

Proof. It follows immediately from Theorem 2.9 and Remark 2.

Analogously, we have

Theorem 2.11. *Let X, Y, D and C as in Theorem* 2.9. *Let $F : D \to 2^Y$ be an upper C-convex multivalued mapping with $F(x)$ being C- bounded and $F(x) + C$ convex for all $x \in$ dom$F$. Then F is locally (-C)-Lipschitz in* ri$D$.

Proof. The proof proceeds similarly as the one of Theorem 9 with G_ξ replaced by g_ξ.

Corollary 2.12. *Let X, Y, D, C and F be as in Theorem* 2.11. *Then F is lower $(-C)$-continuous and upper C-continuous in* ri$D$.

Proof. It follows immediately from Theorem 2.11 and Remark 2.

Especially, we have

Corollary 2.13. *Let X, Y, D and C be as in Theorem* 2.9. *In addition, assume that C has a closed convex bounded base and $f : D \to Y$ is a singlevalued C-convex mapping. Then f is locally Lipschitz in* ri$D$.

Proof. Let $x_0 \in$ ri D. From Proposition 1.8 in [8], we deduce that there exists $\delta > 0$ such that

$$(B_Y(0, \delta) + C) \cap (B_Y(0, \delta) - C) \subseteq B_Y(0, 1). \tag{7}$$

By Theorem 2.11, there exist a neighborhood U of x_0 and a constant $L > 0$ such that

$$f(x) \in f(x') + L\|x - x'\| B_Y(0, 1) - C$$

for all $x, x' \in U \cap D$. This implies

$$f(x) - f(x') \in (L\|x - x'\|B_Y(0,1) + C) \cap (L\|x - x'\|B_Y(0,1) - C)$$

and then

$$f(x) - f(x') \in (\frac{L}{\delta}\|x - x'\|B_Y(0,\delta) + C) \cap (\frac{L}{\delta}\|x - x'\|B_Y(0,\delta) - C) \qquad (8)$$

for all $x, x' \in U$. It follows from (7) that

$$\Big(\frac{L}{\delta}\|x - x'\|B_Y(0,\delta) + C\Big) \cap \Big(\frac{L}{\delta}\|x - x'\|B_Y(0,\delta) - C\Big) \subseteq \frac{L}{\delta}\|x - x'\|B_Y(0,1).$$

Together with (8) we obtain

$$f(x) - f(x') \in \frac{L}{\delta}\|x - x'\|B_Y(0,1)$$

for all $x, x' \in U \cap D$. This shows that f is locally Lipschitz on D and the proof is complete. ■

3. C-Approximation of Vector Multivalued Mappings

It is well-known that a differentiable function can be approximated by its derivative in a neighborhood of any point belonging to its definition domain. Convex and locally Lipschitz functions can be approximated by the set of subdifferentials defined by different authors as Clarke [5], Michel-Penot [11], Ioffe-Morduchovich [14] and Treiman [17]. In [7] Jeyakumar and Luc introduced the notion of a convexificator of nonsmooth functions. From the point of view of optimization and applications, the descriptions of the optimality conditions and calculus rules in terms of derivatives, subdifferentials, convexificator provide sharp results. In this section, we introduce the notion of C-approximations of vector multivalued mappings and show some applications to vector optimization problems.

Let X, D, Y and C be as in Sec. 2. We consider a multivalued mapping $F : D \to 2^Y$. The graph of F is defined by the set

$$\mathrm{Gr}\, F = \{(x, y) \in D \times Y | y \in F(x)\}.$$

Definition 3.1. *The mapping F is said to be upper (lower) C-approximated at $(\overline{x}, \overline{y}) \in$ $Gr F$ if there are a positive homogeneous multivalued mapping $\mathcal{D}_{(\overline{x},\overline{y})}$: $X \to 2^Y$ and a neighborhood U of $\overline{x}$ in X such that*

$$F(x) \subset \overline{y} + \mathcal{D}_{(\overline{x},\overline{y})}(x - \overline{x}) + C$$
$$(F(x) \subset \overline{y} + \mathcal{D}_{(\overline{x},\overline{y})}(x - \overline{x}) - C, \quad respectively)$$

holds for all $x \in U \cap D$.

The multivalued mapping $\mathcal{D}_{(\overline{x},\overline{y})}$ as above is called an upper (a lower) C-approximation of F at $(\overline{x}, \overline{y})$.

Definition 3.2.
1) *Let* $(A_n)_{n\in N}$ *be a sequence of subsets of a metric space* (E,d). *We say that the subset*

$$\limsup_{n\to+\infty} A_n = \{x\in E|\lim_{n\to+\infty}\inf d(x,A_n)=0\}$$

is the upper limit of the sequence (A_n) *and the subset*

$$\liminf_{n\to+\infty} A_n = \{x\in E|\lim_{n\to+\infty} d(x,A_n)=0\}$$

is its lower limit.
2) *A subset* A *is said to be the limit set of the sequence* (A_n) *and denoted by* $A=\lim_{n\to+\infty} A_n$ *if*

$$A=\liminf_{n\to+\infty} A_n = \limsup_{n\to+\infty} A_n.$$

Upper and lower limit are obviously closed. We also see that

$$\liminf_{n\to+\infty} A_n \subset \limsup_{n\to+\infty} A_n$$

and the upper limit and lower limit of subsets A_n *and that of their closures* $\overline{A}_n$ *do coincide, since* $d(x,A_n)=d(x,\overline{A}_n)$.

Definition 3.3. *Let* $K\subset Y$ *be a subset of a normed space* Y *and* $\overline{x}\in K$. *The upper (lower) contingent cone* $T^u_K(\overline{x})$ $(T^l_K(\overline{x}))$ *of the a set* K *at the point* $\overline{x}$ *is defined by*

$$T^u_K(\overline{x}) = \{x\in Y|\exists t_n>0, t_n\to 0, n\to+\infty, x\in \limsup_{n\to+\infty}\frac{K-\overline{x}}{t_n}\},$$
$$(T^l_K(\overline{x}) = \{x\in Y|\forall t_n>0, t_n\to 0, n\to+\infty, x\in \liminf_{n\to+\infty}\frac{K-\overline{x}}{t_n}\}).$$

It is clear that $T^i_K(\overline{x}), i=u,l$ *are always closed cones of tangent directions which are convex when* K *is convex.*

Definition 3.4. *The upper (lower) radial cone* $R^u_K(\overline{x})$ $(R^l_K(\overline{x}))$ *of* K *at* $\overline{x}$ *is defined by*

$$R^u_K(\overline{x}) = \Big\{x\in Y|\exists t_n\in(0,+\infty), x\in \limsup_{n\to+\infty}\frac{K-\overline{x}}{t_n}\Big\},$$
$$\Big(R^l_K(\overline{x}) = \Big\{x\in Y|\forall t_n\in(0,+\infty), x\in \liminf_{n\to+\infty}\frac{K-\overline{x}}{t_n}\Big\}\Big).$$

Remark 1. One can easily verify that

$$R^u_K(\overline{x}) = \overline{\text{cone}}\,(K-\bar{x}),$$

and

$$R^l_K(\overline{x}) = \{u:\bar{x}+tu\in K,\quad \forall t\ge 0\}.$$

Remark 2. The following conclusions hold

1/ $R^l_K(\overline{x}) \subset T^l_K(\overline{x}) \subset T^u_K(\overline{x}) \subset R^u_K(\overline{x})$;

2/ $T^u_K(\overline{x}) = R^u_K(\overline{x})$, whenever K is a convex set.

Definition 3.5. *Let $F : D \to 2^Y$ and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. The upper (lower) contingent derivative $\mathcal{D}^u F(\overline{x}, \overline{y})$ $(\mathcal{D}^l F(\overline{x}, \overline{y}))$ of F at $(\overline{x}, \overline{y})$ is a multivalued mapping from X into Y defined by*

$$y \in \mathcal{D}^u F(\overline{x}, \overline{y})(x) \quad \text{if and only if} \quad (x, y) \in T^u_{GrF}(\overline{x}, \overline{y})$$
$$(y \in \mathcal{D}^l F(\overline{x}, \overline{y})(x) \quad \text{if and only if} \quad (x, y) \in T^l_{GrF}(\overline{x}, \overline{y}), \quad \text{respectively}).$$

These are equivalent to: There exists (for any) a sequence $\{t_n\},\ t_n \to 0^+$ for which one can find sequences $\{x_n\} \subset D,\ \ \{y_n\} \subset Y$ such that

$$\overline{y} + t_n y_n \in F(\overline{x} + t_n x_n) \quad \text{for all} \quad n$$

and $\{x_n\}, \{y_n\}$ have convergent subsequences

$$x_{n_j} \to x, y_{n_j} \to y.$$

Analogously, we define the upper (lower) radial derivative $R^u F(\overline{x}, \overline{y})$ $(R^l F$ $(\overline{x}, \overline{y}))$ of F at $(\overline{x}, \overline{y}) : y \in R^u F(\overline{x}, \overline{y})(x)$ $(y \in R^l F(\overline{x}, \overline{y})(x))$ if and only if $(x, y) \in R^u_{GrF}(\overline{x}, \overline{y})$ $((x, y) \in R^l_{GrF}(\overline{x}, \overline{y})$, respectively).

Remark 3. Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. The following conclusions hold:
i) The multivalued mappings $\mathcal{D}^i F(\overline{x}, \overline{y}), R^i F(\overline{x}, \overline{y}), i = u, l$, are positively homogeneous with closed graphs.
ii) $\operatorname{Gr} R^l F(\overline{x}, \overline{y}) \subset \operatorname{Gr} \mathcal{D}^l F(\overline{x}, \overline{y}) \subset \operatorname{Gr} \mathcal{D}^u F(\overline{x}, \overline{y}) \subset \operatorname{Gr} R^u F(\overline{x}, \overline{y})$.
iii) $\operatorname{Gr} \mathcal{D}^u F(\overline{x}, \overline{y}) = \operatorname{Gr} R^u F(\overline{x}, \overline{y})$, whenever $\operatorname{Gr} F$ is convex in $X \times Y$.

Theorem 3.6. *Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then we have*

$$F(x) \subset \overline{y} + R^u F(\overline{x}, \overline{y})(x - \overline{x}) \tag{9}$$

for all $x \in D$. This implies that $R^u F(\overline{x}, \overline{y})$ is an upper $\{0\}$-approximation of F at $(\overline{x}, \overline{y})$.

Proof. It follows immediately from the definition of $R^u F(\bar{x}, \bar{y})$ and Remark 2. ∎

Theorem 3.7. *If $\dim Y < +\infty$ and $F : D \to 2^Y$ is a locally Lipschitz mapping, then for any $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $\mathcal{D}^l F(\overline{x}, \overline{y})(x) \neq \emptyset$ for all $x \in D$.*

Proof. Let $x \in D$. Take arbitrary $t_n \to 0^+$ and $\{x_n\} \subset D, x_n \to x$. Without loss of generality, we may assume that there exist a neighborhood U_1 of $\overline{x}$ and a constant $\gamma > 0$ such that $\overline{x} + t_n x_n \in U_1$ and $\|x_n\| \leq \gamma$ for all $n \in N$. Since F is locally Lipschitz at $\overline{x}$ there exist a neighborhood U_2 of $\overline{x}$ and a constant $L > 0$ such that

$$F(x') \subset F(x'') + L\|x' - x''\| B_Y$$

holds for all $x', x'' \in U_2$. We may also assume that $U_1 = U_2$.
Since

$$\overline{y} \in F(\overline{x}) \subset F(\overline{x} + t_n x_n) + Lt_n \|x_n\| B_Y,$$

we have

$$\overline{y} = z_n + Lt_n \|x_n\| u_n$$

with $z_n \in F(\overline{x} + t_n x_n)$ and $u_n \in B_Y, n = 1, 2, ...$ This implies

$$z_n = \overline{y} - Lt_n \|x_n\| u_n.$$

For B_Y is compact we deduce that $\{y_n\}$ with $y_n = -L\|x_n\| u_n$ has a convergent subsequence $y_{n_j} \to y$ for some $y \in Y$.
Therefore

$$\overline{y} + t_n y_n = z_n \in F(\overline{x} + t_n x_n).$$

This shows $y \in \mathcal{D}^l(\overline{x}, \overline{y})(x)$ and hence $\mathcal{D}^l F(\overline{x}, \overline{y})(x) \neq \emptyset$ for all $x \in D$. This completes the proof of the theorem. ∎

Further, for given $F : D \to 2^Y$ we define the multivalued mapping $F + C : D \to 2^Y$ by

$$(F + C)(x) = F(x) + C.$$

We have

Proposition 3.8. *Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. *Then*

$$\mathcal{D}^i F(\overline{x}, \overline{y})(x) + C \subset \mathcal{D}^i (F + C)(\overline{x}, \overline{y})(x) \tag{10}$$

and

$$R^i F(\overline{x}, \overline{y})(x) + C \subset R^i (F + C)(\overline{x}, \overline{y})(x)$$

for all $x \in X$, *where* $i = u$ *or* $i = l$.

Proof. We prove (10) for $i = u$. Let $y \in \mathcal{D}^u F(\overline{x}, \overline{y})(x)$ and $c \in C$. By the definition there exist sequences $\{t_n\}, t_n \to 0^+, \{x_n\} \subset X, \{y_n\} \subset Y$ such that

$$\overline{y} + t_n y_n \in F(\overline{x} + t_n x_n) \quad \text{for all} \quad n \in N$$

and $\{x_n\}, \{y_n\}$ have convergent subsequences $x_{n_j} \to x, y_{n_j} \to y$. This implies

$$\overline{y} + t_n(y_n + c) \in F(\overline{x} + t_n x_n) + C, \quad \text{for all} \quad n \in N.$$

Since $y_{n_j} + c \to y + c$ we conclude that

$$y + c \in \mathcal{D}^u (F + C)(\overline{x}, \overline{y})(x).$$

The proofs of the other conclusions proceed similarly.

Proposition 3.9. *Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ *and let the multivalued mapping* F *be upper* C*-convex. Then*

$$F(x) \subset \overline{y} + \mathcal{D}^u (F + C)(\overline{x}, \overline{y})(x - \overline{x}) \quad \textit{for all} \quad x \in D.$$

Proof. Since F is upper C-convex, then $\mathrm{Gr}(F+C)$ is convex in $X\times Y$. Therefore, it follows from Remark 1 that

$$\mathcal{D}^u(F+C)(\overline{x},\overline{y})(x)=R^u(F+C)(\overline{x},\overline{y})(x),\quad \text{for all}\quad x\in D$$

Applying Theorem 3.6 to $F+C$ we obtain

$$F(x)\subset\overline{y}+R^u(F+C)(\overline{x},\overline{y})(x-\overline{x})=\overline{y}+\mathcal{D}^u(F+C)(\overline{x},\overline{y})(x-\overline{x}),$$

for all $x\in D$. This completes the proof of the proposition. ∎

Definition 3.10. *Let $(\overline{x},\overline{y})\in\ Gr F$. The upper S-derivative $S^uF(\overline{x},\overline{y})$ is the multivalued mapping from X into Y defined by:*

$$y\in S^uF(\overline{x},\overline{y})(x)$$

if and ony if there exist sequences $\{t_n\}\subset(0,+\infty),\{x_n\}\subset X,\{y_n\}\subset Y$ such that $\overline{y}+t_ny_n\in F(\overline{x}+t_nx_n)$ for all $n\in\mathbb{N}$ and $\{x_n\},\{y_n\}$ have convergent subsequences $x_{n_j}\to x,y_{n_j}\to y$ and $t_{n_j}x_{n_j}\to 0$.

Remark 4 . It is also easy to see that:
i) The multivalued mapping $S^iF(\overline{x},\overline{y}),i=u$ or $i=l$, are positively homogeneous with closed graphs.
ii) $\mathrm{Gr}\,\mathcal{D}^uF(\overline{x},\overline{y})\subset\ \mathrm{Gr}\,S^uF(\overline{x},\overline{y})\subset\ \mathrm{Gr}\,R^uF(\overline{x},\overline{y})$.

Lemma 3.11. *Let $(\overline{x},\overline{y})\in\ \mathrm{Gr}\,F$. If*

$$S^iF(\overline{x},\overline{y})(0)\cap(-C)=\{0\}$$

and C has a compact base, then

$$\mathcal{D}^i(F+C)(\overline{x},\overline{y})(x)=\mathcal{D}^iF(\overline{x},\overline{y})(x)+C$$

for all $x\in X,i=u$ or $i=l$.

Proof. Using Proposition 3.8, we need only show that

$$\mathcal{D}^u(F+C)(\overline{x},\overline{y})(x)\subset\mathcal{D}^uF(\overline{x},\overline{y})(x)+C$$

for all $x\in X$.

Now, let $x\in X$ be fixed and $y\in\mathcal{D}^u(F+C)(\overline{x},\overline{y})(x)$. Due to the definition, there exist $\{t_n\}\subset R,t_n\to 0^+,\{x_n\}\subset X,\{y_n\}\subset Y$ such that

$$\overline{y}+t_ny_n\in F(\overline{x}+t_nx_n)+C$$

and $\{x_n\},\{y_n\}$ have convergent subsequences $x_{n_j}\to x,y_{n_j}\to y$. We have

$$\overline{y}+t_ny_n=z_n+c_n$$

with

$$z_n\in F(\overline{x}+t_nx_n),c_n\in C.$$

Assume that B is a compact base of C. Then, $c_n=\gamma_nb_n$ with $\gamma_n>0,b_n\in B$. Therefore, it follows that

$$\overline{y} + t_n(y_n - \frac{\gamma_n}{t_n} b_n) = z_n \in F(\overline{x} + t_n x_n). \tag{11}$$

We consider the following cases:
1) There exists an infinite number of j_i such that

$$\left|\frac{\gamma_{nj_i}}{t_{n_{j_i}}}\right| < +\infty \quad \text{for all} \quad i = 1, 2, ...$$

Since B is compact, without loss of generality, we may assume that

$$\frac{\gamma_{nj_i}}{t_{n_{j_i}}} b_{n_{j_i}} \to c \in C,$$

and then

$$y_{n_{j_i}} - \frac{\gamma_{n_{j_i}}}{t_{n_{j_i}}} b_{n_{j_i}} \to y - c.$$

Together with (11) we conclude

$$y - c \in \mathcal{D}^u F(\overline{x}, \overline{y})(x)$$

and so $y \in \mathcal{D}^u F(\overline{x}, \overline{y})(x) + C$.
2) $\frac{\gamma_{n_j}}{t_{n_j}} \to +\infty$ as $j \to +\infty$. It follows from (11) that

$$\overline{y} + \gamma_n \Big(\frac{t_n y_n}{\gamma_n} - b_n\Big) = z_n \in F\Big(\overline{x} + \gamma_n \big(\frac{t_n}{\gamma_n} x_n\big)\Big). \tag{12}$$

We have

$$\frac{t_{n_j} y_{n_j}}{\gamma_{n_j}} \to 0, b_{n_j} \to b \in B$$

for some b, and

$$\frac{t_{n_j}}{\gamma_{n_j}} x_{n_j} \to 0.$$

Together with (12) we conclude $-b \in S^u F(\overline{x}, \overline{y})(0)$, and therefore $-b \in S^u F(\overline{x}, \overline{y})$ $(0) \cap (-C) = \{0\}$. It contradicts the fact $0 \notin B$. This means that it only gives the case 1) and we have the proof of the lemma.

Corollary 3.12. *Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. *If* F *is upper* C*-convex,* $S^u F(\overline{x}, \overline{y})(0) \cap (-C) = \{0\}$ *and* C *has a compact base* B*, then* $\mathcal{D}^u F(\overline{x}, \overline{y})$ *is an upper* C*-approximation of* F *at* $(\overline{x}, \overline{y})$ *i.e.*

$$F(x) \subset \overline{y} + \mathcal{D}^u F(\overline{x}, \overline{y})(x - \overline{x}) + C$$

for all $x \in D$.

Proof. It follows from Proposition 3.9 that

$$F(x) \subset \overline{y} + \mathcal{D}^u (F + C)(\overline{x}, \overline{y})(x - \overline{x})$$

for all $x \in D$. Further, using Lemma 3.11 we conclude the proof of the corollary. ∎

Definition 3.13. *A mapping* $F : D \to 2^Y$ *is said to be compactly approximable at* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ *if for each* $\overline{v} \in X$ *there exist a multivalued mapping* $R : D \to$

$2^Y, R(v) \neq \emptyset$ and compact for all $v \in D$, a neighborhood U of $\overline{x}$ in X and a function $r : (0,1] \times D \to (0,+\infty)$ satisfying

i) $\lim_{(t,v)\to(0^+,\overline{v})} r(t,v) = 0$,

ii) *for each $v \in U$ and $t \in (0,1]$, $\overline{x} + tv \in D$ we have*

$$F(\overline{x} + tv) \subset \overline{y} + t(R(\overline{v}) + r(t,v)B_Y).$$

Theorem 3.14. *Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. If F is compactly approximable at $(\overline{x}, \overline{y})$, then*

$$\mathcal{D}^i(F + C)(\overline{x}, \overline{y})(x) = \mathcal{D}^i F(\overline{x}, \overline{y})(x) + C$$

for all $x \in X, i = u$, or $i = l$.

Proof. Obviously $\mathcal{D}^u F(\overline{x}, \overline{y})(x) + C \subset \mathcal{D}^u(F + C)(\overline{x}, \overline{y})(x)$, so it suffices to prove the reverse inclusion. Let $x \in D$ and $y \in \mathcal{D}^u(F + C)(\overline{x}, \overline{y})(x)$. It follows that there exist sequences $\{t_n\} \subset (0, +\infty), t_n \to 0, \{x_n\} \subset X, \{y_n\} \subset Y$ such that

$$\overline{y} + t_n y_n \in F(\overline{x} + t_n x_n) + C$$

and $\{x_n\}, \{y_n\}$ have convergent subsequences $x_{n_j} \to x, y_{n_j} \to y$. Therefore

$$\overline{y} + t_n x_n = z_n + c_n$$

with

$$z_n \in F(\overline{x} + t_n x_n), c_n \in C.$$

Since F is compactly approximable at $(\overline{x}, \overline{y})$, there exists $\{k_n\} \subset R(x), r_n = r(t_n, x_n), b_n \in B_Y$ such that $r_n \to 0$ and

$$z_n = \overline{y} + t_n(k_n + r_n b_n)$$

for all $n \in N$.

Due to the compactness of $R(x)$, there exists a subsequence

$$\{k_{n_{j_i}}\} \subset R(x), k_{n_{j_i}} \to k \in R(x).$$

Thus, we have

$$k_{n_{j_i}} + r_{n_{j_i}} b_{n_{j_i}} \to k$$

and hence $k \in \mathcal{D}^u F(\overline{x}, \overline{y})(x)$. Since

$$\overline{y} + t_{n_{j_i}} y_{n_{j_i}} = \overline{y} + t_{n_{j_i}}(k_{n_{j_i}} + r_{n_{j_i}} b_{n_{j_i}}) + c_{n_{j_i}},$$

it follows that

$$y_{n_{j_i}} - k_{n_{j_i}} - r_{n_{j_i}} b_{n_{j_i}} = t_{n_{j_i}}^{-1} c_{n_{j_i}} \in C.$$

Taking the limit $i \to +\infty$ we obtain $y - k \in C$ and then

$$y \in k + C \subset \mathcal{D}^u F(\overline{x}, \overline{y})(x) + C.$$

The proof for $\mathcal{D}^l(F + C)$ proceeds similarly. This completes the proof of the theorem. ∎

Corollary 3.15. *Let $(\overline{x}, \overline{y}) \in GrF$ and let F be upper C-convex and compactly approximable at $(\overline{x}, \overline{y})$. Then $\mathcal{D}^u F(\overline{x}, \overline{y})$ is an upper C-approximation of F at $(\overline{x}, \overline{y})$.*

Proof. Since F is upper C-convex, then it follows from Proposition 3.5 that

$$R^u(F+C)(\overline{x},\overline{y})(x-\overline{x}) = \mathcal{D}^u(F+C)(\overline{x},\overline{y})(x-\overline{x}) \tag{13}$$

for all $x \in X$.

It follows from Theorem 3.6 and Proposition 3.8 and (13) that

$$\begin{aligned} F(x) &\subset \overline{y} + R^u F(\overline{x},\overline{y})(x-\overline{x}) \\ &\subset \overline{y} + R^u F(\overline{x},\overline{y})(x-\overline{x}) + C \\ &\subset \overline{y} + R^u (F+C)(\overline{x},\overline{y})(x-\overline{x}) \\ &\subset \overline{y} + \mathcal{D}^u (F+C)(\overline{x},\overline{y})(x-\overline{x}) \end{aligned} \tag{14}$$

for all $x \in D$.

Since F is compactly approximable at $(\overline{x},\overline{y})$, it follows from Theorem 3.14 that

$$\mathcal{D}^u(F+C)(\overline{x},\overline{y})(x-\overline{x}) = \mathcal{D}^u F(\overline{x},\overline{y})(x-\overline{x}) + C$$

for all $x \in D$. This completes the proof of the corollary. ■

Further, we say that $x \leq y$ if $y - x \in C$. We have

Theorem 3.16. *Let* $\dim Y < +\infty$*. Let* $C \subset Y$ *be a cone such that any set* $[0,c] = \{y \in C \mid 0 \leq y \leq c\}$ *is a compact set for any* $c \in C$*. If* $f : D \to Y$ *is a singlevalued locally upper* C*-Lipschitz at* $\overline{x} \in \operatorname{dom} f$*, then*

$$\mathcal{D}^i(f+C)(\overline{x},\overline{y})(x) = \mathcal{D}^i f(\overline{x},\overline{y})(x) + C,$$

for all $x \in D$ *where* $\overline{y} = f(\overline{x}), i = u, l$*.*

Proof. First, we prove for the case $i = u$. It suffcies to show the inclusion $\mathcal{D}^u(f+C)(\overline{x},\overline{y})(x) \subset \mathcal{D}^u f(\overline{x},\overline{y})(x) + C, x \in X$. Indeed, let $y \in \mathcal{D}^u(f+C)(\overline{x},\overline{y})(x)$. It follows that there exist $t_n \to 0^+, x_n \in D, y_n \in Y$ such that

$$\overline{y} + t_n y_n \in f(\overline{x} + t_n x_n) + C$$

and $\{x_n\}, \{y_n\}$ have convergent subsequences $x_{n_j} \to x, y_{n_j} \to y$. We can write

$$\overline{y} + t_{n_j} y_{n_j} = f(\overline{x} + t_{n_j} x_{n_j}) + c^1_{n_j}, \quad \text{for} \quad c^1_{n_j} \in C.$$

Since f is locally upper C-Lipschitz at $\overline{x}$, there exist a neighborhood U of $\overline{x}$ and a constant $L > 0$ such that

$$f(\overline{x} + t_{n_j} x_{n_j}) = f(\overline{x}) + L t_{n_j} \|x_{n_j}\| v_{n_j} + c^2_{n_j},$$

for $v_{n_j} \in B_Y, c^2_{n_j} \in C$ and $\overline{x} + t_{n_j} x_{n_j} \in U \cap D$. Consequently,

$$\overline{y} + t_{n_j} y_{n_j} = \overline{y} + L t_{n_j} \|x_{n_j}\| v_{n_j} + c^1_{n_j} + c^2_{n_j}$$

and then

$$y_{n_j} - L\|x_{n_j}\| v_{n_j} = \frac{c^1_{n_j} + c^2_{n_j}}{t_{n_j}}.$$

Since $y_{n_j} \to y$ and B_Y is compact, without loss of generality, we may assume that $L\|x_{n_j}\| v_{n_j} \to u$ and

$$y_{n_j} - L\|x_{n_j}\|v_{n_j} = \frac{c^1_{n_j} + c^2_{n_j}}{t_{n_j}} \to c \in C, \quad as \quad j \to +\infty.$$

Therefore, for

$$0 \le \frac{c^i_{n_j}}{t_{n_j}} \le \frac{c^1_{n_j} + c^2_{n_j}}{t_{n_j}}, i = 1, 2,$$

we conclude that

$$\frac{c^i_{n_j}}{t_{n_j}} \to c^i \in C, i = 1, 2$$

and $c = c^1 + c^2$. It follows that

$$y_{n_j} = L\|x_{n_j}\|v_{n_j} + \frac{c^1_{n_j} + c^2_{n_j}}{t_{n_j}} \to u + c^1 + c^2.$$

Setting

$$z_{n_j} = L\|x_{n_j}\|v_{n_j} + \frac{c^2_{n_j}}{t_{n_j}}$$

we obtain

$$\overline{y} + t_{n_j} z_{n_j} = \overline{y} + t_{n_j} L\|x_{n_j}\|v_{n_j} + c^2_{n_j} = f(\overline{x} + t_{n_j} x_{n_j}).$$

Since $z_{n_j} \to u + c^2, x_{n_j} \to x$, we conclude $u + c^2 \in \mathcal{D}^u f(\overline{x}, \overline{y})(x)$ and then $y = u + c^1 + c^2 \in \mathcal{D}^u f(\overline{x}, \overline{y})(x) + C$. Thus, we have

$$\mathcal{D}^u(f + C)(\overline{x}, \overline{y})(x) \subset \mathcal{D}^u f(\overline{x}, \overline{y})(x) + C,$$

for all $x \in X$. The proof for $i = l$ proceeds similarly. This completes the proof the theorem.

Corollary 3.17. *Let Y, C be as in Theorem 3.16 and $\dim X < +\infty$. In addition, assume that C is a closed convex cone with $\operatorname{int} C \neq \emptyset$ and $C' = \operatorname{cone}(\operatorname{conv}\{\xi_1, ..., \xi_n\})$ for some $\xi_1, ..., \xi_n \in Y'$ and $0 \notin \operatorname{conv}\{\xi_1, ..., \xi_n\}$. Let $f : D \to Y$ be a C-convex singlevalued mapping. Then*

$$f(x) \in \overline{y} + \mathcal{D}^u f(\overline{x}, \overline{y})(x - \overline{x}) + C$$

holds for all $x \in D$.

Proof. It follows from Corollary 2.13 that f is Lipschitz in $\operatorname{ri} D$. Applying Theorem 3.16 we deduce that

$$\mathcal{D}^u(f + C)(\overline{x}, \overline{y})(x) \subset \mathcal{D}^u f(\overline{x}, \overline{y})(x) + C$$

for all $x \in X$. Since f is C-convex, by Proposition 3.9 we have

$$f(x) \in \overline{y} + \mathcal{D}^u(f + C)(\overline{x}, \overline{y})(x - \overline{x})$$

for all $x \in D$. Consequently, for any $x \in D$ we conclude that

$$f(x) \in \overline{y} + \mathcal{D}^u f(\overline{x}, \overline{y})(x - \overline{x}) + C.$$

This completes the proof of the corollary. ∎

4. Vector Optimization Problems

Let B be a nonempty subset of a topological linear locally convex space with a pointed cone C. A point $\overline{y} \in B$ is said to be a minimizer (a weak minimizer, respectively) of B if

$$(B - \overline{y}) \cap (-C) = \{0\}$$
$$\big((B - \overline{y}) \cap (-intC) = \emptyset, \quad \text{respectively}\big).$$

It is clear that $\overline{y}$ is a minimizer (a weak minimizer) of B if and only if there is no $y \in B, y \neq \overline{y}$ with $\overline{y} \in y + C \quad (\overline{y} \in y + \operatorname{int} C$, respectively). We denote by $\operatorname{Min}(B)(\operatorname{W} \operatorname{Min}(B))$ the set of all (weak, respectively) minimizers of B.

We consider the problem

$$\begin{cases} \text{minimize } \; F(x) \\ \text{subject to } \;\; x \in D. \end{cases} \tag{P}$$

A point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is said to be a local (a weak local, respectively) minimizer of (P), if there exists a neighborhood V of $\overline{x}$ such that $\overline{y} \in \operatorname{Min}(F(V \cap D))$ $(\overline{y} \in W \operatorname{Min}(F(V \cap D))$, respectively).

This means that for all $x \in V \cap D$

$$F(x) \subset \overline{y} + (Y \setminus (-C)) \cup \{0\}$$
$$(F(x) \subset \overline{y} + Y \setminus (-\operatorname{int} C), \quad \text{respectively}).$$

If $V = X$, then we call that $(\overline{x}, \overline{y})$ is a global minimizer (a weak global minimizer). We have

Lemma 4.1. *Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr}(F)$ *with* $\overline{x} \in D$. *If* F *is upper* C*-convex multivalued mapping, then* $(\overline{x}, \overline{y})$ *is a local (weak) minimizer of* (P) *if and only if is a global (weak) minimizer of* (P).

Proof. It is obvious that any global weak minimizer of (P) is a local weak minimizer of (P). Let us prove the converse implication. Suppose that $(\overline{x}, \overline{y})$ is a local weak minimizer of (P). Then there exists a neighborhood V of $\overline{x}$ such that

$$F(x) \subset \overline{y} + Y \setminus (-\operatorname{int} C) \quad \text{for all} \quad x \in V \cap D. \tag{15}$$

Assume that $(\overline{x}, \overline{y})$ is not a global weak minimizer of (P). Then there is some $x_0 \in D, y_0 \in F(x_0)$ such that

$$\overline{y} - y_0 \in \operatorname{int} C. \tag{16}$$

We set $x_\lambda = \lambda x_0 + (1 - \lambda \overline{x})$. Using the upper C-convexity of F we conclude

$$(1-\lambda)\overline{y} + \lambda y_0 \in (1-\lambda) F(\overline{x}) + \lambda F(x_0) \subset F(x_\lambda) + C.$$

For sufficiently small positive $\lambda, \quad x_\lambda \in V$, we have by (15)

$$\lambda(y_0 - \overline{y}) \subset F(x_\lambda) - \overline{y} + C \subset Y \setminus (-\operatorname{int} C) + C \subset Y \setminus (-\operatorname{int} C).$$

It follows that

$$y_0 - \overline{y} \in Y \setminus (-\operatorname{int} C)$$

which contradicts (16) and hence $(\overline{x}, \overline{y})$ is a global weak minimizer of (P). The proof for local minimizer proceeds similarly. This completes the proof of the lemma.

Proposition 4.2. *Let C be a convex cone and let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Assume that $\mathcal{D}$ is an upper C-approximation of F at $(\overline{x}, \overline{y})$ with*

$$\mathcal{D}(x - \overline{x}) \cap (-C) \subset \{0\} \quad (\mathcal{D}(x - \overline{x}) \cap (-\operatorname{int} C) = \emptyset)$$

for all $x \in D$. Then $(\overline{x}, \overline{y})$ is a (weak, respectively) minimizer of (P).

Proof. Since

$$F(x) - \overline{y} \subset \mathcal{D}(x - \overline{x}) + C, \quad \text{for all} \quad x \in D.$$

We conclude

$$(F(x) - \overline{y}) \cap (-C) \subset (\mathcal{D}(x - \overline{x}) + C) \cap (-C).$$

Suppose that $z \in (\mathcal{D}(x - \overline{x}) + C) \cap (-C)$, $z = z_1 + z_2$ with $z_1 \in \mathcal{D}(x - \overline{x}), z_2 \in C$ and $z \in -C$. We have $z_1 = z - z_2 \in -C$. Hence, $z_1 \in \mathcal{D}(x - \overline{x}) \cap (-C) \subset \{0\}$ and hence $(F(x) - \overline{y}) \cap (-C) \subset \{0\}$ for all $x \in D$. This implies

$$F(x) \subset \overline{y} + (Y \setminus (-C)) \cup \{0\}$$

for all $x \in D$. This shows that $(\overline{x}, \overline{y})$ is a minimizer of (P). The proof of weak minimizers proceeds similarly. This completes the proof of the proposition. ∎

Corollary 4.3. *Let C, F, D and $(\overline{x}, \overline{y})$ be as in Proposition* 4.2. *If $\mathcal{D}(x - \overline{x}) \subset C$ $(\mathcal{D}(x - \overline{x}) \subset \operatorname{int} C$ respectively) for all $x \in D$, then $(\overline{x}, \overline{y})$ is a (weak, respectively) minimizer of (P).*

Proof. It follows immediately from the above proposition. ∎

Corollary 4.4. *Let C be a convex cone and f be a singlevalued convex function. If $0 \in \partial f(\overline{x})$, then $(\overline{x}, f(\overline{x}))$ is a minimizer of (P).*

Proof. If $0 \in \partial f(\overline{x})$, then $\mathcal{D}(x) \equiv 0, x \in D$ is a C-approximation of f at $(\overline{x}, f(\overline{x}))$ and $\mathcal{D}(x - \overline{x}) \subset C$ for all $x \in D$. Therefore, applying Corollary 4.3 we obtain the proof of this corollary. ∎

Proposition 4.5. *Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and let F be upper C-convex and compactly approximable at $(\overline{x}, \overline{y})$ with C convex. In addition, assume that*

$$\mathcal{D}^u F(\overline{x}, \overline{y})(x - \overline{x}) \subset Y \setminus (-C \setminus \{0\})$$
$$(\mathcal{D}^u F(\overline{x}, \overline{y})(x - \overline{x}) \subset Y \setminus (-\operatorname{int} C), \quad for\ all \quad x \in D.$$

Then $(\overline{x}, \overline{y})$ is a (weak) minimizer of (P).

Proof. Using Corollary 3.15 we conclude that $\mathcal{D}^u F(\overline{x}, \overline{y})$ is an upper C-approximation of F at $(\overline{x}, \overline{y})$. This implies

$$F(x) \subset \overline{y} + \mathcal{D}^u F(\overline{x},\overline{y})(x-\overline{x}) + C \subset \overline{y} + (Y \setminus (-C \setminus \{0\})) + C. \tag{17}$$

We show that

$$(Y \setminus (-C \setminus \{0\})) + C \subset Y \setminus (-C \setminus \{0\}). \tag{18}$$

Indeed, let $y \in (Y \setminus (-C \setminus \{0\})) + C$. We can write $y = y_1 + y_2$ with $y_1 \in Y \setminus (-C \setminus \{0\}) + C$ and $y_2 \in C$. Assume that $y = y_1 + y_2 \in -C \setminus \{0\}$, then $y_1 = y - y_2 \in -C \setminus \{0\} - C = -C \setminus \{0\}$ and we have a contradiction. Therefore, we conclude $y \notin -C \setminus \{0\}$ and hence $y \in Y \setminus (-C \setminus \{0\})$.

A combination of (17) and (18) yields

$$(F(x) - \overline{y}) \cap (-C) = \{0\} \quad \text{for all} \quad x \in D.$$

This means that $(\overline{x},\overline{y})$ is a minimizer of (P). The proof of the second part proceeds similarly.

This completes the proof of the proposition. ∎

Proposition 4.6. *Let* $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. *If* $(\overline{x},\overline{y})$ *is a local weak minimizer of* (P), *then for all* $v \in T_D^u(\overline{x})$ *we have*

$$\mathcal{D}^u F(\overline{x},\overline{y})(v) \subset Y \setminus (-\operatorname{int} C).$$

Proof. Suppose the contrary: there exist $v \in T_D^u(\overline{x})$ and $y \in \mathcal{D}^u F(\overline{x},\overline{y})(v) \cap (-\operatorname{int} C)$. Choose $t_n \to 0^+, \{v_n\} \subset X, \{y_n\} \subset Y$ such that

$$\overline{y} + t_n y_n \in F(\overline{x} + t_n v_n)$$

and $\overline{x}+t_n v_n \in D$ and $\{v_n\}, \{y_n\}$ have convergent subsequences $v_{n_j} \to v, y_{n_j} \to y$. Since $y \in -\operatorname{int} C$ there exists an integer j_0 such that $y_{n_j} \in -\operatorname{int} C$ for all $j \geq j_0$ and then $\overline{y} + t_{n_j} y_{n_j} \notin \overline{y} + Y \setminus (-\operatorname{int} C)$. This contradicts the fact

$$F(x) \subset \overline{y} + Y \setminus (-\operatorname{int} C), \quad \text{for all} \quad x \in U \cap D$$

for some neighborhood U of $\overline{x}$ in X. This completes the proof of the proposition.

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References

1. T. Amahroq and L. Thibault, On proto-differentiability and strict proto-differentiability of multifunctions of feasible points in perturbed optimization problems, *Numer. Funct. Anal. and Opt.* (to appear).
2. J. P. Aubin and H. Frankowska, *Set-valued Analysis,* Birkhauser, 1990.
3. C.Berge, *Espaces Topologiques et Fontions Multivoques,* Dunord, Paris, 1959.
4. E. Blum and W. Oettly, From optimization and variational inequalities to equilibrium problems, *The Math. Student* **64** (1993) 1–23.
5. F. H. Clarke, *Optimization and Nonsmooth Analysis,* Wiley-Interscience, New York, 1983.
6. G. Bouligan, Sur l'exstence des demi-tangentes à une courbe de Jordan, *Fund. Math.* **15** (1930) 215–218.

7. V. Jeyakumar and D. T. Luc, Nonsmooth calculus, minimality, and monotonicity of convexification, *J. Opt. Theory and Appl.* **101** (1999) 599–621.
8. D. T. Luc, *Theory of Vector Optimization,* Lectures Notes in Economics and Math. Systems 319, Springer - Verlag, Berlin - New York, 1989.
9. D. T. Luc, Contingent derivatives of set-valued maps and application to vector optimizations *Math. Prog.* **50** (1991) 99–111.
10. D. T. Luc and C. Malivert, Invex optimization problems, *Bull. Australian Math. Soc.* **46** (1992) 47–66.
11. P. Michel and J. P. Penot, A generalized derivative for calm and stable functions, *Differential and Integral Equation,* **5** (1992) 433–454.
12. N. B. Minh and N. X. Tan, Some sufficient conditions for the existence of equilibrium points concerning multivalued mappings, *Vietnam J. Math.* **28** (2000) 295–310.
13. N. B. Minh and N. X. Tan, On the continuity of vector convex multivalued functions, *Acta Math. Vietnam.* (to appear).
14. B. S. Morduchovich and Y. Shao, On nonconvex subdifferential calculus in Banach spaces, *J. Convex Analysis* **2** (1995) 211–228.
15. D. S. Shi, Contingent derivative of the perturbation map in multiobjective optimization, *J. Opt. Theory Appl.* **70** (1991) 385–395.
16. A. Taa, Set valued derivative of multifunctions and optimality condition, *Numer. Funct. Anal. and Opt.* **19** (1998) 121–140.
17. J. S. Treiman, The linear nonconvex generalized gradient and Lagrange multipliers, *SIAM J. Opt.* **5** (1995) 670–680.