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On the Second Mixed Boundary Value Problems for Linear Equations with Generalized Right Invertible Operators

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Abstract. The aim of this paper is to study a second mixed boundary value problem (SMBVP) with a given generalized right invertible operator V as follows:

Find all solutions of the problem

$$Q[V]x := \sum_{m=0}^{M}\sum_{n=0}^{N} V^m A_{mn} V^n x = y, \quad y \in X, \tag{0.1}$$

$$\begin{aligned} &F_i x = y_i \quad (i = 0, \dots, K-1),\\ &F_i V^i x = y_i \quad (i = K, \dots, M+N-1), \quad y_i \in \ker V \quad \text{are given,} \end{aligned} \tag{0.2}$$

where $M, N, K \in \mathbb{N}$, $K < M+N$, $A_{mn} \in L_0(X)$, $A_{MN} = I$, $A_{mn}X_{M+N-n} \subset X_m$, $X_i :=$ dom V^i, and $F_0, \dots, F_{M+N-1}$ are right initial operators of V, where $F_0, \dots, F_{K-1}$ have $c(W)$-property.

1. Introduction

The $c(R)$-property for initial operators, induced by a given right invertible operator, has been introduced and applied to solving boundary value problems with right invertible operators by Przeworska–Rolewicz, Mau, Binderman and others (see [1 - 4]). In [5], Mau and Tuan introduced a class of generalized right invertible operators. Recently, the author studied the first mixed boundary value problems for the equation (0.1) (see [7]).

In this paper, we introduce the $c(W)$-property for right initial operators of a given generalized right invertible operator V. We use this property for solving the

SMBVP (0.1) – (0.2). In particular, we shall construct general forms of resolving operator for this problem, which permits to find all solutions of (0.1) – (0.2) in closed forms.

2. Preliminaries and Notations

Let X be a linear space over the field $\mathcal{K}$ of scalars. Denote by $L(X)$ the set of all linear operators with domains and ranges in X and by $L_0(X) = \{A \in L(X) : \operatorname{dom} A = X\}$. We denote by $R(X)$ ($\Lambda(X)$ or $W(X)$) the set of all right (left or generalized) invertible operators belonging to $L(X)$, and by $\mathcal{R}_D$ ($\mathcal{L}_\Delta$ or $\mathcal{W}_V$) the set of all right (left or genezalized) inverses of D (Δ or V), respectively (see [1 - 3]).

Definition 1. [5] *An operator $V \in W(X)$ is said to be generalized right invertible* (*for short:* GR-*invertible*) *if there exists a $W \in \mathcal{W}_V$ such that* $\operatorname{Im}(VW - I) \subset \ker V$, *i.e.* $VWV = V, \quad V^2W = V$.

Denote by $R_1(X)$ the set of all GR-invertible operators. For a $V \in R_1(X)$, denote by $\mathcal{R}^1_V$ the set of all generalized inverses of V, by $\mathcal{F}_V$ and $\mathcal{G}_V$ the set of all right and left initial operators of V, i.e.

$$\begin{aligned}
\mathcal{R}^1_V &= \{W \in L(X) : \operatorname{Im} V \subset \operatorname{dom} W,\ \operatorname{Im} W \subset \operatorname{dom} V,\ VWV = V,\ V^2W = V\},\\
\mathcal{F}_V &= \{F \in L(X) : F^2 = F, \operatorname{Im} F = \ker V \text{ and } \exists W \in \mathcal{R}^1_V : FW = 0 \text{ on } \operatorname{dom} W\},\\
\mathcal{G}_V &= \{G \in L(X) : G^2 = G,\ GV = 0 \text{ on } \operatorname{dom} V \text{ and } \exists W \in \mathcal{R}^1_V : \operatorname{Im} G = \ker W\}.
\end{aligned} \tag{2.1}$$

It is easy to see that $R(X) \subset R_1(X) \subset W(X)$, $\Lambda(X) \subset W(X)$.

Lemma 1. [6] *For every $V \in R_1(X)$ there exists $W \in \mathcal{R}^1_V$ such that*

$$WVW = W, \quad VW^2 = W \quad \text{on } \operatorname{dom} W.$$

Write

$$\mathcal{R}^{(1)}_V = \{W \in \mathcal{R}^1_V : WVW = W, \quad VW^2 = W\}. \tag{2.2}$$

Lemma 2. [7] *Suppose that $V \in R_1(X)$, $W_1, W_2 \in \mathcal{R}^{(1)}_V$, $\operatorname{Im} W_2 \subset \operatorname{dom} W_1$. Then*

$$VW_1W_2 = W_2. \tag{2.3}$$

Theorem 1. [4] *A necessary and sufficient condition for an operator $F \in L(X)$ (or $G \in L(X)$) to be a right (or left) initial operator for $V \in R_1(X)$ corresponding to a $W \in \mathcal{R}^{(1)}_V$ is that*

$$F = I - WV \text{ on } \operatorname{dom} V \quad (\text{or} \quad G = I - VW \text{ on } \operatorname{dom} W). \tag{2.4}$$

Theorem 2. [4] (Taylor-Gontcharov formula)

Suppose that $V \in R_1(X)$ *and* $\mathcal{F}_V = \{F_\beta\}_{\beta\in\Gamma}$ *is a family of right initial operators corresponding to* $\{W_\beta\}_{\beta\in\Gamma} \subset \mathcal{R}_V^{(1)}$. *Let* $\{\beta_n\} \subset \Gamma,\ n \in \mathbb{N}$ *be an arbitrary sequence of indices such that* $\operatorname{Im} W_{\beta_j} \subset \operatorname{dom} W_{\beta_{j-1}},\ j = 1, \ldots, N-1$. *Then for every positive integer* N *the following identity holds on* $\operatorname{dom} V^N$.

$$I = F_{\beta_0} + \sum_{j=1}^{N-1} W_{\beta_0} \cdots W_{\beta_{j-1}} F_{\beta_j} V^j + W_{\beta_0} \cdots W_{\beta_{N-1}} V^N. \tag{2.5}$$

Corollary 1. [7] *Suppose that* $V \in R_1(X)$, $W_j \in \mathcal{R}_V^{(1)}$, $\operatorname{Im} W_j \subset \operatorname{dom} W_{j-1}$, $j = 1, \ldots, N-1$. *For every* $N \in \mathbb{N}$, *we have*

$$\ker V^N = \Big\{x \in X : x = \sum_{j=1}^{N-1} W_0 \cdots W_{j-1} z_j + z_0,\ z_0, \ldots, z_{N-1} \in \ker V\Big\}. \tag{2.6}$$

Puting $W_0 = \cdots = W_{K-1} = W$, we obtain

$$\ker V^N = \Big\{x \in X : x = \sum_{j=0}^{K-1} W^j z_j + \sum_{j=K}^{N-1} W^K W_K \cdots W_{j-1} z_j,\ z_0, \ldots, z_{N-1} \in \ker V\Big\}. \tag{2.7}$$

Theorem 3. [4] *Let* $A, B \in L(X)$, $\operatorname{Im} A \subset \operatorname{dom} B$ *and* $\operatorname{Im} B \subset \operatorname{dom} A$. *Then* $I + AB$ *is right invertible (left invertible, generalized invertible or invertible) if and only if so is* $I + BA$. *Moreover, if we denote by* R_{AB} $(L_{AB},\ W_{AB}$ *or* $(I + AB)^{-1})$ *a right inverse (left inverse, generalized inverse or inverse) of* $I + AB$ *then there exists* $R_{BA} \in \mathcal{R}_{I+BA}$ $(L_{BA} \in \mathcal{L}_{I+BA},\ W_{BA} \in \mathcal{W}_{I+BA}$ *or* $(I + BA)^{-1} \in \mathcal{R}_{I+BA} \cap \mathcal{L}_{I+BA})$ *such that*

$$\begin{aligned} R_{AB} &= I - AR_{BA}B, \quad R_{BA} = I - BR_{AB}A, \\ L_{AB} &= I - AL_{BA}B, \quad L_{BA} = I - BL_{AB}A, \\ W_{AB} &= I - AW_{BA}B, \quad W_{BA} = I - BW_{AB}A, \\ \text{or} \quad (I + AB)^{-1} &= I - A(I + BA)^{-1}B,\ (I + BA)^{-1} = I - B(I + AB)^{-1}A \end{aligned} \tag{2.8}$$

respectively.

3. $C(W)$- Property

Definition 2. *Let* $V \in R_1(X)$ *and* $W \in R_V^1$. *An operator* $F_0 \in \mathcal{F}_V$ *possesses the* $c(W)$- *property if there exist scalars* d_k *such that*

$$F_0 W^k z = d_k z \quad for\ all \quad z \in \ker V,\ k \in \mathbb{N}. \tag{3.1}$$

(*where we admit* $d_k = 0$ *for all* $k \in \mathbb{N}$ *if* F_0 *is a right initial operator for* V *corresponding to* W).

We denote by $\mathcal{F}_{V,W}$ the set of all right initial operators possessing the $c(W)$ property.

Lemma 1. *Let* $\dim \ker V = s < +\infty$*. Then a right initial operator* F_0 *for* V *has the* $c(W)$*- property for a right inverse* W *if and only if*

$$F_0 W^k e_j = d_k e_j, \quad d_k \in \mathcal{K}, \ \ k \in \mathbb{N} \ \ j = 1, \dots, s; \tag{3.2}$$

where $\{e_1, \dots, e_s\}$ *is a basis of* $\ker V$*.*

The proof follows directly from Definition 2. ■

Theorem 4. *If* $\dim \ker V = 1$ *then* $\mathcal{F}_{V,W} = \mathcal{F}_V$*.*

Example 1. Let $X = \mathbb{R}^2$ and

$$V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ W = \begin{pmatrix} 1/3 & 1/8 \\ 1/6 & 3/8 \end{pmatrix}, \ W_0 = \begin{pmatrix} 7/40 & 7/40 \\ 13/40 & 13/40 \end{pmatrix},$$
$$F_0 = \begin{pmatrix} 13/20 & -7/20 \\ -13/20 & 7/20 \end{pmatrix}.$$

It is easy to check that $V \in R_1(X)$, $W, W_0 \in R_V^1$, $F_0 \in \mathcal{F}_V$, $\ker V = \operatorname{lin}\{e\}$, where $e = (1, -1)$, $\dim \ker V = 1$ and

$$F_0 W^k e = \left(\frac{5}{24}\right)^k e, \quad k = 0, 1, \dots .$$

Hence, F_0 has the $c(W)$ property.

Suppose $\{F_0, \dots, F_{N-1}\} \subset \mathcal{F}_{V,W}$, i.e.

$$F_i W^k z = d_{ik} z \ \text{ for } \ i = 0, \dots, N-1; \quad k \in \mathbb{N}; \ \ z \in \ker V, \ d_{ik} \in \mathcal{K}. \tag{3.3}$$

Denote

$$V_N := \det(d_{ik})_{i,k=0,\dots,N-1}; \tag{3.4}$$
$$\widehat{F}_i := (F_i, F_i W, \dots, F_i W^{N-1}), \quad i = 0, \dots, N-1; \tag{3.5}$$
$$d_i := (d_{i0}, d_{i1}, \dots, d_{iN-1}), \quad i = 0, \dots, N-1. \tag{3.6}$$

Lemma 4. *Let* $\widehat{F}_i$ *and* d_i *be defined by* (3.5) *and* (3.6), *respectively. Then the system of vectors* $\{\widehat{F}_0, \dots, \widehat{F}_{N-1}\}$ *is linearly independent of* $\ker V$ *if and only if the system* $\{d_0, d_1, \dots, d_{N-1}\}$ *is linearly independent.*

Proof. Let $\{\widehat{F}_0, \dots, \widehat{F}_{N-1}\}$ be linearly independent of $\ker V$ and let

$$\sum_{i=0}^{N-1} \beta_i d_i = 0, \quad \text{i.e.,} \sum_{i=0}^{N-1} \beta_i d_{ik} = 0, \quad k = 0, \dots, N-1.$$

Then

$$\sum_{i=0}^{N-1} \beta_i d_{ik} z = 0, \quad \text{for all} \quad z \in \ker V, \ \ k = 0, \dots, N-1.$$

i.e.

$$\sum_{i=0}^{N-1} \beta_i F_i W^k z = 0.$$

Hence

$$\sum_{i=0}^{N-1} \beta_i \widehat{F}_i z = 0 \quad \text{for all} \quad z \in \ker V.$$

By the assumption, it implies that $\beta_i = 0 \quad$ for $\quad i = 0, \dots, N-1.$

Conversely, if $\{d_0, \dots, d_{N-1}\}$ is linearly independent, then $\{\widehat{F}_0, \dots, \widehat{F}_{N-1}\}$ is linearly independent of $\ker V$. ■

Corollary 2. *Let $\{F_0, \dots, F_{N-1}\}$ be a system of right initial operators for V having the c(W)-property. Then $V_N \neq 0$ if and only if the system $\{F_0 W^k, \dots, F_{N-1} W^k\}$ is linearly independent of* $\ker V$ *for every $k \in \{0, \dots, N-1\}$.*

Proof. By Lemma 4, $F_0 W^k, \dots, F_{N-1} W^k$ are linearly independent of $\ker V$ for each fixed k if and only if the vectors $d_0, \dots, d_{N-1}$ given by (3.6) are linearly independent, i.e. $V_N = \det(d_{ik})_{i,k=0,\dots,N-1} \neq 0$. ■

Theorem 5. *$V_N \neq 0$ if and only if the system $\{F_0, \dots, F_{N-1}\}$ is linearly independent of $P_N(W)$, where*

$$P_N(W) = \text{lin}\{W^k z, \quad z \in \ker V,\ k = 0, \dots, N-1\}. \tag{3.7}$$

Proof. By Corollary 2, $V_N \neq 0$ if and only if for every $k \in \{0, \dots, N-1\}$, the system $\{F_0 W^k, \dots, F_{N-1} W^k\}$ is linearly independent on $\ker V$, i.e. the equality

$$\sum_{i=0}^{N-1} \alpha_i F_i W^k z = 0 \quad \text{for all} \quad z \in \ker V,\ \ \alpha_i \in \mathcal{K},$$

implies $\alpha_i = 0$ for $i = 0, \dots, N-1$. It means that

$$\sum_{k=0}^{N-1} \beta_k \sum_{i=0}^{N-1} \alpha_i F_i W^k z = 0 \quad \text{for all} \quad \beta_k \in \mathcal{K},$$

if and only if

$$\sum_{k=0}^{N-1} \alpha_i F_i \Big(\sum_{i=0}^{N-1} \beta_k W^k z \Big) = 0, \ \text{ i.e.,} \quad \sum_{i=0}^{N-1} \alpha_i F_i x = 0, \ \ \forall x \in P_N(W).$$

Thus, the system $\{F_0, \dots, F_{N-1}\}$ is linearly independent of $P_N(W)$. The proof is complete. ■

4. The SMBVP with Generalized Right Invertible Operators

Let $V \in R_1(X)$, $W \in \mathcal{R}_V^{(1)}$, and let $F_0, \dots, F_{M+N-1}$ and $G_0, \dots, G_{M+N-1}$ be right and left initial operators for V corresponding to $W_0, \dots, W_{M+N-1} \in \mathcal{R}_V^{(1)}$, $\text{Im } W_j \subset \text{ dom } W_{j-1}$ for $j = 1, \dots, M+N-1$. Moreover, let $F_0, \dots, F_{K-1}$

possess the $c(W)$-property and be linearly independent of the $P_K(W)$, where $P_K(W)$ is of the form (3.7), $K < M + N$.

Hence there exist scalars d_{ij} such that

$$F_i W^j z = d_{ij} z, \quad \forall z \in \ker V, \;\; i, \; j = 0, \dots, K-1. \tag{4.1}$$

By the assumption and Theorem 5, the matrix

$$\Delta_K =: (d_{ij})_{i,j=0,\dots,K-1} \tag{4.2}$$

is invertible, i.e. Δ_K^{-1} exists. Write

$$\Delta_K^{-1} =: (d'_{ij})_{i,j=0,\dots,K-1}. \tag{4.3}$$

To begin with, we consider the SMBVP for the operator V^N : Find all solutions of the problem

$$V^N x = y, \quad y \in X, \tag{4.4}$$

$$F_i x = y_i \quad (i = 0, \dots, K-1),$$

$$F_i V^i x = y_i, \quad (i = K, \dots, N-1); \;\; y_i \in \ker V \;\text{ are given.} \tag{4.5}$$

Theorem 6. *Suppose that* $V \in R_1(X)$, dim ker$V \neq 0$, dim coker $V \neq 0$. *Let* $F_0, \dots, F_{N-1}$ and $G_0, \dots, G_{N-1}$ *be right and left initial operators for* V *corresponding to the right inverses* $W_0, \dots, W_{N-1}$, *respectively* (*where we admit* Im $W_i \subset$ dom W_{i-1}, $i = 1, \dots, N-1$). *Moreover, suppose that* $F_0, \dots, F_{K-1}$ *possess the* $c(W)$*-property and they are linearly independent of* $P_K(W)$. *Then the problem* (4.4) – (4.5) *has solutions if and only if*

$$y \in \text{Im } V^N, \;\; G_{i-1} y_i = 0, \;\; i = K \dots, N-1.$$

If this is the case, then the SMBVP has a unique solution

$$x = \sum_{j=0}^{K-1} \sum_{k=0}^{K=1} d'_{jk} W^j y_k + \Big(I - \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} W^j F_k \Big) y_N, \tag{4.6}$$

where $P_K(W)$ *and* d'_{jk} *are denoted by* (3.7) *and* (4.3), *respectively,*

$$y_N = W^K W_K \cdots W_{N-1} y + \sum_{j=K}^{N-1} W^K W_K \cdots W_{j-1} y_j.$$

Proof. Note that, if the problem (4.4) – (4.5) has solutions, then $y \in \text{Im } V^N$. By formulas (2.3) and (2.4), we have

$$\begin{aligned} G_{i-1} y_i &= G_{i-1} F_i V^i x = (I - V W_{i-1})(I - W_i V) V^i x \\ &= (I - V W_{i-1} - W_i V + V W_{i-1} W_i V) V^i x = 0, \quad i = K, \dots, N-1. \end{aligned}$$

Conversely, if $y \in \text{Im } V^N$, then there is $y_1 \in$ dom V^N such that $y = V^N y_1$. Hence, (4.4) can be written in the form $V^N x = V^N y_1$. Since $V^N = V^N W^K W_K \dots W_{N-1} V^N$, the last equation is equivalent to $V^N(x - W^K W_K \cdots W_{N-1} y) = 0$. From Corollary 1, it follows that

$$x = W^K W_K..W_{N-1}y + \sum_{j=0}^{K-1} W^j z_j + \sum_{j=K}^{N-1} W^K W_K..W_{j-1}z_j,\ z_0,.., z_{N-1} \in \ker V.$$

For $i = K, \dots, N-1$, we have

$$y_i = F_i V^i x = F_i W_i \cdots W_{N-1}y + \sum_{j=0}^{K-1} F_i V^{i-j} z_j + \sum_{j=K}^{i-1} F_i V^{i-j} z_j$$
$$+ \sum_{j=i+1}^{N-1} F_i W_i \cdots W_{j-1} z_j + F_i V W_{i-1} z_i = V W_{i-1} z_i.$$

Since $G_{i-1}y_i = 0$, we have $y_i = VW_{i-1}y_i$. Hence

$$W_{i-1}z_i = W_{i-1}y_i + t_i, \quad t_i \in \ker V.$$

$$x = W^K W_K \cdots W_{N-1}y + \sum_{j=0}^{K-1} W^j z_j$$
$$+ \sum_{j=K}^{N-1} W^K W_K \cdots W_{j-1}y_j + \sum_{j=K}^{N-1} W^K W_K \cdots W_{j-2}t_j$$

(where we admit $W_{-1} = I$). On the other hand, we have

$$F_i V^i x = F_i V W_{i-1} y_i + F_i V W_{i-1} t_{i+1} = y_i + V W_{i-1} t_{i+1} = y_i$$

thus $VW_{i-1}t_{i+1} = 0$. Thus, $W_{i-1}t_{i+1} \in \ker V$ and $F_{i-1}W_{i-1}t_{i+1} = W_{i-1}t_{i+1} = 0$. Hence

$$x = W^K W_K \cdots W_{N-1}y + \sum_{j=0}^{K-1} W^j z_j + \sum_{j=K}^{N-1} W^K W_K \cdots W_{j-1}y_j.$$

For $i = 0, \dots, K-1$ and $F_i \in \mathcal{F}_{V,W}$ we have

$$y_i = F_i x = F_i\Big(W^K W_K \cdots W_{N-1}y + \sum_{j=K}^{N-1} W^K W_K \cdots W_{j-1}y_j\Big) + \sum_{j=0}^{K-1} F_i W^j z_j,$$

$$y_i - F_i y_N = \sum_{j=0}^{K-1} d_{ij} z_j. \tag{4.7}$$

Since $F_0, \dots, F_{K-1}$ are linearly independent of $P_K(W)$, the system (4.7) has a unique solution

$$z_j = \sum_{k=0}^{K-1} d'_{jk}(y_k - F_k y_N),$$

which implies the required formula (4.6). ∎

Definition 3. (cf. Przeworska-Rolewicz [1])

(i) *The SMBVP* (0.1) − (0.2) *is called well-posed if it has a unique solution for every* $y \in Q[V]X_{M+N}$, $y_0, \dots, y_{M+N-1} \in \ker V$, $G_{j-1}y_j = 0$, $j = K, \dots, M+N-1$.

(ii) *The SMBVP* (0.1) − (0.2) *is called ill-posed if either there exists* $y \in Q[V]X_{M+N}$, $y_0, \dots, y_{M+N-1} \in \ker V$, $G_{j-1}y_j = 0$, $j = K, \dots, M+N-1$ *such that this problem has no solutions or the corresponding homogeneous problem induced by* (0.1) − (0.2) *(i.e.* $y = y_0 = \cdots = y_{M+N-1} = 0$*) has non-trivial solutions.*

Write

$$S := I - \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu - \sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk} W^j F_k \Big(I - \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu \Big), \tag{4.8}$$

$$H := SH_0, \quad H_0 := \sum_{m=0}^{M}\sum_{n=0}^{N} W^K W_K \cdots W_{M+N-m-1} A'_{mn} V^n, \tag{4.9}$$

where $W^K W_K \cdots W_{M+N-m-1} = W^{M+N-m}$ if $M+N-m \le K$,

$$A'_{mn} := \begin{cases} 0 & \text{if } m = M,\ n = N, \\ A_{mn} & \text{otherwise} \end{cases} \tag{4.10}$$

It is easy to check that dom $S = X_M$, dom $H_0 = X_{M+N}$ and $SX_{M+N} \subset X_{M+N}$, $(I + H_0)X_{M+N} \subset X_{M+N}$ i.e. $S \in L_0(X_{M+N})$ and $I + H_0 \in L_0(X_{M+N})$.

Lemma 5. *Let* S *and* H *be defined by* (4.8) *and* (4.9)*, respectively. Then*

$$\begin{aligned} &\text{(i) } F_i S = 0 \quad (i = 0, \dots, K-1),\\ &\text{(ii) } F_i V^i S = 0 \quad (i = K, \dots, M+N-1),\\ &\text{(iii) } F_i(I+H) = F_i \quad (i = 0, \dots, K-1),\\ &\text{(iv) } F_i V^i (I+H) = F_i V^i \quad (i = K, \dots, M+N-1). \end{aligned} \tag{4.11}$$

Proof. By the assumption, $\{F_0, \dots, F_{K-1}\} \subset \mathcal{F}_{V,W}$, i.e., $VF_k = 0$ and

$$F_i W^j F_k = d_{ij} F_k \quad (i, j = 0, \dots, K-1;\ k = 0, \dots, M+N-1;\ d_{ij} \in \mathcal{K}).$$

(i) For $i = 0, \dots, K-1$, we have

$$\begin{aligned} F_i S = F_i &- \sum_{\mu=K}^{M+N-1} F_i W^K W_K \cdots W_{\mu-1} F_\mu V^\mu \\ &- \sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk} F_i W^j F_k \Big(I - \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu \Big) \\ = F_i &- \sum_{\mu=K}^{M+N-1} F_i W^K W_K \cdots W_{\mu-1} F_\mu V^\mu \\ &- \sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk} d_{ij} F_k \Big(I - \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu \Big) \end{aligned}$$

$$= F_i - \sum_{\mu=K}^{M+N-1} F_i W^K W_K \cdots W_{\mu-1} F_\mu V^\mu$$
$$- \sum_{k=0}^{K-1} \delta_{ik} F_k \Big(I - \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu \Big)$$
$$= F_i - \sum_{\mu=K}^{M+N-1} F_i W^K W_K \cdots W_{\mu-1} F_\mu V^\mu$$
$$- F_i \Big(I - \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu \Big) = 0.$$

(ii) For $i = K, \dots, M+N-1$, we have $V^i W^j F_k = V^{i-j} F_k = 0$, $F_i V W_{i-1} F_i V^i = F_i V^i$, and

$$F_i V^i S = F_i V^i - \sum_{\mu=K}^{M+N-1} F_i V^i W^K W_K \cdots W_{\mu-1} F_\mu V^\mu$$
$$- \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} F_i V^{i-j} F_k \Big(I - \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu \Big)$$
$$= F_i V^i - \sum_{\mu=K}^{i-1} F_i V^{i-\mu} F_\mu V^\mu - F_i V^i W_{i-1} F_i V^i - \sum_{\mu=i+1}^{M+N-1} F_i W_i \cdots W_{\mu-1} F_\mu V^\mu$$
$$= F_i V^i - F_i V^i = 0.$$

(iii) By (i), we have

$$F_i(I+H) = F_i(I+SH_0) = F_i + F_i S H_0 = F_i \quad (i = 0, \dots, K-1).$$

(iv) By (ii), for $i = 0, \dots, K-1$, we have

$$F_i V^i (I+H) = F_i V^i (I+SH_0) = F_i V^i + F_i V^i S H_0 = F_i V^i.$$

■

Write

$$W'_j := \begin{cases} W & \text{if} \quad j = 0, \dots, K-1, \\ W_j & \text{if} \quad j = K, \dots, M+N-1. \end{cases} \tag{4.12}$$

$$T_m := I - \sum_{\mu=M+N-m}^{M+N-1} W'_{M+N-m} \cdots W'_\mu F_\mu V^{\mu+m-M-N+1} W'_{M+N-m-1}, \tag{4.13}$$

$$H_1 := I - \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} W'_0 \cdots W'_{j-1} F_k, \tag{4.14}$$

$$H_2 := \sum_{m=0}^{M} \sum_{n=0}^{N} W'_0 \cdots W'_{M+N-m-1} T_m A'_{mn} V^n. \tag{4.15}$$

It is easy to see that the operator H given by (4.9) can be written in the form $H = H_1H_2$.

Lemma 6. *Write*

$$B_{mn} := \begin{cases} A'_{mn} & \text{if } m = 0, \dots, M;\ n \le \min\,(N, K-1), \\ 0 & \text{if } m = 0, \dots, M;\ n > \min\,(N, K-1), \end{cases} \tag{4.16}$$

$$H'_0 := \sum_{m=,0}^{M}\sum_{m=0}^{N} W'_0 \ldots W'_{M+N-m-1}T_m \times \Big(A'_{mn}V^n - B_{mn}\sum_{j=n}^{K-1}\sum_{k=0}^{K-1} d'_{jk}VW'_{n-1}\ldots W'_{j-1}F_k\Big). \tag{4.17}$$

Then $I+H$ is right invertible (left invertible, generalized invertible or invertible) if and only if so is $I + H'_0$. Moreover, if $R_{H'_0} \in \mathcal{R}_{I+H'_0}$ ($L_{H'_0} \in \mathcal{L}_{I+H'_0}$, $W_{H'_0} \in \mathcal{W}_{I+H'_0}$ or $(I + H'_0)^{-1} \in \mathcal{R}_{I+H'_0} \cap \mathcal{L}_{I+H'_0}$), then

$$\begin{aligned} R_H &:= I - H_1R_{H'_0}H_2 \in \mathcal{R}_{I+H}, \quad L_H := I - H_1L_{H'_0}H_2 \in \mathcal{L}_{I+H}, \\ W_H &:= I - H_1W_{H'_0}H_2 \in \mathcal{W}_{I+H}, \quad (I + H)^{-1} := I - H_1(I + H'_0)^{-1}H_2, \end{aligned} \tag{4.18}$$

where H, H_1 and H_2 are defined by (4.9), (4.14) *and* (4.15)*, respectively.*

Proof. We have

$$H_2H_1 = \sum_{m=0}^{M}\sum_{n=0}^{N} W'_0 \cdots W'_{M+N-m-1}T_mA'_{mn}V^n\Big(I - \sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk}W'_0 \ldots W'_{j-1}F_k\Big). \tag{4.19}$$

On the other hand

$$\begin{aligned} &\sum_{m=0}^{M}\sum_{n=0}^{N} W'_0 \cdots W'_{M+N-m-1}T_mA'_{mn}V^n \sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk}W'_0 \cdots W'_{j-1}F_k \\ = &\sum_{m=0}^{M}\sum_{n=0}^{N} W'_0 \cdots W'_{M+N-m-1}T_mA'_{mn} \sum_{j=n}^{K-1}\sum_{k=0}^{K-1} d'_{jk}VW_{n-1} \cdots W'_{j-1}F_k \\ = &\sum_{m=0}^{M}\sum_{n=0}^{N} W'_0 \cdots W'_{M+N-m-1}T_mB_{mn} \sum_{j=n}^{K-1}\sum_{k=0}^{K-1} d'_{jk}VWn - 1 \cdots W'_{j-1}F_k. \end{aligned}$$

These equalities and (4.17), (4.19) together imply $H'_0 = H_2H_1$. Since $I + H = I + H_1H_2$ and $I + H'_0 = I + H_2H_1$, Theorem 3 implies (4.18). ■

Lemma 7. *Let H'_0 be defined by* (4.17)*. Write*

$$H' := \sum_{m=0}^{M}\sum_{n=0}^{N} W'_N \cdots W'_{M+N-m-1}B'_{mn}W'_n \cdots W'_{N-1}, \tag{4.20}$$

$$H_1' := \sum_{m=0}^{M}\sum_{n=0}^{N} W_N' .. W_{M+N-m-1}' T_m \Big(A_{mn}' V^n - B_{mn} \sum_{j=n}^{K-1}\sum_{k=0}^{K-1} d_{jk}' V W_{n-1}' .. W_{j-1}' F_k \Big) \tag{4.21}$$

where

$$B_{mn}' := T_m \Big(A_{mn}' V W_{n-1}' - B_{mn} \sum_{j=n}^{K-1}\sum_{k=0}^{K-1} d_{jk}' V W_{n-1}' \cdots W_{j-1}' F_k W_0' \ldots W_{n-1}' \Big), \tag{4.22}$$

T_m, A_{mn}', B_{mn} *and* W_j' *are defined by* (4.10), (4.12), (4.13) *and* (4.16), *respectively. Then* $I + H_0'$ *is right invertible (left invertible, generalized invertible or invertible) if and only if so is* $I + H'$. *Moreover, if* $R_{H'} \in \mathcal{R}_{I+H'}$ ($L_{H'} \in \mathcal{L}_{I+H'}$, $W_{H'} \in \mathcal{W}_{I+H'}$ *or* $(I+H')^{-1} \in \mathcal{R}_{I+H'} \cap \mathcal{L}_{I+H'}$), *then*

$$\begin{aligned} R_{H_0'} &:= I - W_0' \cdots W_{N-1}' R_{H'} H_1' \in \mathcal{R}_{I+H_0'}, \\ L_{H_0'} &:= I - W_0' \cdots W_{N-1}' L_{H'} H_1' \in \mathcal{L}_{I+H_0'}, \\ W_{H_0'} &:= I - W_0' \cdots W_{N-1}' W_{H'} H_1' \in \mathcal{W}_{I+H_0'}, \\ (I+H_0')^{-1} &:= I - W_0' \cdots W_{N-1}' (I+H')^{-1} H_1'. \end{aligned} \tag{4.23}$$

Proof. It is easy to check that $H_0' = W_0' \cdots W_{N-1}' H_1'$ and $H' = H_1' W_0' \cdots W_{N-1}'$. Hence, the lemma is an immediate consequence of Theorem 3. ■

Lemmas 6 and 7 together imply the following

Corollary 3. *Let* H *and* H' *be defined by* (4.9) *and* (4.20), *respectively. Then* $I+H$ *is right invertible (left invertible, generalized invertible or invertible) if and only if so is* $I + H'$. *Moreover, if* $R_{H'} \in \mathcal{R}_{I+H'}$ ($L_{H'} \in \mathcal{L}_{I+H'}$, $W_{H'} \in \mathcal{W}_{I+H'}$ *or* $(I+H')^{-1} \in \mathcal{R}_{I+H'} \cap \mathcal{L}_{I+H'}$), *then*

$$\begin{aligned} R_H &:= I - H_1(I - W_0' \cdots W_{N-1}' R_{H'} H_1') H_2 \in \mathcal{R}_{I+H}, \\ L_H &:= I - H_1(I - W_0' \cdots W_{N-1}' L_{H'} H_1') H_2 \in \mathcal{L}_{I+H}, \\ W_H &:= I - H_1(I - W_0' \cdots W_{N-1}' W_{H'} H_1') H_2 \in \mathcal{W}_{I+H}, \\ (I+H)^{-1} &:= I - H_1(I - W_0' \cdots W_{N-1}' (I+H')^{-1} H_1') H_2, \end{aligned} \tag{4.24}$$

where H_1, H_2 *and* H_1' *are defined by* (4.14), (4.15) *and* (4.21), *respectively.*

Definition 4. *Let* H' *be given by* (4.20). *Then the operator* $I + H'$ *is called the resolving operator for the SMBVP* (0.1) – (0.2).

Theorem 7. *The SMBVP* (0.1) – (0.2) *is well-posed if and only if its resolving operator* $I+H'$ *is invertible. If this is the case, the unique solution of the SVBVP* (0.1) – (0.2) *is*

$$x = \big(I - S_0\big(I - W_0' \cdots W_{N-1}' (I+H')^{-1} H_1'\big) H_2\big) \Big(\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d_{jk}' W^j y_k + S_0 y_{M+N} \Big), \tag{4.25}$$

where

$$S_0 := I - \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} W^j F_k, \tag{4.26}$$

$$y_{M+N} := W^K W_K \cdots W_{M+N-1} y + \sum_{j=K}^{M+N-1} W^K W_K \cdots W_{j-1} y_j, \tag{4.27}$$

W'_j, H'_1 *and* H_2 *are defined by* (4.12), (4.21) *and* (4.15), *respectively.*

Proof. We have

$$\sum_{m=0}^{M} \sum_{n=0}^{N} V^m A_{mn} V^n x = y, \quad y \in Q[V] X_{M+N},$$

$$\Big(V^{M+N} + \sum_{m=1}^{M} \sum_{n=0}^{N} V^m A'_{mn} V^n\Big) x = y - \sum_{n=0}^{N} A'_{0n} V^n x,$$

$$\begin{aligned}
V^{M+N}\Big(I + &\sum_{m=1}^{M} \sum_{n=0}^{N} W^K W_K \cdots W_{M+N-m-1} A'_{mn} V^n \\
&- \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu H_0 \\
&- \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} W^j F_k \Big(I - \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu\Big) H_0\Big) x \\
=&y - \sum_{n=0}^{N} A'_{0n} V^n x \\
&\Big(I + \sum_{m=0}^{M} \sum_{n=0}^{N} W^K W_K \cdots W_{M+N-m-1} A'_{mn} V^n \\
&- \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu H_0 \\
&- \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} W^j F_k \Big(I - \sum_{\mu=K}^{M+N-1} W^K W_K \cdots W_{\mu-1} F_\mu V^\mu\Big) H_0\Big) x \\
=&W^K W_K \cdots W_{M+N-1} y + \sum_{j=0}^{K-1} W^j z_j + \sum_{j=K}^{M+N-1} W^K W_K \cdots W_{j-1} z_j \\
&\Big(I + \Big(I - \sum_{\mu=K}^{M+N-1} W^K W_K ... W_{\mu-1} F_\mu V^\mu - \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} W^j F_k \\
&\times \Big(I - \sum_{\mu=K}^{M+N-1} W^K W_K ... W_{\mu-1} F_\mu V^\mu\Big) H_0\Big)\Big) x \\
=&W^K W_K ... W_{M+N-1} y + \sum_{j=0}^{K-1} W^j z_j + \sum_{j=K}^{M+N-1} W^K W_K ... W_{j-1} z_j
\end{aligned}$$

$$(I+H)x = W^K W_K \cdots W_{M+N-1} y + \sum_{j=0}^{K-1} W^j z_j + \sum_{j=K}^{M+N-1} W^K W_K \cdots W_{j-1} z_j.$$

The formulae (4.11), (4.6) and the last equation together imply

$$(I+H)x = \sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk} W^j y_k + \Big(I - \sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk} W^j F_k\Big) y_{M+N},$$

$$(I+H)x = \sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk} W^j y_k + S_0 y_{M+N}. \tag{4.28}$$

Therefore the problem (0.1) − (0.2) is well-posed if and only if (4.28) has a unique solution, i.e., $I+H$ is invertible. On the other hand, by Corollary 3, $I+H$ is invertible if and only if $I+H'$ is invertible, and

$$(I+H)^{-1} = I - S_0\Big(I - W'_0 \cdots W'_{N-1}(I+H')^{-1}H'_1\Big)H_2. \tag{4.29}$$

(4.28) and (4.29) together imply (4.25). The proof is complete. ■

Now we consider ill-posed cases of the SMBVP (0.1) – (0.2).

Theorem 8. *Let $V \in R_1(X)$, $W \in \mathcal{R}_V^{(1)}$, $y \in Q[V]X_{M+N}$ and let $F_i \in \mathcal{F}_V$ and $G_i \in \mathcal{G}_V$ be right and left initial operators corresponding to $W_i \in \mathcal{R}_V^{(1)}$,* $\operatorname{Im} W_i \subset \operatorname{dom} W_{i-1}$, $(i = 1, \ldots, M+N-1)$. *Moreover, suppose that $F_0, \ldots, F_{K-1}$ possess the $c(W)$-property and they are linearly independent of $P_K(W)$ and $G_{i-1}y_i = 0$ for $i = K, \ldots, M+N-1$. Suppose that H, H', H'_1, H_2, S_0, y_{M+N} and W'_j are given by* (4.9), (4.20), (4.21), (4.15), (4.26), (4.27) *and* (4.12), *respectively.*

(i) *If the resolving operator $I+H'$ is right invertible and* $\dim\ker(I+H') \neq 0$, *then the SMBVP* (0.1) − (0.2) *is ill-posed and its solutions are*

$$x = \big(I - S_0(I - W'_0 \cdots W'_{N-1} R_{H'} H'_1)H_2\big)\Big(\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk} W^j y_k + S_0 y_{M+N}\Big) + z, \tag{4.30}$$

where $R_{H'} \in \mathcal{R}_{I+H'}$, and $z \in \ker(I+H)$ is arbitrary.

(ii) *If the resolving operator $I+H'$ is left invertible and* $\dim \operatorname{coker}(I+H') \neq 0$, *then the SMBVP* (0.1) − (0.2) *is ill-posed and it has a solution under the following necessary and sufficient condition*

$$\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk} W^j y_k + S_0 y_{M+N} \in (I+H)X_{M+N}. \tag{4.31}$$

If this is the case, then a unique solution of the SMBVP (0.1) − (0.2) *is given by*

$$x = \big(I - S_0\big(I - W'_0 \cdots W'_{N-1} L_{H'} H'_1\big)H_2\big)\Big(\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk} W^j y_k + S_0 y_{M+N}\Big), \tag{4.32}$$

where $L_{H'} \in \mathcal{L}_{I+H'}$.

(iii) *If the resolving operator* $I+H'$ *is generalized invertible,* $\dim\ker(I+H') \neq 0$ *and* $\dim \operatorname{coker}\,(I+H') \neq 0$, *then the SMBVP* (0.1) − (0.2) *is ill-posed and it has solutions if and only if the condition* (4.31) *is satisfied. If this is the case, then all solutions of the SMVBP are given by*

$$x = \big(I - S_0(I - W_0' \cdots W_{N-1}' W_{H'} H_1')H_2\big)\Big(\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk} W^j y_k + S_0 y_{M+N}\Big) + z, \tag{4.33}$$

where $W_{H'} \in \mathcal{W}_{I+H'}$, *and* $z \in \ker(I+H)$ *is arbitrary.*

Proof. By Theorem 6, the FMBVP (0.1) − (0.2) is equivalent to the equation (4.28). Corollary 3 implies that
(i) If $I+H'$ is right invertible then $I+H$ is right invertible on X_{M+N}. Formula (4.24) and the equation (4.28) together imply (4.30);
(ii) If $I+H'$ is left invertible then $I+H$ is now left invertible only. This implies that the problem (0.1) − (0.2) is solvable if and only if the condition (4.31) is satisfied. Formula (4.24) and the equation (4.28) together imply (4.32);
(iii) If $I+H'$ is generalized invertible but not one-sided invertible then $I+H$ is generalized invertible only. Hence, from (4.28) we conclude that the problem (0.1) − (0.2) is ill-posed and has solutions if and only if the condition (4.31) is satisfied. Formula (4.24) and the equation (4.28) imply (4.33). ■

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