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Total Stability and the Existence of Almost Periodic Integrals for Almost Periodic General Processes

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Abstract. A concept of total stability of the bounded integral for general processes is introduced. Total stability implies the existence of the almost periodic integral for almost periodic general processes. Furthermore, for linear processes, the equivalence relationship between total stability and uniform asymptotic stability are studied.

1. Introduction

The notion of processes introduced in [2–3] and [5–8] is a useful tool in the study of mathematical analysis for some phenomena whose dynamics is described by the equations which contains the derivative with respect to time variable. Indeed, Dafermos [2], Hale [5] and the authors [6–7] derived some stability properties for processes and applied those to get stability results and the existence of almost periodic solutions for some kinds of equations, including functional differential equations, partial differential equations and evolution equations.

To ensure the existence of almost periodic solutions of almost periodic systems, the concept of total stability (for this definition, see [11]) of a bounded

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solution plays an important role.

For general processes (equal to the concept of "general dynamical systems" in [1]), Bondi and Moauro [1] gave a definition of total stability of subset M of a metric space $\mathcal{X}$ and proved that uniform asymptotic stability of M implies total stability of M. For example, this definition implies the usual definition of total stability of the constant solution for systems, when the unperturbed is generated by an ordinary differential equation in $\mathbb{R}^n$ with the right hand side satisfying a Lipschitz condition. For details, see [1]. In this means, the concepts of the total stability of differential equations and of general processes are slightly different.

In this paper, at first, we extend the concept of total stability of a subset M of a metric space $\mathcal{X}$ given by Bondi and Moauro to the concept of the total stability of a bounded integral $\mu(t)$ of a general process $w(t,s,x)$ on a metric space $\mathcal{X}$ and show that the total stability of an integral $\mu(t)$ of an almost periodic general process $w(t,s,x)$ implies the existence of an almost periodic integral of the general process $w(t,s,x)$. Secondly, we show that uniform asymptotic stability of an integral $\mu(t)$ of a (not general) process $w(t,s,x)$ implies total stability of the integral $\mu(t)$. By Kato and Sibuya's example [9], it is known that the above result does not hold even for almost periodic general prosesses. Finally, for a linear process, we show that total stability of the equilibrium point zero implies uniform asymptotic stability of the equilibrium point zero, which corresponds to Massera's theorem [10] for ordinary differential equations.

2. Processes and Definitions of Total Stabilities

In this section, we shall give the concept of general processes and total stability property for general processes. Suppose that $\mathcal{X}$ is a separable metric space with metric d and let $w : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{X} \mapsto \mathcal{X}$, $\mathbb{R}^+ := [0,\infty)$ and $\mathbb{R} := (-\infty,\infty)$, be a function satisfying the following properties for all $t,\tau \in \mathbb{R}^+, s \in \mathbb{R}$ and $x \in \mathcal{X}$:

(p1) $w(0,s,x) = x$.

(p2) $w(t+\tau,s,x) = w(t,\tau+s,w(\tau,s,x))$.

(p3) the mapping $w(t,s,x) : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{X} \mapsto \mathcal{X}$ is continuous in t.

Then we call the mapping w *a general process* on $\mathcal{X}$.

If the condition (p3) is replaced by the condition

(p3') the mapping $w : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{X} \mapsto \mathcal{X}$ is continuous in (t,s,x),

then we call the mapping w *a processes* on $\mathcal{X}$ (cf., [2–3, 5–8]).

Denote by W the set of all general processes on $\mathcal{X}$. For $\tau \in \mathbb{R}$ and $w \in W$, we define the translation $\sigma(\tau)w$ of w by

$$(\sigma(\tau)w)(t,s,x) = w(t,\tau+s,x), \qquad (t,s,x) \in \mathbb{R}^+ \times \mathbb{R} \times \mathcal{X},$$

and set $\gamma_\sigma(w) = \bigcup_{t\in\mathbb{R}} \sigma(t)w$. Clearly $\gamma_\sigma(w) \subset W$.

We denote by $H_\sigma(w)$ the set of all functions $v : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{X} \mapsto \mathcal{X}$ such that for some sequence $\{\tau_n\} \subset \mathbb{R}$, $\{\sigma(\tau_n)w\}$ converges to v pointwise on $\mathbb{R}^+ \times \mathbb{R} \times \mathcal{X}$, that is, $\lim_{n\to\infty}(\sigma(\tau_n)w)(t,s,x) = v(t,s,x)$ for any $(t,s,x) \in \mathbb{R}^+ \times \mathbb{R} \times \mathcal{X}$. The set $H_\sigma(w)$ is considered as a topological space and it is called the hull of w.

Consider a general process w on $\mathcal{X}$ satisfying (p4) $H_\sigma(w) \subset W$.

Clearly, $H_\sigma(w)$ is invariant with respect to the translation $\sigma(\tau), \tau \in \mathbb{R}$. Now we suppose that $H_\sigma(w)$ is sequentially compact. Let $\Omega_\sigma(w)$ be the ω-limit set of w with respect to the translation semigroup $\sigma(t)$.

A continuous function $\mu : \mathbb{R}^+ \mapsto \mathcal{X}$ is called *an integral* on $\mathbb{R}^+$ of the general process w, if $w(t, s, \mu(s)) = \mu(t+s)$ for all $t, s \in \mathbb{R}^+$ (cf. [4, p.80]). In the following, we suppose that there exists an integral μ on $\mathbb{R}^+$ of the general process w satisfying

(p5) $\mu(t)$ is uniformly continuous on $\mathbb{R}^+$ and $O^+(\mu) = \{\mu(t) : t \in \mathbb{R}^+\}$ is relatively compact in $\mathcal{X}$.

For any $x_0 \in \mathcal{X}$ and $\varepsilon > 0$, we set $V_\varepsilon(x_0) = \{x \in \mathcal{X} : d(x, x_0) < \varepsilon\}$. We shall give the definition of total stability for the integral μ of the general process w which is given by Bondi and Moauro[1] for total stability of some set $M \subset \mathcal{X}$.

Definition 1. *The integral $\mu : \mathbb{R}^+ \mapsto \mathcal{X}$ of the general process w is said to be total stability* (P-TS) *if for any $\varepsilon > 0$ and $\bar{t} > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $s \in \mathbb{R}^+, w^* \in W$ and $w^*(t, s, V_{\delta_1}(\mu(s))) \in V_{\delta_2}(\mu(t+s))$ for $t \in [0, \bar{t}]$, then $w^*(t, s, V_{\delta_1}(\mu(s))) \in V_\varepsilon(\mu(t+s))$ for $t \geq 0$.*

3. Almost Periodic Integrals for Almost Periodic General Processes

In this section, we shall discuss an existence theorem for an almost periodic integral of almost periodic general processes. A general process $w : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{X} \mapsto \mathcal{X}$ is said to be almost periodic if $w(t, s, x)$ is almost periodic in s uniformly with respect to t, x in bounded sets. Let w be an almost periodic general process on $\mathcal{X}$. Bochner's theorem implies that $\Omega_\sigma(w) = H_\sigma(w)$ is a minimal set. Also, for any $v \in H_\sigma(w)$, there exists a sequence $\{\tau_n\} \subset \mathbb{R}^+$ such that $w(t, s + \tau_n, x) \to v(t, s, x)$ as $n \to \infty$, uniformly in $s \in \mathbb{R}$ and (t, x) in bounded sets of $\mathbb{R}^+ \times \mathcal{X}$. Consider a metric ρ on $H_\sigma(w)$ defined by

$$\rho(u, v) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\rho_n(u, v)}{1 + \rho_n(u, v)}$$

with $\rho_n(u, v) = \sup\{d(u(t, s, x), v(t, s, x)) : 0 \leq t \leq n, s \in \mathbb{R}, d(x, x_0) \leq n\}$, where x_0 is a fixed element in $\mathcal{X}$. Then $v \in H_\sigma(w)$ means that $\rho(\sigma(\tau_n)w, v) \to 0$ as $n \to \infty$, for some sequence $\{\tau_n\} \subset \mathbb{R}^+$.

Definition 2. *An integral $\mu(t)$ of the almost periodic general process $w(t, s, x)$ on $\mathbb{R}^+$ is said to be asymptotically almost periodic if it is a sum of a continuous almost periodic function $\phi(t)$ and a continuous function $\psi(t)$ defined on $\mathbb{R}^+$ which tends to zero as $t \to \infty$, that is*

$$\mu(t) = \phi(t) + \psi(t).$$

Let $\mu(t)$ be an integral on $\mathbb{R}^+$ which satisfies the condition (p5). From Ascoli-Arzéla's theorem and the sequential compactness of $H_\sigma(w)$, it follows that for any sequence $\{\tau'_n\} \subset \mathbb{R}^+$, there exist a subsequence $\{\tau_n\}$ of $\{\tau'_n\}$, a

$v \in H_\sigma(w)$ and a function $\nu : \mathbb{R}^+ \mapsto \mathcal{X}$ such that $\lim_{n\to\infty} \sigma(\tau_n)w = v$ and $\lim_{n\to\infty} \mu(t+\tau_n) = \nu(t)$ uniformly on any compact interval in $\mathbb{R}^+$. In this case, we write as

$$(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, v) \quad \text{compactly on} \quad \mathbb{R}^+,$$

for simplicity. Denote by $H(\mu, w)$ the set of all (ν, v) such that $(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, v)$ compactly on $\mathbb{R}^+$ for some sequence $\{\tau_n\} \subset \mathbb{R}^+$. Clearly, ν is an integral on $\mathbb{R}^+$ of v for any $(\nu, v) \in H(\mu, w)$. Likewise, for any sequence $\{\tau_n'\} \subset \mathbb{R}^+$ with $\tau_n' \to \infty$ as $n \to \infty$, there exist a subsequence $\{\tau_n\}$ of $\{\tau_n'\}$, a $v \in \Omega_\sigma(w)$ and a function $\nu : \mathbb{R} \to \mathcal{X}$ such that $\lim_{n\to\infty} \sigma(\tau_n)w = v$ and $\lim_{n\to\infty} \mu(t+\tau_n) = \nu(t)$ uniformly on any compact interval in $\mathbb{R}$. In this case, we write as

$$(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, v) \quad \text{compactly on} \quad \mathbb{R},$$

for simplicity. Denote by $\Omega(\mu, w)$ the set of all (ν, v) such that $(\mu^{\tau_n}, \sigma(\tau_n)w) \to (\nu, v)$ compactly on $\mathbb{R}$ for some sequence $\{\tau_n\} \subset \mathbb{R}^+$ with $\tau_n \to \infty$ as $n \to \infty$.

As noted in [7], $\mu(t)$ is asymptotically almost periodic if and only if it satisfies the following property:

(L) For any sequence $\{t_n'\}$ such that $t_n' \to \infty$ as $n \to \infty$ there exists a subsequence $\{t_n\}$ of $\{t_n'\}$ for which $\mu(t+t_n)$ converges uniformly on $\mathbb{R}^+$.

The following proposition is shown in [7] for processes. We can easily check that it holds good for general processes.

Proposition 1. *If the integral $\mu(t)$ on $\mathbb{R}^+$ of the almost periodic general process w is asymptotically almost periodic, then there exists an almost periodic integral of the general process w.*

The following theorem corresponds to [11, Corollary 16.1] which is shown for ordinary differential equations.

Theorem 1. *Suppose that $w(t, s, x)$ is an almost periodic general processes and $\mu : \mathbb{R}^+ \to \mathcal{X}$ is an integral of w on $\mathbb{R}^+$ which satisfies the condition* (p5). *If $\mu(t)$ is* P-TS, *then it is asymptotically almost periodic. Consequently there exists an almost periodic integral of the almost periodic general process w.*

Proof. Let $\{\tau_k\}$ be any sequence such that $\tau_k \to \infty$ as $k \to \infty$. It is sufficient to show that the sequence $\{\mu(\tau_k+t)\}$ has a subsequence which converges uniformly on $\mathbb{R}^+$.

There exists a $(\mu^*, w^*) \in \Omega(\mu, w)$ such that

$$(\mu^{\tau_k}, \sigma(\tau_k)w) \to (\mu^*, w^*) \quad \text{compactly on} \quad \mathbb{R}^+.$$

Let $\varepsilon > 0$, $\bar{t} > 0$ $\delta_1 > 0$ and $\delta_2 > 0$ be ones given for P-TS of $\mu(t)$. Then there exists a $k_0 > 0$ such that if $m \geq n \geq k_0$, then

$$d(\mu(\tau_m + t), \mu^*(t)) < \frac{\min(\delta_1, \delta_2)}{2} \quad \text{for} \quad t \in [0, \bar{t}]$$

and

$$d(\mu(\tau_n + t), \mu^*(t)) < \frac{\min(\delta_1, \delta_2)}{2} \quad \text{for} \quad t \in [0, \bar{t}].$$

Since $\mu^*(t) = \omega^*(t, 0, \mu^*(0))$ and μ is P-TS, we have

$$\omega^*(t, 0, \mu^*(0)) \in V_\varepsilon(\mu(\tau_m + t)) \quad \text{for} \quad t \geq 0$$

and

$$\omega^*(t, 0, \mu^*(0)) \in V_\varepsilon(\mu(\tau_n + t)) \quad \text{for} \quad t \geq 0.$$

That is,

$$d(\mu^*(t), \mu(\tau_m + t)) < \varepsilon \quad \text{for} \quad t \geq 0$$

and

$$d(\mu^*(t), \mu(\tau_n + t)) < \varepsilon \quad \text{for} \quad t \geq 0.$$

This implies

$$d(\mu(\tau_m + t), \mu(\tau_n + t)) < \varepsilon \quad \text{for all} \quad t \geq 0,$$

if $m \geq n \geq k_0$. This proves that $\mu(\tau_k + s)$ is uniformly convergent on $\mathbb{R}^+$ as $k \to \infty$. Thus $\mu(t)$ is asymptotically almost periodic. The existence of an almost periodic integral follows from Proposition 1, directly. ■

4. Total Stability and Uniform Asymtotic Stability

In this section, we assume that $w(t, s, x)$ is a process, that is, $w(t, s, x)$ satisfies conditions (p1), (p2) and (p3') and that condition (p4) is satisfies. For any $t \in \mathbb{R}^+$, we consider a function $\pi(t) : \mathcal{X} \times H_\sigma(w) \mapsto \mathcal{X} \times H_\sigma(w)$ defined by

$$\pi(t)(x, v) = (v(t, 0, x), \sigma(t)v)$$

for $(x, v) \in \mathcal{X} \times H_\sigma(w)$. $\pi(t)$ is called *the skew product flow of the process* w, if the following property holds true:

(p6) $\pi(t)(x, v)$ is continuous in $(t, x, v) \in \mathbb{R}^+ \times \mathcal{X} \times H_\sigma(w)$.

The skew product flow $\pi(t)$ is said to be *strongly asymptotically smooth* if, for any nonempty closed bounded set $B \subset \mathcal{X} \times H_\sigma(w)$, there exists a compact set $J \subset \mathcal{X} \times H_\sigma(w)$ with the property that $\{\pi(t_n)(y_n, \chi_n)\}$ has a subsequence which approaches to J whenever sequences $\{t_n\} \subset \mathbb{R}^+$ and $\{(y_n, \chi_n)\} \subset B$ satisfy $\lim_{n\to\infty} t_n = \infty$ and $\pi(t)(y_n, \chi_n) \in B$ for all $t \in [0, t_n]$. The strong asymptotic smoothness of $\pi(t)$ implies the asymptotic smoothness of $\pi(t)$ introduced in [4–5].

From (p6) we see that $\pi(\delta)(\mu(s), \sigma(s)w) = (w(\delta, s, \mu(s)), \sigma(s+\delta)w) = (\mu(s+\delta), \sigma(s+\delta)w)$ tends to $(\mu(s), \sigma(s)w)$ as $\delta \to 0^+$, uniformly for $s \in \mathbb{R}^+$; consequently, the integral μ on $\mathbb{R}^+$ must be uniformly continuous on $\mathbb{R}^+$.

In [7, Lemma 1], it is known that the process w satisfies the following condition

$(*)$ For any $\varepsilon > 0$ and $T > 0$ there exists a $\delta(\varepsilon) > 0$ with the property that $s \in \mathbb{R}^+, d(x, x') < \delta$ such that $d(w(t, s, x), w(t, s, x')) < \varepsilon$ for any $t \in [0, T]$.

Hence the following proposition follows from Bondi and Moauro's results [1, Theorem 2.4].

Proposition 2. *Assume that the process w satisfies the following conditions:*
$(**)$ *If for any $\varepsilon > 0$ there exist $\bar{t} > 0, \delta_1 > 0$ and $\delta_2 > 0$ such that if $s \in \mathbb{R}^+, w^* \in W$ and $w^*(t, s, V_{\delta_1}(\mu(s))) \in V_{\delta_2}(\mu(t+s))$ for $t \in [0, \bar{t}]$, then $w^*(t, s, V_{\delta_1}(\mu(s))) \in V_\varepsilon(\mu(t+s))$ for $t \geq 0$.*

Then the integral $\mu(t)$ of the process w is P-TS.

We shall give the definition of stabilities for the integral μ of the process w.

Definition 3. *The integral μ of the process w is said to be:*

(i) *uniformly stable* (US) (*resp. uniformly stable in $\Omega_\sigma(w)$) if for any $\varepsilon > 0$, there exists a $\delta := \delta(\varepsilon) > 0$ such that $w(t, s, V_\delta(\mu(s))) \subset V_\varepsilon(\mu(t+s))$ for $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$* (resp. *$\chi(t, s, V_\delta(\nu(s))) \subset V_\varepsilon(\nu(t+s))$ for $(\nu, \chi) \in \Omega_\sigma(\mu, w)$ and $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$*);

(ii) *uniformly asymptotically stable* (UAS) (*resp. uniformly asymptotically stable in $\Omega_\sigma(w)$), if it is* US (*resp.* US *in $\Omega_\sigma(w)$) and there exists a $\delta_0 > 0$ with the property that for any $\varepsilon > 0$, there is a $t_0 > 0$ such that $w(t, s, V_{\delta_0}(\mu(s))) \subset V_\varepsilon(\mu(t+s))$ for $t \geq t_0, \ s \in \mathbb{R}^+$ (resp. $\chi(t, s, V_{\delta_0}(\nu(s))) \subset V_\varepsilon(\nu(t+s))$ for $(\nu, \chi) \in \Omega_\sigma(\mu, w)$ and $t \geq t_0, \ s \in \mathbb{R}^+$*);

(iii) *attractive* (*resp. attractive in $\Omega_\sigma(w)$) if there is a $\delta_0 > 0$ such that for $y \in V_{\delta_0}(\mu(0))$ (resp. $y \in V_{\delta_0}(\nu(0))$ and $(\nu, \chi) \in \Omega_\sigma(\mu, w)$), $d(w(t, 0, y), \mu(t)) \to 0$ (resp. $d(\chi(t, 0, y), \nu(t)) \to 0$) as $t \to \infty$;*

(iv) *weakly uniformly asymptotically stable* (WUAS) *in $\Omega_\sigma(w)$ if it is US in $\Omega_\sigma(w)$ and attractive in $\Omega_\sigma(w)$.*

The following proposition is the special case of Theorem 1 in [6].

Proposition 3. *Suppose that w is a process on $\mathcal{X}$ for which $H_\sigma(w)$ is compact and that the skew product flow $\pi(t) : \mathcal{X} \times H_\sigma(w) \mapsto \mathcal{X} \times H_\sigma(w)$ of the process w is strongly asymptotically smooth. Also, suppose that $\mu : \mathbb{R}^+ \mapsto \mathcal{X}$ is an integral of w on $\mathbb{R}^+$ such that $\{\mu(t) : t \in \mathbb{R}^+\}$ is a relative compact subset of $\mathcal{X}$. Then the following statements are equivalent:*

(i) *The integral μ is* UAS.

(ii) *The integral μ is* US *and attractive in $\Omega_\sigma(w)$.*

(iii) *The integral μ is* UAS *in $\Omega_\sigma(w)$.*

(iv) *The integral μ is* WUAS *in $\Omega_\sigma(w)$.*

Theorem 2. *Suppose that w is a process on $\mathcal{X}$ for which $H_\sigma(w)$ is sequentially compact and that the skew product flow $\pi(t) : \mathcal{X} \times H_\sigma(w) \mapsto \mathcal{X} \times H_\sigma(w)$ is strongly asymptotically smooth. Also, suppose that $\mu : \mathbb{R}^+ \to \mathcal{X}$ is an integral of w on $\mathbb{R}^+$ such that $O^+(\mu)$ is a relatively compact subset of $\mathcal{X}$. Then, if the integral μ is* UAS, *then it is* P-TS.

Proof. We assume that the integral μ of the process w is UAS, but does not satisfy the condition $(**)$. Then there exist an $\varepsilon > 0$, $\varepsilon < \delta_0$, and a sequence $\{(\bar{t}_n, t_n, s_n, y_n, w_n)\}$ in $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{X} \times W$, such that $d(y_n, \mu(s_n)) \to 0$ and $\bar{t}_n \to \infty$ as $n \to \infty$,

$$d(w_n(t,s_n,y_n),\mu(t+s_n)) < \frac{1}{n}, \quad t \in [0,\bar{t}_n], \tag{1}$$

$d(w_n(t_n,s_n,y_n),\mu(t_n+s_n)) = \varepsilon$ and $d(w_n(t,s_n,y_n),\mu(t+s_n)) < \varepsilon$ for $t \in [0,t_n)$, where δ_0 is the one given for the attractivity in $\Omega_\sigma(w)$ of the integral μ of the process w. Take a positive constant γ, $\gamma < \varepsilon$, so that $\chi(t,s,V_\gamma(\nu(s)) \subset V_\varepsilon(\nu(t+s))$ for $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $(\nu,\chi) \in \Omega(\mu,w)$, which is possible by the uniform stability in $\Omega_\sigma(w)$ of the integral μ by Proposition 3. Since $d(y_n,\mu(s_n)) \to 0$ as $n \to \infty$, there exists a sequence $\{\tau_n\}$, $0 < \tau_n < t_n$, such that $d(w_n(\tau_n,s_n,y_n),\mu(\tau_n+s_n)) = \gamma/2$ and $d(w_n(t,s_n,y_n),\mu(t+s_n)) \geq \gamma/2$ for all $t \in [\tau_n,t_n]$.

We assert that $\tau_n \to \infty$ as $n \to \infty$. Suppose that the assertion is false. Then, without loss of generality, we may assume that $\tau_n \to \tau_0, \tau_0 < \bar{t}_n$ and $(\mu^{s_n},\sigma(s_n)w) \to (\tilde{\mu},\tilde{w})$ as $n \to \infty$, for some $\tau_0 < \infty$ and $(\tilde{\mu},\tilde{w}) \in H_\sigma(\mu,w)$. From (p6) it follows that $\pi(\tau_n)(y_n,\sigma(s_n)w)$ tends to $\pi(\tau_0)(\tilde{\mu}(0),\tilde{w})$ in $\mathcal{X} \times H_\sigma(w)$ as $n \to \infty$, which implies that $\sigma(s_n)w(\tau_n,0,y_n) = w(\tau_n,s_n,y_n)$ tends to $\tilde{w}(\tau_0,0,\tilde{\mu}(0)) = \tilde{\mu}(\tau_0)$ as $n \to \infty$. On the other hand, since $d(w_n(\tau_n,s_n,y_n),\mu(\tau_n+s_n)) = \gamma/2$ and

$$\begin{aligned}
&d(\tilde{w}(\tau_0,0,\tilde{\mu}(0)),\tilde{\mu}(\tau_0))\\
&\geq d(\tilde{w}(\tau_0,0,\tilde{\mu}(0)),w_n(\tau_n,s_n,y_n)) - d(w_n(\tau_n,s_n,y_n),\mu(\tau_n+s_n))\\
&\quad - d(\mu(\tau_n+s_n),\tilde{\mu}(0))\\
&\geq d(w_n(\tau_n,s_n,y_n),\mu(\tau_n+s_n)) - d(w(\tau_n,s_n,y_n),\tilde{w}(\tau_0,0,\tilde{\mu}(0)))\\
&\quad - d(\mu(\tau_n+s_n),\tilde{\mu}(\tau_0)) - d(w_n(\tau_n,s_n,y_n),\mu(\tau_n+s_n))\\
&\quad - d(\mu(\tau_n+s_n),\tilde{\mu}(0))
\end{aligned}$$

we must get $d(\tilde{w}(\tau_0,0,\tilde{\mu}(0)),\tilde{\mu}(\tau_0)) = r/2$ by (1), a contradiction.

Now we may assume that $(\mu^{\tau_n+s_n},\sigma(\tau_n+s_n)w) \to (\nu,\chi)$ as $n \to \infty$, for some $(\nu,\chi) \in \Omega_\sigma(\mu,w)$. Notice that $\pi(t)(y_n,\sigma(s_n)w) = (w(t,s_n,y_n),\sigma(t+s_n)w) \in V_{\delta_1}(\mu(t+s_n)) \times H\sigma(w)$ for $t \in [0,\tau_n]$. Since $\pi(t)$ is strongly asymptotically smooth, taking a subsequence if necessary, we can assume that $w(\tau_n,s_n,y_n) \to \tilde{y}$ for some $\tilde{y} \in \mathcal{X}$ as $n \to \infty$. Note that $\tilde{y} \in V_\gamma(\nu(0))$. We first consider the case where the sequence $\{t_n - \tau_n\}$ has a convergent subsequence. Without loss of generality, we can assume that $t_n - \tau_n \to \tilde{t}$ as $n \to \infty$, for some $\tilde{t} < \infty$. Then (p6) implies that $\pi(t_n-\tau_n)(w(\tau_n,s_n,y_n),\sigma(\tau_n+s_n)w) = (w(t_n,s_n,y_n),\sigma(t_n+s_n)w)$ tends to $\pi(\tilde{t})(\tilde{y},\chi)$ in $\mathcal{X} \times H_\sigma(w)$ as $n \to \infty$. Since

$$\begin{aligned}
&d((\sigma(\tau_n+s_n)w)(t_n-\tau_n,0,w(\tau_n,s_n,y_n)),\mu(t_n+s_n))\\
&= d(w(t_n-\tau_n,s_n+\tau_n,w(\tau_n,s_n,y_n)),\mu(t_n+s_n))\\
&= d(w(t_n,s_n,y_n),\mu(t_n+s_n)) = \varepsilon,
\end{aligned}$$

$$\begin{aligned}
d(\mu(t_n+s_n),\chi(\tilde{t},0,\tilde{y})) &\leq d(\mu(t_n-\tau_n+s_n+\tau_n),\mu(\tilde{t}+s_n+\tau_n))\\
&\quad + d(\mu(\tilde{t}+s_n+\tau_n),\nu(\tilde{t})) \to 0
\end{aligned}$$

and

$$\begin{aligned}
&d(w_n(t_n,s_n,y_n),\chi(\tilde{t},0,\tilde{y}))\\
&\leq d(w_n(t_n-\tau_n+\tau_n,s_n,y_n),w(t_n-\tau_n+\tau_n,s_n,y_n))
\end{aligned}$$

$$
\begin{aligned}
&\quad + d(w(t_n - \tau_n + \tau_n, s_n, y_n), \chi(\tilde{t}, 0, \tilde{y})) \\
&= d(w_n(t_n - \tau_n + \tau_n, s_n, y_n), w(t_n - \tau_n + \tau_n, s_n, y_n)) \\
&\quad + d(w(t_n - \tau_n, \tau_n + s_n, w(\tau_n, s_n, y_n)), \chi(\tilde{t}, 0, \tilde{y})) \\
&= d(w_n(t_n - \tau_n + \tau_n, s_n, y_n), w(t_n - \tau_n + \tau_n, s_n, y_n) \\
&\quad + d(\sigma(\tau_n + s_n)w(t_n - \tau_n, 0, w(\tau_n, s_n, y_n)), \chi(\tilde{t}, 0, \tilde{y})) \to 0
\end{aligned}
$$

as $n \to \infty$ by (p4) and (1), we get $d(\chi(\tilde{t}, 0, \tilde{y}), \nu(\tilde{t})) = \varepsilon$. This is a contradiction, because of $\chi(\tilde{t}, 0, \tilde{y}) \in \chi(\tilde{t}, 0, V_\gamma(\nu(0))) \subset V_\varepsilon(\nu(\tilde{t}))$. Thus we must have $\lim_{n\to\infty}(t_n - \tau_n) = \infty$. Now, letting $n \to \infty$ in the relation $d(\sigma(s_n + \tau_n)w(t, 0, w(\tau_n, s_n, y_n)), \mu(t + \tau_n + s_n)) = d(w(t + \tau_n, s_n, y_n), \mu(t + \tau_n + s_n)) \leq \varepsilon$ for $t \in [0, t_n - \tau_n]$, we get $d(\chi(t, 0, \tilde{y}), \nu(t)) \leq \varepsilon < \delta_0$ for all $t \geq 0$. Then, from the attractivity in $\Omega_\sigma(w)$ of the integral μ, it follows that $d(\chi(t, 0, \tilde{y}), \nu(t)) \to 0$ as $t \to \infty$. On the other hand, since $d(w(t + \tau_n, s_n, y_n), \mu(t + \tau_n + s_n)) \geq \gamma/2$ for $t \in [0, t_n - \tau_n]$, we must get $d(\chi(t, 0, \tilde{y}), \nu(t)) \geq \gamma/2$ for all $t \geq 0$; hence $d(\chi(t, 0, \tilde{y}), \nu(t)) \not\to 0$ as $t \to \infty$, a contradiction. Hence we have the conclusion by Proposition 2. ■

Finally, let $\mathcal{X}$ be a normed space with norm $|\,.\,|_{\mathcal{X}}$ and we consider a linear process, that is, we assume the following condition on the prosess w:

(p7) the process $w(t, s, x)$ is linear in x.

In this case, $w(t, s, 0) = 0$ for $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$. We say that 0 is an equilibrium point zero for the process w.

The following theorem corresponds to the results of Massera's [10] for ordinary differential equations.

Theorem 3. *Suppose that the condition* (p7) *is satisfied. If the equilibrium point zero of the process w is* P-TS, *then it is* UAS.

Proof. Let $\gamma > 0$. Put

$$w^*(t, s, x) = e^{\gamma t} w(t, s, x).$$

Since $w^*(0, s, x) = w(0, s, x) = x$ and $w^*(t + \tau, s, x) = e^{\gamma(t+\tau)} w(t + \tau, s, x) = e^{\gamma(t+\tau)} w(t, \tau{+}s, w(\tau, s, x)) = e^{\gamma t} w(t, \tau{+}s, e^{\gamma\tau} w(\tau, s, x)) = w^*(t, \tau{+}s, w^*(\tau, s, x))$ by the linearity of w, $w^*(t, s, x)$ is a process. It is easily seen that the condition (p7) and the uniform stability of equilibrium point zero of the process w imply that there exists a constant (independent of s) $K > 0$ such that

$$|w(t, s, x)|_{\mathcal{X}} \leq K|x|_{\mathcal{X}} \quad \text{for} \quad (t, s) \in \mathbb{R}^+ \times \mathbb{R}^+, x \in \mathcal{X}.$$

For any $\varepsilon > 0$, we can take $\eta_1 = \min(\delta_1, \delta_2/2Ke^{\gamma t})$, because for $x \in V_{\eta_1}(0)$ and $t \in [0, \overline{t}]$

$$
\begin{aligned}
w^*(t, s, x)|_{\mathcal{X}} &= w(t, s, e^{\gamma t} x) \\
&\leq \mathrm{K} e^{\gamma t} |x|_{\mathcal{X}} \leq K e^{\gamma \overline{t}} \eta_1 = \frac{\delta_2}{2} < \delta_2.
\end{aligned}
$$

Since the equilibrium point zero of the process w is P-TS, we have $|w^*(t,s,x)|_{\mathcal{X}} = |e^{\gamma t}w(t,s,x)|_{\mathcal{X}} < \varepsilon$ for $t \geq 0$. Hence $|w(t,s,x)|_{\mathcal{X}} \leq \varepsilon e^{-\gamma t}$ for $t \geq 0$. This completes the proof. ∎

By combining Theorems 2 and 3, we have the following.

Theorem 4. *Suppose that w is a process on $\mathcal{X}$ for which $H_\sigma(w)$ is sequentially compact and that the skew product flow $\pi(t) : \mathcal{X} \times H_\sigma(w) \mapsto \mathcal{X} \times H_\sigma(w)$ is strongly asymptotically smooth. Also, the condition* (p7) *is satisfied. Then, the equilibrium point zero of the process w is* UAS *if and only if it is* P-TS.

5. Application

Consider the equation

$$\begin{cases} u_{tt} = u_{xx} - u_t + f(t,x,u), & t > 0,\ 0 < x < 1, \\ u(0,t) = u(1,t) = 0, & t > 0, \end{cases} \tag{2}$$

where $f(t,x,u)$ is a function continuous in $(t,x,u) \in \mathbb{R} \times (0,1) \times \mathbb{R}$, and almost periodic in t uniformly with respect to x and u which satisfies

$$|f(t,x,u)| \leq \frac{1}{6}|u| + J.$$

and

$$|f(t,x,u_1) - f(t,x,u_2)|\frac{1}{6}|u_1 - u_2|$$

for all $(t,x,u),(t,x,u_1),(t,x,u_2), \in \mathbb{R} \times (0,1) \times \mathbb{R}$ and some constant $J > 0$. We consider a Banach space X given by $X = H_0^1(0,1) \times L^2(0,1)$ equipped with the norm $\|(u,v)\| = \{\|u_x\|_{L^2}^2 + \|v\|_{L^2}^2\}^{1/2} = \{\int_0^1 (u_x^2 + v^2)dx\}^{1/2}$. Then (2) can be considered as an abstract equation

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(t,x,u) \end{pmatrix} \tag{3}$$

in X, where A is a (unbounded) linear operator in X defined by

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} - v \end{pmatrix}$$

for $(u,v) \in H^2(0,1) \times H_0^1(0,1)$. It is well known that A generates a C_0 -semigroup of bounded linear operator on X. In [7], we have shown that each (mild) solution of (3) is bounded in the future, and that *the null solution of* (3) is UAS and that each solution of (3) has a compact orbit in X. The solution $\begin{pmatrix} u(t,s,u_1) \\ v(t,s,v_1) \end{pmatrix}$ is sail to be totally stable, if for any $\varepsilon > 0$ there exists a $\delta'(\varepsilon) > 0$ such that if $|g(t,x)| < \delta'(\varepsilon)$ on $[s,\infty) \times (0,1)$ and if $\|(u(s),v(s)) - (\overline{u}(s),\overline{v}(s))\| < \delta'(\varepsilon)$ for an $s \in \mathbb{R}^+$ then $\|(u(t),v(t)) - (\overline{u}(t),\overline{v}(t))\| < \varepsilon$ for all $t \geq s$, where $\begin{pmatrix} \overline{u}(t,s,\overline{u}_2) \\ \overline{v}(t,s,\overline{v}_2) \end{pmatrix}$ is a solution of

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(t,x,u)+g(t,x) \end{pmatrix} \tag{4}$$

through $\left(s, \begin{pmatrix} \overline{u}_2 \\ \overline{v}_2 \end{pmatrix}\right)$. Putting $w(t,s,x_1) = \begin{pmatrix} u(t+s,s,u_1) \\ v(t+s,s,v_1) \end{pmatrix}$ and $w^*(t,s,x_2) = \begin{pmatrix} \overline{u}(t+s,s,\overline{u}_2) \\ \overline{v}(t+s,s,\overline{v}_2) \end{pmatrix}$, w and w^* are processes generated by (3) and (4), respectively.

We shall show that the solution $\begin{pmatrix} u(t,s,u_1) \\ v(t,s,v_1) \end{pmatrix}$ of (3) is totally stable. One can apply Theorem 2 to the process $w(t,s,x_1)$ and conclude that the process $w(t,s,x_1)$ is P-TS. Let $s \in \mathbb{R}^+, \varepsilon > 0$, $\overline{t} > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ be the ones given by P-TS of $w(t,s,x_1)$. Take $\delta' = \delta_1$ and $\delta_2 = \frac{7C}{3}\delta' e^{\frac{1}{6}\overline{t}}$, where $C = \sup_{0\le t\le \overline{t}} \|e^{At}\|$. For $|g(t,x)| < \delta'(\varepsilon)$ on $[s,\infty)\times(0,1)$, $\|(u_1,v_1)-(\overline{u}_2,\overline{v}_2)\| < \delta'(\varepsilon)$ and $t \in [0,\overline{t}]$, we have

$$\|w(t,s,x_1) - w^*(t,s,x_2)\| = \left\| \begin{pmatrix} u(t+s,s,u_1) \\ v(t+s,s,v_1) \end{pmatrix} - \begin{pmatrix} \overline{u}(t+s,s,\overline{u}_2) \\ \overline{v}(t+s,s,\overline{v}_2) \end{pmatrix} \right\|$$

$$\le \|e^{At}\| \left\| \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} \overline{u}_1 \\ \overline{v}_2 \end{pmatrix} \right\| + \int_s^{t+s} \|e^{A(t+s-r)}\|\{f(r,x,u(r+s,s,u_1))$$

$$- f(r,x,\overline{u}(r+s,s,\overline{u}_2)) + |g(r,x)|\}dr$$

$$\le C\delta_1 + \frac{C}{6}\int_s^{t+s} \{|u(r+s,s,u_1) - \overline{u}(r+s,s,\overline{u}_2)| + \delta_1\}dr$$

$$\le C\Big\{\delta_1 + \frac{1}{6}\int_s^{t+s} \{\|w(r+s,s,x_1) - w^*(r+s,s,w^*,x_2)\| + \delta_1\}dr\Big\}$$

for $t \in [0,\overline{t}]$. It follows from Gronwall's inequality that

$$\|w(t,s,x_1) - w^*(t,s,x_2)\| \le C\frac{7}{6}\delta_1 e^{\frac{\overline{t}}{6}} = \frac{\delta_2}{2} < \delta_2.$$

Hence if $\|x_1 - x_2\| < \delta_2'$ and $|g(t,x)| < \delta_2'$, we have

$$\|w(t,s,x_1) - w^*(t,s,x_2)\| < \varepsilon$$

for $t \ge 0$, by P-TS of the process $w(t,s,x_1)$. That is $\begin{pmatrix} u(t,s,u_1) \\ v(t,s,v_1) \end{pmatrix}$ is the totally stable solution of (3).

References

1. P. Bondi and V. Moauro, Total stability for general dynamical systems, *Ricerche di Matematica* **25** (1976) 163–175.

2. C. M. Dafermos, An invariance principle for compact processes, *J. Differential Equations* **9** (1971) 239–252.
3. C. M. Dafermos, Almost periodic processes and almost periodic solutions of evolution equations, *Proceedings of University of Florida International Symposium*, Academic Press, New York, 1977, 43–57.
4. J. K. Hale, Theory of Functional Differential Equations, *Applied Math. Sciences* 3, Springer-Verlag, New York, 1977.
5. J. K. Hale, Asymptotic Behavior of Dissipative Systems, *Amer. Math. Soc.,* Providence, Rhode Island, 1988.
6. Y. Hino and S. Murakami, A generalization of processes and stabilities in abstract functional differential equations, *Funkcial. Ekvac.* **41** (1998) 235–255.
7. Y. Hino and S. Murakami, Almost periodic processes and the existence of almost periodic solutions, *Electronic J. of QTDE* **3** (1998) 1–19.
8. Y. Hino, T. Naito, N. V. Minh, and J. S. Shin, *Almost Periodic Solution of Differential Equations in Banach Spaces,* Taylor and Francis, 2002.
9. J. Kato and Y. Sibuya, Catastrophic deformation of a flow and non-existence of almost periodic solutions, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **24** (1977) 267–279.
10. J. L. Massera, Contributions to stability theory, *Ann. of Math.* **64** (1956) 182–206.
11. T. Yoshizawa, Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions, Springer-Verlag, New York, Heidelberg, Berlin, 1975.