Short Communication

Parametrical Characterizations for Pseudo and Sequentially Cohen-Macaulay Modules

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1. Introduction

Throughout this note, let $(A, \mathfrak{m})$ be a commutative Noetherian and $M$ a finitely generated $A$-module with $\dim_A M = d$. We denote by $Q_M(\underline{x})$ the submodule of $M$ defined by

$$Q_M(\underline{x}) = \bigcup_{t > 0} \left( (x_1^{t+1}, \ldots, x_d^{t+1})M : x_1^t \cdots x_d^t \right),$$

where $\underline{x} = (x_1, \ldots, x_d)$ is a system of parameters on $M$. Consider the difference

$$J_{M, x}(\underline{n}) = n_1 \cdots n_d c(\underline{x}; M) - \ell_A \left( M/Q_M(\underline{x}(\underline{n})) \right)$$

as a function in $\underline{n}$, where $\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_d^{n_d})$ and $\underline{n} = (n_1, \ldots, n_d)$ is a $d$-tuple of positive integers. Then it is known (see [5]) that in general $J_{M, x}(\underline{n})$ is not a polynomial for enough large $\underline{n}$. But, it was proved in [4] that the last degree of all polynomials in $\underline{n}$ bounding above the function $J_{M, x}(\underline{n})$ is independent of the choice of $\underline{x}$. This invariant of the module $M$ is denoted by $\text{pf}(M)$. Following [5], a module $M$ is called a pseudo Cohen-Macaulay module if $\text{pf}(M) = -\infty$, where we stipulate that the degree of the zero polynomial equals $-\infty$. Then it is clear that a Cohen-Macaulay module is pseudo Cohen-Macaulay. Moreover a sequentially Cohen-Macaulay module, which is defined by Stanley [10] for graded modules and by Schenzel for the non graded case, is also a pseudo Cohen-Macaulay module (see [5]).

The purpose of this short note is to characterize pseudo Cohen-Macaulay and sequentially Cohen-Macaulay of modules in terms of systems of parameters. The readers can find all detailed proofs for the results in this note in [6].
2. Pseudo Cohen-Macaulay Modules

Keep all notations as above, we begin with the following definition.

**Definition 2.1** (see [5]). $M$ is said to be a pseudo Cohen-Macaulay module if $pf(M) = -\infty$.

**Remark 2.2.**
(i). Let $\hat{M}$ be the $m$-adic completion of $M$. We have $pf_A(M) = pf_A(\hat{M})$ (see [4]). Thus $M$ is a pseudo Cohen-Macaulay $A$-module if and only if $\hat{M}$ is a pseudo Cohen-Macaulay $\hat{A}$-module.
(ii). For a non-negative integer $i$, we set $D_i(M) = \text{Hom}_A(H^i_m(M); E)$ the Matlis’ dual of $H^i_m(M)$, where $E = E_A(A/m)$ is the injective hull of $A/m$. It is well-known that $D_i(M)$ is a finitely generated $\hat{A}$-module for all $i \geq 0$. It was proved in [3] that $M$ is pseudo Cohen-Macaulay if either $D_i(M) = 0$ or $D_i(M)$ is a Cohen-Macaulay $\hat{A}$-module for all $i = 1, \ldots, d - 1$.
(iii). The notation $p = p(M)$ means the polynomial type of $M$ (see [2]).

We have some properties of the pseudo Cohen-Macaulay modules as follows.

**Proposition 2.3.** Assume that $M$ is a pseudo Cohen-Macaulay $A$-module with $\text{dim } M = d$. Then the following statements are true:
(i) For any $p \in \text{Supp } M$, then $\text{dim } A/p + \text{dim } M_p$ is either $d$ or smaller than $p(M) + 1$.
(ii) We have $\text{depth } D_p(M) \geq \min\{2, p\}$.
(iii) $M$ is a Cohen-Macaulay module if and only if depth $M > p(M)$.

**Proposition 2.4.** Let $M$ be a pseudo Cohen-Macaulay $A$ module with $\text{dim } M = d$. Assume that $p = p(M) > 0$ and for all $p \in \text{Supp } M \setminus \{m\}$, $M_p$ is a pseudo Cohen-Macaulay module. Then for each $q \in \text{Supp } D_p(M) \setminus \{\hat{m}\}$, one of the following two statements are true:
(i) $\text{dim } M_q + \text{dim } A/q = d$.
(ii) $p(M_q) + \text{dim } A/q = p$ or $\text{dim } M_q + \text{dim } A/q = p$.

Following [2], a subset of an s.o.p. $(x_1, \ldots, x_j)$ of $M$ is called a reducing sequence if it holds: $x_i \notin p$ for all $p \in \text{Ass}(M/(x_1, \ldots, x_{i-1})M)$ with $\text{dim } A/p \geq d - i, i = 1, \ldots, j$. Note that if $\underline{x} = (x_1, \ldots, x_d)$ is an s.o.p on $M$ and $x_1, \ldots, x_{d-1}$ form a reducing sequence, then $\underline{x}$ is just a reducing s.o.p which has been introduced in [1].

**Definition 2.5.** Let $\underline{x} = (x_1, \ldots, x_t)$ be a sequence of elements of $m$. We denote $M_i = M/(x_1, \ldots, x_i)M$ for all $i = 0, \ldots, t$. A sequence $\underline{x}$ is called a pseudo-coregular sequence, if $x_i$ is an $H^d_{\text{m}}(M_{i-1})$-coregular element for all $i = 1, \ldots, t$.

Using this notion, we can show some characterizations of the pseudo Cohen-Macaulay modules as follows.
Theorem 2.6. Let \((A, \mathfrak{m})\) be a commutative Noetherian local ring and \(M\) a finitely generated \(A\)-module with \(\dim M = d\). Then the following statements are equivalent.

(i) \(M\) is a pseudo Cohen-Macaulay module,
(ii) Any reducing s.o.p. on \(M\) is a pseudo-coregular sequence,
(iii) \(M\) has a reducing s.o.p. which is a pseudo-coregular sequence,
(iv) \(M\) has an s.o.p. which is a pseudo-coregular sequence.

Let \(\underline{x} = (x_1, \ldots, x_d)\) be an s.o.p on \(M\). Set
\[
M_i = M/(x_1, \ldots, x_i)M
\]
for all \(i = 0, \ldots, d\).

Theorem 2.7. Let \(M\) be a finitely generated \(A\)-module with \(\dim M = d\). Suppose that \(p = p(M) > 0\). Then the following statements are equivalent:

(i) \(M\) is a pseudo Cohen-Macaulay module,
(ii) \(H^i_{\mathfrak{m}}(M) = 0\) for all \(i = p + 1, \ldots, d - 1\) and there exists an s.o.p. \((x_1, \ldots, x_p)\) on \(M\) such that \(x_i\) is an \(H^{p-i+1}_{\mathfrak{m}}(M_{i-1})\)-coregular element for all \(i = 1, \ldots, p\).

3. Sequentially Cohen-Macaulay Modules

Let \(M\) be a finitely generated \(A\)-module with \(\dim A = d \geq 1\), where \((A, \mathfrak{m})\) is a Noether local ring. For integer \(0 \leq i \leq d\), let \(N_i\) denote the largest submodules of such that \(\dim N_i \leq i\). Because \(M\) is a Noetherian \(A\)-module, the submodules \(N_i\) of \(M\) are well-defined. Moreover it follows that \(N_{i-1} \subseteq N_i\) for all \(1 \leq i \leq d\). The increasing filtration \(M = \{N_i\}_{0 \leq i \leq d}\) of submodules of \(M\) is called the dimension filtration of \(M\) (see \([8, 2.1]\)).

Definition 3.1 (see \([8, 4.1]\)). A finitely generated \(A\)-module is called a sequentially Cohen-Macaulay module if \(N_i/N_{i-1}\) is either zero or an \(i\)-dimensional Cohen-Macaulay module for all \(0 \leq i \leq \dim A\).

Note that this definition has been first introduced by Stanley for modules (see \([10, \text{Chapter III, 2.9}]\)). Then, Schenzel extended it to finitely generated modules over Noetherian local rings (see \([8]\)).

We have a property of sequentially Cohen-Macaulay modules.

Proposition 3.1. Assume that \(M\) is a sequentially Cohen-Macaulay \(A\)-module with \(\dim A = d\). Then, we have either \(H^i_{\mathfrak{m}}(M) = 0\) or \(H^i_{\mathfrak{m}}(M)\) is a co-Cohen-Macaulay module for all \(0 \leq i \leq d\).

Suppose that \(A\) has a dualizing complex \(D_A\). Recall the notation in \([7]\):
\[
K^i(M) = H^{-i}(\text{Hom}_A(M, D_A)), \quad i \in \mathbb{Z}.
\]
Note that \(K^i(M) = 0\) for all \(i < 0\) or \(i > d\) and the \(K^i(M), i \in \mathbb{Z}\) are finitely generated \(A\)-modules.
We have a characterization of sequentially Cohen-Macaulay modules as follows.

**Theorem 3.2.** Let \((A, \mathfrak{m})\) be a Noetherian local ring. Suppose that \(A\) has a dualizing complex. Let \(M\) be a finitely generated \(A\)-module with \(\dim M = d\). Then the following statements are equivalent:

(i) \(M\) is a sequentially Cohen-Macaulay module,

(ii) Every s.o.p. \(\underline{x} = (x_1, \ldots, x_d)\) on \(M\) which is a filter regular sequence such that \(x_i\) is a regular element of \(K^j(M_{i-1})\) for all \(j = 1, \ldots, d - i\) and \(i = 1, \ldots, d - 1\),

(iii) There exists an s.o.p. \(\underline{x} = (x_1, \ldots, x_d)\) on \(M\) which is a filter regular sequence such that \(x_i\) is a regular element of \(K^j(M_{i-1})\) for all \(j = 1, \ldots, d - i\) and \(i = 1, \ldots, d - 1\),

(iv) There exists an s.o.p. \(\underline{x} = (x_1, \ldots, x_d)\) on \(M\) such that \(x_i\) is a regular element on \(K^j(M_{i-1})\) for all \(j = 1, \ldots, d - i\) and \(i = 1, \ldots, d - i\).

**References**


