Vietnam Journal of Mathematics **30**:4 (2002) 417-420

Vietnam Journal
of
MATHEMATICS

Short Communication

Parametrical Characterizations for Pseudo and Sequentially Cohen-Macaulay Modules

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Received December 25, 2001

1. Introduction

Throughout this note, let $(A, \mathfrak{m})$ be a commutative Noetherian and M a finitely generated A-module with $\dim_A M = d$. We denote by $Q_M(\underline{x})$ the submodule of M defined by

$$Q_M(\underline{x}) = \bigcup_{t>0} \Big((x_1^{t+1}, ..., x_d^{t+1})M : x_1^t \cdots x_d^t \Big),$$

where $\underline{x} = (x_1, ..., x_d)$ is a system of parameters on M. Consider the difference

$$J_{M,\underline{x}}(\underline{n}) = n_1 \cdots n_d \, e(\underline{x}; M) - \ell_A \Big(M/Q_M(\underline{x}(\underline{n})) \Big)$$

as a function in $\underline{n}$, where $\underline{x}(\underline{n}) = (x_1^{n_1}, ..., x_d^{n_d})$ and $\underline{n} = (n_1, ..., n_d)$ is a d-tuple of positive integers. Then it is known (see [5]) that in general $J_{M,\underline{x}}(\underline{n})$ is not a polynomial for enough large $\underline{n}$. But, it was proved in [4] that the last degree of all polynomials in $\underline{n}$ bounding above the function $J_{M,\underline{x}}(\underline{n})$ is independent of the choice of $\underline{x}$. This invariant of the module M is denoted by $pf(M)$. Following [5], a module M is called a *pseudo Cohen-Macaulay module* if $pf(M) = -\infty$, where we stupilate that the degree of the zero polynomial equals $-\infty$. Then it is clear that a Cohen-Macaulay module is pseudo Cohen-Macaulay. Moreover a sequentially Cohen-Macaulay module, which is defined by Stanley [10] for graded modules and by Schenzel for the non graded case, is also a pseudo Cohen-Macaulay module (see [5]).

The purpose of this short note is to characterize pseudo Cohen-Macaulay and sequentiall Cohen-Macaulay of modules in terms of systems of parameters. The readers can find all detailed proofs for the results in this note in [6].

2. Pseudo Cohen-Macaulay Modules

Keep all notations as above, we begin with the following definition.

Definition 2.1 (see [5]). *M is said to be a pseudo Cohen-Macaulay module if $pf(M) = -\infty$.*

Remark 2.2.
(i). Let $\widehat{M}$ be the $\mathfrak{m}$-adic completion of M. We have $pf_A(M) = pf_{\widehat{A}}(\widehat{M})$ (see [4]). Thus M is a pseudo Cohen-Macaulay A-module if and only if $\widehat{M}$ is a pseudo Cohen-Macaulay $\widehat{A}$-module.
(ii). For a non-negative integer i, we set

$$D^i(M) = \mathrm{Hom}_A(H^i_{\mathfrak{m}}(M); E)$$

the Matlis'dual of $H^i_{\mathfrak{m}}(M)$, where $E = E_A(A/\mathfrak{m})$ is the injective hull of $A/\mathfrak{m}$. It is well-known that $D^i(M)$ is a finitely generated $\widehat{A}$-module for all $i \geq 0$. It was proved in [3] that M is pseudo Cohen-Macaulay if either $D^i(M) = 0$ or $D^i(M)$ is a Cohen-Macaulay $\widehat{A}$-module for all $i = 1, ..., d-1$.
(iii). The notation $p = p(M)$ means the *polynomial type* of M (see [2]).

We have some properties of the pseudo Cohen-Macaulay modules as follows.

Proposition 2.3. *Assume that M is a pseudo Cohen-Macaulay A-module with* $\dim M = d$. *Then the following statements are true:*
(i) *For any $\mathfrak{p} \in \mathrm{Supp}\, M$, then* $\dim A/\mathfrak{p} + \dim M_{\mathfrak{p}}$ *is either d or smaller than $p(M)+1$.*
(ii) *We have* $\mathrm{depth}\, D^p(M) \geq \min\{2, p\}$.
(iii) *M is a Cohen-Macaulay module if and only if* $\mathrm{depth}\, M > p(M)$.

Proposition 2.4. *Let M be a pseudo Cohen-Macaulay A module with* $\dim M = d$. *Assume that $p = p(M) > 0$ and for all $\mathfrak{p} \in \mathrm{Supp}\, M \setminus \{\mathfrak{m}\}$, $M_{\mathfrak{p}}$ is a pseudo Cohen-Macaulay module. Then for each $\mathfrak{q} \in \mathrm{Supp}\, D^p(M) \setminus \{\widehat{\mathfrak{m}}\}$, one of the following two statements are true:*
(i) $\dim M_{\mathfrak{q}} + \dim A/\mathfrak{q} = d$,
(ii) $p(M_{\mathfrak{q}}) + \dim A/\mathfrak{q} = p$ *or* $\dim M_{\mathfrak{q}} + \dim A/\mathfrak{q} = p$.

Following [2], a subset of an s.o.p. $(x_1, ..., x_j)$ of M is called a reducing sequence if it holds: $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M/(x_1, ..., x_{i-1})M)$ with $\dim A/\mathfrak{p} \geq d - i,\ i = 1, ..., j$. Note that if $\underline{x} = (x_1, ..., x_d)$ is an s.o.p on M and $x_1, ..., x_{d-1}$ form a reducing sequence, then $\underline{x}$ is just a reducing s.o.p which has been introduced in [1].

Definition 2.5. *Let $\underline{x} = (x_1, ..., x_t)$ be a sequence of elements of $\mathfrak{m}$. We denote $M_i = M/(x_1, ..., x_i)M$ for all $i = 0, ..., t$. A sequence $\underline{x}$ is called a pseudo-coregular sequence, if x_i is an $H^{d-i}_{\mathfrak{m}}(M_{i-1})$-coregular element for all $i = 1, ..., t$.*

Using this notion, we can show some characterizations of the pseudo Cohen-Macaulay modules as follows.

Theorem 2.6. *Let $(A, \mathfrak{m})$ be a commutative Noetherian local ring and M a finitely generated A-module with* $\dim M = d$. *Then the following statements are equivalent.*
(i) *M is a pseudo Cohen-Macaulay module,*
(ii) *Any reducing s.o.p. on M is a pseudo-coregular sequence,*
(iii) *M has a reducing s.o.p. which is a pseudo-coregular sequence,*
(iv) *M has an s.o.p. which is a pseudo-coregular sequence.*

Let $\underline{x} = (x_1, ..., x_d)$ be an s.o.p on M. Set

$$M_i = M/(x_1, ..., x_i)M$$

for all $i = 0, ..., d$.

Theorem 2.7. *Let M be a finitely generated A-module with* $\dim M = d$. *Suppose that $p = p(M) > 0$. Then the following statements are equivalent:*
(i) *M is a pseudo Cohen-Macaulay module,*
(ii) *$H^i_{\mathfrak{m}}(M) = 0$ for all $i = p+1, ..., d-1$ and there exists an s.o.p. $(x_1, ..., x_p)$ on M such that x_i is an $H^{p-i+1}_{\mathfrak{m}}(M_{i-1})$-coregular element for all $i = 1, ..., p$.*

3. Sequentially Cohen-Macaulay Modules

Let M be a finitely generated A-module with $\dim_A M = d \geq 1$, where $(A, \mathfrak{m})$ is a Noether local ring. For integer $0 \leq i \leq d$, let N_i denote the largest submodules of such that $\dim N_i \leq i$. Because M is a Noetherian A-module, the submodules N_i of M are well-defined. Moreover it follows that $N_{i-1} \subseteq N_i$ for all $1 \leq i \leq d$. The increasing filtration $M = \{N_i\}_{0 \leq i \leq d}$ of submodules of M is called the *dimension filtration* of M (see [8, 2.1]).

Definition 3.1 (see [8, 4.1]). *A finitely generated A-module is called a sequentially Cohen-Macaulay module if N_i/N_{i-1} is either zero or an i-dimensional Cohen-Macaulay module for all* $0 \leq i \leq \dim_A M$.

Note that this definition has been first introduced by Stanley for modules (see [10, Chapter III, 2.9]). Then, Schenzel extended it to finitely generated modules over Noetherian local rings (see [8]).

We have a property of sequentially Cohen-Macaulay modules.

Proposition 3.1. *Assume that M is a sequentially Cohen-Macaulay A-module with* $\dim_A M = d$. *Then, we have either $H^i_{\mathfrak{m}}(M) = 0$ or $H^i_{\mathfrak{m}}(M)$ is a co-Cohen-Macaulay module for all* $0 \leq i \leq d$.

Suppose that A has a dualizing complex $D^{\cdot}_A$. Recall the notation in [7]:

$$K^i(M) = H^{-i}(\mathrm{Hom}_A(M, D^{\cdot}_A)), \quad i \in \mathbb{Z}.$$

Note that $K^i(M) = 0$ for all $i < 0$ or $i > d$ and the $K^i(M),\ i \in \mathbb{Z}$ are finitely generated A-modules.

We have a characterization of sequentially Cohen-Macaulay modules as follows.

Theorem 3.2. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring. Suppose that* A *has a dualizing complex. Let* M *be a finitely generated* A*-module with* $\dim M = d$. *Then the following statements are equivalent:*
(i) *M is a sequentially Cohen-Macaulay module,*
(ii) *Every s.o.p.* $\underline{x} = (x_1, ..., x_d)$ *on* M *which is a filter regular sequence such that* x_i *is a regular element of* $K^j(M_{i-1})$ *for all* $j = 1, ..., d-i$ *and* $i = 1, ..., d-1$,
(iii) *There exists an s.o.p.* $\underline{x} = (x_1, ..., x_d)$ *on* M *which is a filter regular sequence such that* x_i *is a regular element of* $K^j(M_{i-1})$ *for all* $j = 1, ..., d-i$ *and* $i = 1, ..., d-1$.
(iv) *There exists an s.o.p.* $\underline{x} = (x_1, ..., x_d)$ *on* M *such that* x_i *is a regular element on* $K^j(M_{i-1})$ *for all* $j = 1, ..., d-i$ *and* $i = 1, ..., d-i$.

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