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Some Invariant Manifolds for Abstract Functional Differential Equations and Linearized Stabilities

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Abstract. For abstract nonlinear autonomous functional differential equations, the existence of some invariant manifolds such as stable manifold and unstable manifold is established by using the variation-of-constants formula in the phase space which has recently been established in [12]. As an immediate consequence, a stability result on the zero solution of the nonlinear equation is derived, together with an instability result.

1. Introduction

In this paper, we are concerned with the abstract functional differential equation

$$\dot{u}(t) = Au(t) + L(u_t) + f(u_t), \tag{1}$$

where A is the infinitesimal generator of a strongly continuous compact semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$, u_t is an element of $\mathcal{B}$ defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in (-\infty, 0]$, $L : \mathcal{B} \mapsto \mathbb{X}$ is a bounded linear operator and $f \in C^1(\mathcal{B}; \mathbb{X})$ with $f(0) = f'(0) = 0$; here $\mathcal{B} = \mathcal{B}((-\infty, 0]; \mathbb{X})$ is the phase space for Eq. (1) which satisfies some fundamental axioms.

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The main purpose of this paper is to establish the existence of some invariant manifolds for Eq. (1) such as stable manifold and unstable manifold, by using the variation-of-constants formula in the phase space for Eq. (1) which has recently been established by Hino, Murakami, Naito and Minh in the paper [12]. On the existence of some invariant manifolds for Eq. (1), there are many research papers. Among them, we refer the reader to [1, 2, 4-7, 10, 14, 18] and references therein for more informations.

As a corollary of our main result, one can derive a stability result on the zero solution of Eq. (1) which is often called as the principle of linearized stability in the theory of ODEs; indeed, the zero solution of Eq. (1) is uniformly asymptotically stable if the zero solution of the linearized equation

$$\dot{u}(t) = Au(t) + L(u_t) \tag{2}$$

is uniformly asymptotically stable.

Furthermore, establishing the existence of the center-unstable manifold for Eq. (1), an instability result on the zero solution of Eq. (1) is derived under the situation that the characterisitic operator for Eq. (2) possesses a characteristic root with positive real part.

2. Phase Spaces and Some Preparatory Results

Throughout this paper, we will use the following notation: $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively. Also, $C(J, \mathbb{X})$ denotes the space of all $\mathbb{X}$-valued continuous functions on J, and $BC(J, \mathbb{X})$ denotes the subspace of $C(J, \mathbb{X})$ consisting of all bounded and continuous functions on J.

2.1. Phase Spaces

Now we will explain the phase space $\mathcal{B}$ employed throughout this paper. Let us denote the norm of $\mathbb{X}$ by $\|\cdot\|_{\mathbb{X}}$. For any function $x : (-\infty, a) \to \mathbb{X}$ and $t < a$, we define a function $x_t : \mathbb{R}^- := (-\infty, 0] \to \mathbb{X}$ by $x_t(s) = x(t+s)$ for $s \in \mathbb{R}^-$. A Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ which consists of functions $\psi : (-\infty, 0] \to \mathbb{X}$ is called a *fading memory space* if it satisfies the following axioms:

(A1) There exist a positive constant N and locally bounded functions $K(\cdot)$ and $M(\cdot)$ on $\mathbb{R}^+$ with the property that if $x : (-\infty, a) \mapsto \mathbb{X}$ is continuous on $[\sigma, a)$ with $x_\sigma \in \mathcal{B}$ for some $\sigma < a$, then for all $t \in [\sigma, a)$,

(i) $x_t \in \mathcal{B}$,

(ii) x_t is continuous in t (w.r.t. $\|\cdot\|_{\mathcal{B}}$),

(iii) $N\|x(t)\|_{\mathbb{X}} \le \|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup_{\sigma \le s \le t}\|x(s)\|_{\mathbb{X}} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,

(A2) If $\{\phi^k\}$, $\phi^k \in \mathcal{B}$, converges to ϕ uniformly on any compact set in $\mathbb{R}^-$ and if $\{\phi^k\}$ is a Cauchy sequence in $\mathcal{B}$, then $\phi \in \mathcal{B}$ and $\phi^k \to \phi$ in $\mathcal{B}$.

A fading memory space $\mathcal{B}$ is called a *uniform fading memory space*, if it satisfies (A1) and (A2) with $K(\cdot) \equiv K$ (a constant) and $M(\beta) \to 0$ as $\beta \to \infty$ in (A1). A typical example of uniform fading memory spaces is the following one:

$$C_\gamma := C_\gamma(\mathbb{X}) = \{\phi \in C(\mathbb{R}^-; \mathbb{X}) : \lim_{\theta \to -\infty} \frac{\|\phi(\theta)\|_{\mathbb{X}}}{e^{\gamma\theta}} = 0\}$$

which is equipped with the norm $\|\phi\|_{C_\gamma} = \sup_{\theta \le 0} \|\phi(\theta)\|_{\mathbb{X}}/e^{\gamma\theta}$, where γ is a negative constant.

It is known [8, Lemma 3.2] that if $\mathcal{B}$ is a uniform fading memory space, then $\mathrm{BC} := BC(\mathbb{R}^-; \mathbb{X}) \subset \mathcal{B}$ and the inclusion map from BC into $\mathcal{B}$ is continuous. For other properties of fading memory spaces and uniform fading memory spaces, we refer the reader to the book [11].

2.2. A Variation of Constants Formula for FDE

We consider the perturbed functional differential equation

$$\dot{u}(t) = Au(t) + L(u_t) + h(t) \tag{3}$$

of (2), where $h \in C(\mathbb{R}; \mathbb{X})$. Throughout the paper we shall assume that $L : \mathcal{B} \mapsto \mathbb{X}$ is a bounded linear operator with the form:

$$L(\phi) = \int_{-\infty}^{0} [d_\theta \eta(\theta)]\phi(\theta), \quad \phi \in C_{00},$$

where $\eta(\theta)$ is an $\mathcal{L}(\mathbb{X})$-valued function of locally bounded variation on $\mathbb{R}^-$; here C_{00} denotes the subspace of $C(\mathbb{R}^-, \mathbb{X})$ consisting of functions with compact support, and $\mathcal{L}(\mathbb{X})$ is the space of all bounded linear operators on $\mathbb{X}$.

For any $(\sigma, \phi) \in \mathbb{R} \times \mathcal{B}$, there exists a (unique) function $u : \mathbb{R} \mapsto \mathbb{X}$ such that $u_\sigma = \phi$, u is continuous on $[\sigma, \infty)$ and the following relation holds:

$$u(t) = T(t-\sigma)\phi(0) + \int_{\sigma}^{t} T(t-s)\{L(u_s) + h(s)\}ds, \quad t \ge \sigma.$$

The function u is called a *(mild) solution* of (3) through (σ, ϕ) on $[\sigma, \infty)$, and denoted by $u(\cdot, \sigma, \phi; h)$. When J is an interval in $\mathbb{R}$, a function v is called a solution of Eq. (3) on J, if $v_t \in \mathcal{B}$ is defined for all $t \in J$ and if it satisfies $u(t, \sigma, v_\sigma; f) = v(t)$ for all t and σ in J with $t \ge \sigma$. Also, a solution v of Eq. (3) on J is said to be *J-bounded* if $\sup_{t \in J} \|v_t\|_{\mathcal{B}} < \infty$.

For any $t \ge 0$, we define an operator $V(t)$ on $\mathcal{B}$ by

$$V(t)\phi = u_t(0, \phi; 0), \quad \phi \in \mathcal{B}.$$

We can easily see that $(V(t))_{t \ge 0}$ is a strongly continuous semigroup of bounded linear operators on $\mathcal{B}$, which is called the *solution semigroup* of (2).

For any $n \in \mathbb{N}$ we consider a function Γ^n defined by

$$\Gamma^n(\theta) = \begin{cases} (n\theta + 1)I, & -1/n \le \theta \le 0 \\ 0, & \theta < -1/n, \end{cases}$$

where I is the identity operator on $\mathbb{X}$. It follows from (A1) that if $x \in \mathbb{X}$, then $\Gamma^n x \in \mathcal{B}$ with $\|\Gamma^n x\|_{\mathcal{B}} \le K(1)\|x\|_{\mathbb{X}}$.

The following theorem yields a representation formula for solutions of (3) in the phase space $\mathcal{B}$, which plays an important role in the establishment of some invariant manifolds for (1) in the next section:

Theorem 2.1. [12] *The segment* $u_t(\sigma,\phi;h)$ *of solution* $u(\cdot,\sigma,\phi,h)$ *of* (3) *satisfies the following relation in* $\mathcal{B}$*:*

$$u_t(\sigma,\phi;h) = V(t,\sigma)\phi + \lim_{n\to\infty}\int_\sigma^t V(t,s)\Gamma^n h(s)ds, \quad t \geq \sigma.$$

3. Some Invariant Manifolds for Functional Differential Equations

Throughout the rest of this paper we will make as a *standing assumption* that $\mathcal{B}$ is a uniform fading memory space, A is the generator of a compact semigroup $(T(t))_{t\geq 0}$, $L : \mathcal{B} \mapsto \mathbb{X}$ is a bounded linear operator satisfying the assumption stated in Sec. 2 and $f \in C^1(\mathcal{B};\mathbb{X})$ with $f(0) = f'(0) = 0$.

We say that the zero solution of the linearized equation

$$\dot{u}(t) = Au(t) + L(u_t) \tag{4}$$

is *hyperbolic*, if for any $\lambda \in \mathbb{C}$ with $\Re\lambda = 0$, the null space of the characteristic operator $A-\lambda I-L(e^{\lambda\cdot}I)$ of (4) is trivial (equivalently, the characteristic operator $A - \lambda I - L(e^{\lambda\cdot}I)$ is invertible in $\mathcal{L}(\mathbb{X})$). In this section, we will establish some (local) invariant manifolds for the functional differential equation

$$\dot{u}(t) = Au(t) + L(u_t) + f(u_t) \tag{5}$$

in each case where the zero solution of (4) is hyperbolic or it is not always hyperbolic.

We now consider the set

$$\Sigma = \{\Re\lambda \geq 0 : A - \lambda I - L(e^{\lambda\cdot}I) \quad \text{has a nontrivial null space}\}.$$

As was explained in [12, Sec. 5], correponding to the set Σ the space $\mathcal{B}$ is decomposed as a direct sum

$$\mathcal{B} = S \oplus CN \oplus U,$$

where S, CN and U are some closed subspaces of $\mathcal{B}$ which are invariant under the solution semigroup $(V(t))_{t\geq 0}$. In particular, CN and U are finite dimensional subspaces of $\mathcal{B}$ which correspond to the subset $\Sigma^{CN} = \{\lambda \in \Sigma : \Re\lambda = 0\}$ and $\Sigma^U = \{\lambda \in \Sigma : \Re\lambda > 0\}$, respectively. Henceforth, we use the notation

$$V(t)|_S = V^S(t), V(t)|_{CN} = V^C(t),$$
$$V(t)|_U = V^U(t), V(t)|_{CN\oplus U} = V^{C,U}(t),$$

and denote by Π^S (or Π^C, Π^U, $\Pi^{C,U}$) the projection from $\mathcal{B}$ onto S (or CN, U, $CN \oplus U$, respectively) along to the above decomposition. The operator norm of $V^S(t)$ decays exponentially as $t \to \infty$, and $V^{C,U}(t)$ are extendable for all $t \in \mathbb{R}$

as a group of linear operators on the finite dimensional space $CN \oplus U$. The operators $V^S(t)$, $V^C(t)$ and $V^U(t)$ satisfy the following estimates:

$$\exists\, C \geq 1,\ \alpha > \epsilon > 0 : \begin{cases} \|V^S(t)\| \leq Ce^{-\alpha t} & (\forall t \geq 0) \\ \|V^U(t)\| \leq Ce^{\alpha t} & (\forall t \leq 0) \\ \|V^C(t)\| \leq Ce^{\epsilon|t|} & (\forall t \in \mathbb{R}) \end{cases}$$

Henceforth, we set

$$C_1 = \|\Pi^S\| + \|\Pi^{C,U}\|.$$

3.1. Stable Manifold and Unstable Manifold

In this subsection, we assume that the zero solution of (4) is hyperbolic, and hence $CN = \{0\}$.

As in Sec. 2, the function u which satisfies

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)\{L(u_s) + f(u_s)\}ds,$$
$$u_0 = \phi \in \mathcal{B}$$

is called a (mild) solution of (5) and is denoted by $u(0, \phi; f)$. For any positive constant δ, we consider the set defined by

$$W^s(\delta) = \{\phi \in \mathcal{B} : \|\Pi^S\phi\|_{\mathcal{B}} < \delta/(2C), \quad \|u_t(0, \phi; f)\|_{\mathcal{B}} < \delta \quad (\forall t \geq 0)\}.$$

The set $W^s(\delta)$ is called the *local stable manifold* of (5). In fact, as the following theorem shows, $W^s(\delta)$ for some δ is a manifold which is homeomorphic to an open subset of S, and moreover $W^s(\delta)$ is tangent to S at zero; that is,

$$\lim_{\phi \in W^s(\delta),\ \phi \to 0} \frac{\|\Pi^U\phi\|_{\mathcal{B}}}{\|\Pi^S\phi\|_{\mathcal{B}}} = 0.$$

In what follows, for any $a > 0$ we set $\mathcal{B}_a = \{\phi \in \mathcal{B} : \|\phi\|_{\mathcal{B}} < a\}$.

Theorem 3.1. *Assume that the zero solution of* (4) *is hyperbolic and that* $f \in C^1(\mathcal{B}; \mathbb{X})$ *with* $f(0) = f'(0) = 0$. *Then there is a positive constant* δ *with the following properties:*

(i) *The map* Π^S *is a Lipschitz homeomorphism from* $W^s(\delta)$ *to* $\Pi^S(\mathcal{B}) \cap \mathcal{B}_{\delta/(2C)}$,

(ii) $W^s(\delta)$ *is tangent to* S *at zero,*

(iii) *there are positive constants* M, β *such that*

$$\|u_t(0, \phi; f)\|_{\mathcal{B}} \leq Me^{-\beta t}\|\phi\|_{\mathcal{B}}, \quad t \geq 0,\ \phi \in W^s(\delta),$$

(iv) $W^s(\delta)$ *is locally positively invariant for* (5); *that is,* $u_\tau(0, \phi; f) \in W^s(\delta)$ *whenever* $\phi \in W^s(\delta)$ *and* $\Pi^S u_\tau(0, \phi; f) \in \mathcal{B}_{\delta/(2C)}$ *for some* $\tau \geq 0$.

In order to establish the theorem, we need the following lemma.

Lemma 3.2. *Let* $\psi \in S$*. If* u *is an* $\mathbb{R}^+$*-bounded solution of* (5) *with* $\Pi^S u_0 = \psi$ *and* $\sup_{t\geq 0} \|f(u_t)\|_{\mathbb{X}} < \infty$*, then*

$$u_t = V(t)\psi + \lim_{n\to\infty} \int_0^t V(t-s)\Pi^S \Gamma^n f(u_s) ds + \lim_{n\to\infty} + \int_\infty^t V(t-s)\Pi^U \Gamma^n f(u_s) ds, \quad t \geq 0.$$

Conversely, if $y \in BC(\mathbb{R}^+;\mathcal{B})$ *satisfies the relation* $\sup_{t\geq 0} \|f(y(t)\|_{\mathbb{X}} < \infty$ *and*

$$y(t) = V(t)\psi + \lim_{n\to\infty} \int_0^t V(t-s)\Pi^S \Gamma^n f(y(s)) ds + \lim_{n\to\infty} + \int_\infty^t V(t-s)\Pi^U \Gamma^n f(y(s)) ds, \qquad t \geq 0,$$

then the function ξ *defined by*

$$\xi(t) = \begin{cases} [y(t)](0), & t \geq 0 \\ [y(0)](t), & t < 0 \end{cases}$$

is an $\mathbb{R}^+$*-bounded solution of* (5) *with* $\Pi^S \xi_0 = \psi$ *and* $\xi_t = y(t)$ *in* $\mathcal{B}$ *for all* $t \geq 0$*.*

Proof. First, we shall establish the former part of the lemma. Set $h(t) = f(u_t)$ for $t \geq 0$. Then $h \in BC(\mathbb{R}^+;\mathbb{X})$, and u is a solution of (3) on $\mathbb{R}^+$. By virtue of Theorem 2.1, we get

$$u_t = V(t-\sigma)u_\sigma + \lim_{n\to\infty} \int_\sigma^t V(t-s)\Gamma^n h(s) ds, \quad \forall t \geq \sigma \geq 0.$$

Therefore

$$\Pi^S u_t = V(t-\sigma)\Pi^S u_\sigma + \lim_{n\to\infty} \int_\sigma^t V(t-s)\Pi^S \Gamma^n h(s) ds,$$

and hence

$$\Pi^S u_t = V(t)\psi + \lim_{n\to\infty} \int_0^t V(t-s)\Pi^S \Gamma^n h(s) ds, \quad \forall t \geq 0.$$

Also, we get

$$\Pi^U u_t = V(t-\sigma)\Pi^U u_\sigma + \lim_{n\to\infty} \int_\sigma^t V(t-s)\Pi^U \Gamma^n h(s) ds,$$

or

$$\Pi^U u_\sigma = V(\sigma - t)\Big[\Pi^U u_t - \lim_{n\to\infty}\int_\sigma^t V(t-s)\Pi^U\Gamma^n h(s)ds\Big]$$
$$= V(\sigma - t)\Pi^U u_t - \lim_{n\to\infty}\int_\sigma^t V(\sigma - s)\Pi^U\Gamma^n h(s)ds, \quad \forall t \ge \sigma \ge 0.$$

Since $\|V(\sigma - t)\Pi^U u_t\|_{\mathcal{B}} \le CC_1 e^{\alpha(\sigma - t)}\sup_{t\ge 0}\|u_t\|_{\mathcal{B}} \to 0$ as $t \to \infty$, it follows that

$$\Pi^U u_\sigma = -\lim_{t\to\infty}\lim_{n\to\infty}\int_\sigma^t V(\sigma - s)\Pi^U\Gamma^n h(s)ds, \quad \forall \sigma \ge 0.$$

Observe that if $t_1 \ge t_2 \ge \sigma$, then

$$\Big\|\int_\sigma^{t_1} V(\sigma - s)\Pi^U\Gamma^n h(s)ds - \int_\sigma^{t_2} V(\sigma - s)\Pi^U\Gamma^n h(s)ds\|_{\mathcal{B}}$$
$$= \Big\|\int_{t_1}^{t_2} V(\sigma - s)\Pi^U\Gamma^n h(s)ds\|_{\mathcal{B}} \le (CC_1K_0/\alpha)\|h\|e^{\alpha(\sigma - t_2)} \to 0$$

as $t_2 \to \infty$, where $K_0 = K(1)$ and $\|h\| = \sup_{t\ge 0}\|h(t)\|_{\mathbb{X}}$. Therefore the limit

$$\lim_{t\to\infty}\int_\sigma^t V(\sigma - s)\Pi^U\Gamma^n h(s)ds = \int_\sigma^\infty V(\sigma - s)\Pi^U\Gamma^n h(s)ds$$

converges in $\mathcal{B}$ uniformly for n. On the other hand, the sequence $\{\int_\sigma^\infty V(t-s)\Pi^U\Gamma^n h(s)ds\}_{n\ge 1}$ must converge in $\mathcal{B}$, because it is a Cauchy sequence in $\mathcal{B}$ since

$$\limsup_{n,\ m\to\infty}\Big\|\int_\sigma^\infty V(\sigma - s)\Pi^U\Gamma^n h(s)ds - \int_\sigma^\infty V(\sigma - s)\Pi^U\Gamma^m h(s)ds\Big\|_{\mathcal{B}}$$
$$\le (2CC_1K_0/\alpha)\|h\|e^{\alpha(\sigma - t)}$$
$$+ \limsup_{n,\ m\to\infty}\Big\|\Pi^U\Big(\int_\sigma^t V(\sigma - s)\Gamma^n h(s)ds - \int_\sigma^t V(\sigma - s)\Gamma^m h(s)ds\Big)\Big\|_{\mathcal{B}}$$
$$= (2CC_1K_0/\alpha)\|h\|e^{\alpha(\sigma - t)}$$

for any $t > \sigma$. Hence

$$\lim_{n\to\infty}\int_\sigma^\infty V(\sigma-s)\Pi^U\Gamma^n h(s)ds = \lim_{n\to\infty}\lim_{t\to\infty}\int_\sigma^t V(\sigma-s)\Pi^U\Gamma^n h(s)ds$$
$$= \lim_{n\to\infty}\lim_{n\to infty}\int_\sigma^t V(\sigma-s)\Pi^U\Gamma^n h(s)ds$$
$$= -\Pi^U u_\sigma$$

or

$$\Pi^U u_t = \lim_{n\to\infty}\int_\infty^t V(t-s)\Pi^U\Gamma^n h(s)ds, \quad \forall t\geq 0.$$

Thus

$$\begin{aligned} u_t &= \Pi^S u_t + \Pi^U u_t \\ &= V(t)\psi + \lim_{n\to\infty}\int_0^t V(t-s)\Pi^S\Gamma^n h(s)ds \\ &\quad + \lim_{n\to\infty}\int_\infty^t V(t-s)\Pi^U\Gamma^n h(s)ds, \qquad \forall t\geq 0, \end{aligned}$$

which proves the former part of the lemma.

Next, we shall prove the latter part of the lemma. Set $g(t) = f(y(t))$ for $t\geq 0$. Then $g\in BC(\mathbb{R}^+;\mathbb{X})$, and moreover by the same reasoning as in the proof of the former part of the lemma, the limit $\lim_{n\to\infty}\int_\infty^t V(t-s)\Pi^U\Gamma^n g(s)ds$ converges in $\mathcal{B}$ for each $t\geq 0$. Let $v(t)$ be the solution of Eq. (3) with $h = g$ on $\mathbb{R}^+$ and $v_0 = y(0)$. By virtue of Theorem 2.1, we get

$$\begin{aligned} v_t &= V(t)y(0) + \lim_{n\to\infty}\int_0^t V(t-s)\Gamma^n g(s)ds \\ &= V(t)\{\psi + \lim_{n\to\infty}\int_\infty^0 V(-s)\Pi^U\Gamma^n g(s)ds\} + \lim_{n\to\infty}\int_0^t V(t-s)\Gamma^n g(s)ds \\ &= V(t)\psi + \lim_{n\to\infty}\int_\infty^0 V(t-s)\Pi^U\Gamma^n g(s)ds + \lim_{n\to\infty}\int_0^t V(t-s)(\Pi^S+\Pi^U)\Gamma^n g(s)ds \\ &= V(t)\psi + \lim_{n\to\infty}\int_0^t V(t-s)\Pi^S\Gamma^n g(s)ds + \lim_{n\to\infty}\int_\infty^t V(t-s)\Pi^U\Gamma^n g(s)ds \\ &= y(t) \end{aligned}$$

for $t \geq 0$. In particular, we get

$$v(t) = [y(t)](0) = \xi(t), \qquad \forall t \geq 0,$$

and hence ξ is continuous on $\mathbb{R}^+$. It follows from (A1) that $\xi_t \in \mathcal{B}$ and $\|\xi_t - y(t)\|_{\mathcal{B}} = 0$ for all $t \geq 0$ because of

$$\begin{aligned}\|\xi_t - y(t)\|_{\mathcal{B}} &= \|\xi_t - v_t\|_{\mathcal{B}} \\ &\leq K \sup_{0 \leq s \leq t} \|\xi(s) - v(s)\|_X + M(t)\|\xi_0 - y(0)\|_{\mathcal{B}} \\ &= 0.\end{aligned}$$

Also, it follows that

$$\Pi^S \xi_0 = \Pi^S y(0) = \Pi^S(\psi + \lim_{n\to\infty} \int_{\infty}^{0} V(-s)\Pi^U \Gamma^n h(s) ds) = \psi.$$

Moreover, we get

$$\xi(t) = v(t) = u(t, 0, \xi_0; h), \quad \forall t \geq 0,$$

and hence ξ is a solution on $\mathbb{R}^+$ of (3) with $h = g$. Since $g(t) = f(y(t)) = f(\xi_t)$ for $t \geq 0$, we see that ξ is a solution of (5) on $\mathbb{R}^+$, which completes the proof of the latter part of the lemma. ■

Now we are in a position to prove Theorem 3.1. Since $f \in C^1(\mathcal{B}; \mathbb{X})$ with $f(0) = f'(0) = 0$, there exist a constant $\delta_0 > 0$ and a nonincreasing continuous function $\zeta : [0, \delta_0] \mapsto [0, \infty)$ with the property that $\zeta(0) = 0$ and

$$\|f(\phi) - f(\psi)\|_{\mathbb{X}} \leq \zeta(\sigma)\|\phi - \psi\|_{\mathcal{B}}, \qquad \forall \phi,\ \psi \in \mathcal{B}_\sigma.$$

Choose a constant $\delta \in (0, \delta_0)$ so small that

$$8C^2 C_1 K_0 \zeta(\delta) < \alpha.$$

Let $\psi \in S$ with $\|\psi\|_{\mathcal{B}} < \delta/(2C)$ be given. We claim that there is a unique $\mathbb{R}^+$-bounded solution u of (5) satisfying $\Pi^S u_0 = \psi$ and $\sup_{t\geq 0} \|u_t\|_{\mathcal{B}} < \delta$. To check the claim, we define a closed subset $\mathcal{S}(\psi, \delta)$ of the Banach space $BC(\mathbb{R}^+; \mathcal{B})$ (equipped with the supremum norm $\|\cdot\|$) by

$$\mathcal{S}(\psi, \delta) = \{y \in BC(\mathbb{R}^+; \mathcal{B}) : \Pi^S y(0) = \psi \quad \text{and} \quad \|y\| \leq \delta\},$$

and consider a mapping $\mathcal{T} : \mathcal{S}(\psi, \delta) \mapsto BC(\mathbb{R}^+; \mathcal{B})$ defined by

$$\begin{aligned}(\mathcal{T}y)(t) = V(t)\psi &+ \lim_{n\to\infty} \int_0^t V(t-\tau)\Pi^S \Gamma^n f(y(\tau)) d\tau \\ &+ \lim_{n\to\infty} \int_{\infty}^{t} V(t-\tau)\Pi^U \Gamma^n f(y(\tau)) d\tau, \quad t \geq 0.\end{aligned}$$

We assert that $\mathcal{T}$ is a contraction on $\mathcal{S}(\psi,\delta)$. Let $y \in (\mathbb{R}^+;\mathcal{B})$. By the same reasoning as in the proof of the latter part of Lemma 3.2, we see that $(\mathcal{T}y)(t) = u_t(0,(\mathcal{T}y)(0);h)$ with $h(t) = f(y(t))$, and hence $\mathcal{T}y \in C(\mathbb{R}^+;\mathcal{B})$. Also, since $\|f(y(t)\|_{\mathbb{X}} \le \zeta(\delta)\|y(t)\|_{\mathcal{B}}$ and $\|\Gamma^n x\|_{\mathcal{B}} \le K_0\|x\|_{\mathbb{X}}$ for $x \in \mathbb{X}$, we get

$$\begin{aligned}\mathcal{T}y)(t)\|_{\mathcal{B}} &= Ce^{-\alpha t}\|\psi\|_{\mathcal{B}} + CC_1K_0\zeta(\delta)\int_0^t e^{-\alpha(t-\tau)}\|y(\tau)\|_{\mathcal{B}}d\tau \\ &\quad + CC_1K_0\zeta(\delta)\int_t^\infty e^{\alpha(t-\tau)}\|y(\tau)\|_{\mathcal{B}}d\tau \\ &\le C\|\psi\|_{\mathcal{B}} + CC_1K_0\zeta(\delta)(2\delta/\alpha) \\ &< \delta(1/2 + 2CC_1K_0\zeta(\delta)/\alpha) \le (3/4)\delta,\end{aligned}$$

and hence $\mathcal{T}y \in \mathcal{S}(\psi,\delta)$. Thus $\mathcal{T}(\mathcal{S}(\psi,\delta)) \subset \mathcal{S}(\psi,\delta)$.

Next, let y and z be the elements in $\mathcal{S}(\psi,\delta)$. Then

$$\begin{aligned}\|(\mathcal{T}y)(t) - (\mathcal{T}z)(t)\|_{\mathcal{B}} &= CC_1K_0\zeta(\delta)\int_0^t e^{-\alpha(t-\tau)}\|y(\tau)-z(\tau)\|_{\mathcal{B}}d\tau \\ &\quad + CC_1K_0\zeta(\delta)\int_t^\infty e^{\alpha(t-\tau)}\|y(\tau)-z(\tau)\|_{\mathcal{B}}d\tau \\ &\le \left(2CC_1K_0\zeta(\delta)/\alpha\right)\|y-z\| \le (1/4)\|y-z\|,\end{aligned}$$

or

$$\|\mathcal{T}y - \mathcal{T}z\| \le (1/4)\|y-z\|.$$

Thus, $\mathcal{T}$ is a contraction on $\mathcal{S}(\psi,\delta)$. By virtue of the contraction mapping theorem, the mapping $\mathcal{T}$ has a unique fixed point in $\mathcal{S}(\psi,\delta)$ which we denote by $y^*(\cdot,\psi)$. Then it follows from Lemma 3.2 that the function u defined by $u(t) = [y^*(t,\psi)](0)$ if $t \ge 0$, and $u(t) = [y^*(0,\psi)](t)$ if $t < 0$, is a unique $\mathbb{R}^+$-bounded solution of (5) satisfying $\Pi^S u_0 = \psi$ and $\sup_{t\ge 0}\|u_t\|_{\mathcal{B}} < \delta$.

For each $\psi \in S$ with $\|\psi\|_{\mathcal{B}} < \delta/(2C)$, let $y^*(\cdot,\psi)$ be the one ensured in the above paragraph. It is obvious that $y^*(t,0) \equiv 0$. We will establish the following estimate:

$$\|y^*(t,\psi) - y^*(t,\bar\psi)\|_{\mathcal{B}} \le 2Ce^{-(\alpha/2)t}\|\psi-\bar\psi\|_{\mathcal{B}}, \ \ \psi,\ \bar\psi \in S\cap\mathcal{B}_{\delta/(2C)},\ \ t\ge 0. \quad (6)$$

Indeed, it follows that

$$\begin{aligned}&\|y^*(t,\psi) - y^*(t,\bar\psi)\|_{\mathcal{B}} = \|(\mathcal{T}y^*(\cdot,\psi))(t) - (\mathcal{T}y^*(\cdot,\bar\psi))(t)\|_{\mathcal{B}} \\ &= \|V(t)(\psi-\bar\psi) + \lim_{n\to\infty}\int_0^t V(t-\tau)\Pi^S\Gamma^n(f(y^*(\tau,\psi)) - f(y^*(\tau,\bar\psi)))d\tau \\ &\quad + \lim_{n\to\infty}\int_\infty^t V(t-\tau)\Pi^U\Gamma^n(f(y^*(\tau,\psi)) - f(y^*(\tau,\bar\psi)))d\tau\|_{\mathcal{B}}\end{aligned}$$

$$\le Ce^{-\alpha t}\|\psi-\bar{\psi}\|_{\mathcal{B}} + CC_1K_0\zeta(\delta)\int_0^t e^{-\alpha(t-\tau)}\|y^*(\tau,\psi)-y^*(\tau,\bar{\psi})\|_{\mathcal{B}}d\tau$$
$$+ CC_1K_0\zeta(\delta)\int_t^\infty e^{\alpha(t-\tau)}\|y^*(\tau,\psi)-y^*(\tau,\bar{\psi})\|_{\mathcal{B}}d\tau,$$

and hence letting $\mu := 2CC_1\zeta(\delta)K_0/\alpha$ $(< 1/4)$ we get by [5, Lemma 6.2, p.110] that

$$\begin{aligned}\|y^*(t,\psi)-y^*(t,\bar{\psi})\|_{\mathcal{B}} &\le [1/(1-\mu)]C\|\psi-\bar{\psi}\|_{\mathcal{B}} \times e^{-(\alpha - CC_1\zeta(\delta)K_0/(1-\mu))|t|} \\ &\le 2Ce^{-(\alpha/2)t}\|\psi-\bar{\psi}\|_{\mathcal{B}}, \quad t \ge 0,\end{aligned}$$

which is the desired estimate.

Now we set

$$G\psi = y^*(0,\psi) = \psi + \lim_{n\to\infty}\int_\infty^0 V(-\tau)\Pi^U\Gamma^n f(y^*(\tau,\psi))d\tau$$

for $\psi \in S \cap B_{\delta/(2C)}$. G is a mapping from $S\cap B_{\delta/(2C)}$ to $\mathcal{B}_\delta$. In fact, from Lemma 3.2 and the claim in this proof we see that the range of G exactly equals the set $W^s(\delta)$. Also, it follows from (6) that $\|G\psi - G\bar{\psi}\|_{\mathcal{B}} \le 2C\|\psi-\bar{\psi}\|_{\mathcal{B}}$, and hence G is Lipschitz continuous.

We will establish the following estimate:

$$\|\psi-\bar{\psi}\|_{\mathcal{B}} \le 2\|G\psi - G\bar{\psi}\|_{\mathcal{B}}, \quad \psi,\ \bar{\psi} \in S \cap \mathcal{B}_{\delta/(2C)}. \tag{7}$$

Indeed, by using (6) we get

$$\begin{aligned}&\|G\psi - G\bar{\psi}\|_{\mathcal{B}} \\ &\ge \|\psi-\bar{\psi}\|_{\mathcal{B}} - CC_1K_0\zeta(\delta)\int_0^\infty e^{-\alpha\tau}\|y^*(\tau,\psi)-y^*(\tau,\bar{\psi})\|_{\mathcal{B}}d\tau \\ &\ge \|\psi-\bar{\psi}\|_{\mathcal{B}}\{1 - CC_1K_0\zeta(\delta)\int_0^\infty 2Ce^{-3\alpha\tau/2}d\tau\} \\ &\ge \|\psi-\bar{\psi}\|_{\mathcal{B}}\{1 - 4C^2C_1K_0\zeta(\delta)/(3\alpha)\} \ge (1/2)\|\psi-\bar{\psi}\|_{\mathcal{B}},\end{aligned}$$

which implies the estimate (7).

From the estimate (7) we see that G is a Lipschitz homeomorphism from $S\cap\mathcal{B}_{\delta/(2C)}$ onto $W^s(\delta)$. Since $\Pi^S G\psi = \psi$ for $\psi \in S\cap\mathcal{B}_{\delta/(2C)}$, it follows that Π^S equals the inverse mapping G^{-1} of G. Thus Property (i) holds true.

Set $\psi = G^{-1}\phi$ for $\phi \in W^s(\delta)$. Since

$$(1/2)\|\psi\|_{\mathcal{B}} \le \|G\psi\|_{\mathcal{B}} \le 2C\|\psi\|_{\mathcal{B}}$$

by (6), (7) and the fact that $G0 = 0$, we get

$$\begin{aligned}\|u_t(0,\phi;f)\|_{\mathcal{B}} &= \|y^*(t,\psi)\|_{\mathcal{B}} \le 2Ce^{-(\alpha/2)t}\|\psi\|_{\mathcal{B}} \\ &\le 4Ce^{-(\alpha/2)t}\|G\psi\|_{\mathcal{B}} = 4Ce^{-(\alpha/2)t}\|\phi\|_{\mathcal{B}},\end{aligned}$$

which implies Property iii) with $M = 4C$ and $\beta = \alpha/2$.

Observe that $\phi\ (\in W^s(\delta)) \to 0$ if and only if $\psi := G^{-1}\phi \in (S \cap \mathcal{B}_{\delta/(2C)}) \to 0$. Therefore, if $\phi\ (\in W^s(\delta)) \to 0$, then

$$\begin{aligned}\|\Pi^U\phi\|_{\mathcal{B}}/\|\Pi^S\phi\|_{\mathcal{B}} &= \|\Pi^U G\psi\|_{\mathcal{B}}/\|\Pi^S G\psi\|_{\mathcal{B}} \\ &= \|\lim_{n\to\infty}\int_{\infty}^{0} V(-\tau)\Pi^U\Gamma^n f(y^*(\tau,\psi))d\tau\|_{\mathcal{B}}/\|\psi\|_{\mathcal{B}} \\ &\le (2C^2C_1K_0/\alpha)\zeta(2C\|\psi\|_{\mathcal{B}}) \to 0,\end{aligned}$$

where we used the estimate (6). Thus Property (ii) holds true.

Finally, we will check Property (iv). Set $\chi = \Pi^S u_\tau(0,\phi;f)$. Then $\chi \in S \cap \mathcal{B}_{\delta/(2C)}$, and $z(t) := u(t+\tau,0,\phi;f)$ is an $\mathbb{R}^+$-bounded solution of (5). By the first claim in this proof, one can easily see that $z_t = y^*(t,\chi)$ for all $t \ge 0$, and hence $u_\tau(0,\phi;f) = z_0 = y^*(0,\chi) = G\chi \in W^s(\delta)$, as required. This completes the proof of Theorem 3.1. ■

Next we shall establish the local unstable manifold of Eq. (5). For any positive δ, we consider the set defined by

$$\begin{aligned}W^u(\delta) = \{\phi \in \mathcal{B} : \|\Pi^U\phi\|_{\mathcal{B}} < \delta/(2C),\ \text{and}\ \ \exists\mathbb{R}^-\text{ -bounded solution } u \\ \text{of Eq. (5) such that}\ \ u_0 = \phi\ \ \text{and}\ \ \|u_t\|_{\mathcal{B}} < \delta\ \ (\forall t \le 0)\}.\end{aligned}$$

The set $W^u(\delta)$ is called the *local unstable manifold* of (5). In fact, if δ is sufficiently small, for any $\phi \in \mathcal{B}$ with $\|\Pi^U\phi\|_{\mathcal{B}} < \delta/(2C)$, Eq. (5) possesses one and only one solution u with the property that $\mathbb{R}^-$-bounded, $u_0 = \phi$ and $\|u_t\|_{\mathcal{B}} < \delta$ for all $t \le 0$. In what follows, we denote the solution by $u(\cdot,0,\phi;f)$, again.

Theorem 3.3. *Assume that the zero solution of* (4) *is hyperbolic and that* $f \in C^1(\mathcal{B};\mathbb{X})$ *with* $f(0) = f'(0) = 0$. *Then there is a positive constant* δ *with the following properties:*

(i) *The map* Π^U *is a Lipschitz homeomorphism from* $W^u(\delta)$ *to* $\Pi^U(\mathcal{B})\cap\mathcal{B}_{\delta/(2C)}$,

(ii) $W^u(\delta)$ *is tangent to* U *at zero,*

(iii) *there are positive constants* M, β *such that*

$$\|u_t(0,\phi;f)\|_{\mathcal{B}} \le Me^{\beta t}\|\phi\|_{\mathcal{B}}, \quad t \le 0,\ \phi \in W^u(\delta),$$

(iv) $W^u(\delta)$ *is locally negatively invariant for* (5); *that is,* $u_\tau(0,\phi;f) \in W^u(\delta)$ *whenever* $\phi \in W^u(\delta)$ *and* $\Pi^U u_\tau(0,\phi;f) \in \mathcal{B}_{\delta/(2C)}$ *for some* $\tau \le 0$.

Indeed, by utilizing the following lemma which is a counterpart of Lemma 3.2, one can prove the theorem by modifying slightly the proof of Theorem 3.1.

So, we omit the proof of the above theorem, together with the proof of the lemma.

Lemma 3.4. *Let $\psi \in U$. If u is an $\mathbb{R}^-$-bounded solution of* (5) *with $\Pi^U u_0 = \psi$ and $\sup_{t\geq 0} \|f(u_t)\|_{\mathbb{X}} < \infty$, then*

$$u_t = V(t)\psi + \lim_{n\to\infty} \int_0^t V(t-s)\Pi^U \Gamma^n f(u_s)ds$$
$$+ \lim_{n\to\infty} \int_{-\infty}^t V(t-s)\Pi^S \Gamma^n f(u_s)ds, \quad t \leq 0.$$

Conversely, if $y \in BC(\mathbb{R}^-; \mathcal{B})$ satisfies the relation $\sup_{t\leq 0} \|f(y(t))\|_{\mathbb{X}} < \infty$ and

$$y_t = V(t)\psi + \lim_{n\to\infty} \int_0^t V(t-s)\Pi^U \Gamma^n f(y(s))ds$$
$$+ \lim_{n\to\infty} \int_{-\infty}^t V(t-s)\Pi^S \Gamma^n f(y(s))ds, \quad t \leq 0,$$

then the function ξ defined by

$$\xi(t) = [y(t)](0), \quad t \leq 0$$

is an $\mathbb{R}^-$-bounded solution of (5) *with $\Pi^U \xi_0 = \psi$ and $\xi_t = y(t)$ in $\mathcal{B}$ for all $t \leq 0$.*

3.2. Center-Unstable Manifold

In this subsection, we will treat the case where the zero solution of (4) is not always hyperbolic.

Let $d = \dim(CN \oplus U)$. Take a basis $\{\phi_1(s), \cdots, \phi_d(s)\}$ in $CN \oplus U$, and set $\Phi := (\phi_1(s), \cdots, \phi_d(s))$. Then there exist d elements $\psi_1, \cdots, \psi_d$ in $\mathcal{B}^*$ (the dual space of $\mathcal{B}$) such that $\langle \psi_i, \phi_j \rangle = 1$ if $i = j$, 0 if $i \neq j$, and that $\psi_i = 0$ on S. Here and hereafter, $\langle\ ,\ \rangle$ denotes the canonical pairing between the dual space and the original space. Denote by Ψ the transpose of $(\psi_1, \cdots, \psi_d)$ to use the expression $\langle \Psi, \Phi \rangle = I_d$ (here I_d is the $d \times d$ unit matrix). Then the projection operator $\Pi^{C,U}$ is given by

$$\Pi^{C,U}\phi = \Phi\langle \Psi, \phi \rangle, \quad \phi \in \mathcal{B}.$$

Since $(V^{C,U}(t))_{t\geq 0}$ is a strongly continuous semigroup on the finite dimensional space $CN \oplus U$, there is a $d \times d$ matrix G such that

$$V^{C,U}(t)\Phi = \Phi e^{Gt} \quad (\forall t \geq 0),$$

where $\sigma(G) = \Sigma$. As shown in [12, Proposition 4.2], the $(CN \oplus U)$-component of the solution of Eq. (5) relates to a solution of some ordinary differential equation. Indeed, if $u(t)$ is a solution of Eq. (5), then the function $y(t)$ determined by $\Phi y(t) = \Pi^{C,U} u_t$ is a solution of the ordinary differential equation

$$\dot{y}(t) = Gy(t) + \langle x^*, f(\Phi y(t) + \Pi^S u_t)\rangle, \tag{8}$$

where x^* is a d-column vector in $\mathbb{X}^*$ which is determined by Ψ through the relation

$$\langle x^*, x\rangle = \lim_{n\to\infty} \langle \Psi, \Gamma^n x\rangle, \quad x \in \mathbb{X};$$

also, refer to [13].

The set $W^{cu}(V)$ which we will give in the next theorem is called the *local center-unstable manifold* of (5).

Theorem 3.5. *Assume that $f \in C^1(\mathcal{B}; \mathbb{X})$ with $f(0) = f'(0) = 0$. Then there is an open neighborhood V of 0 in $\mathcal{B}$ and a Lipschitz continuous mapping $H : CN \oplus U \mapsto S$, $H(0) = 0$, with the following properties:*

(i) *The set $W^{cu}(V) := \{\xi = \phi + H(\phi) : \phi \in (CN \oplus U) \cap V\}$ is homeomorphic to $(CN \oplus U) \cap V$,*

(ii) *$W^{cu}(V)$ is tangent to $CN \oplus U$ at zero,*

(iii) *$W^{cu}(V)$ is locally invariant for Eq. (5); that is,*

(a) *for any $\xi \in W^{cu}(V)$ there is some $t_\xi > 0$ such that $u_t(0, \xi; f) \in W^{cu}(V)$ for $|t| \le t_\xi$,*

(b) *if $\xi \in W^{cu}(V)$ and $u_t(0, \xi; f) \in V$ on $[0, b]$, then $u_t(0, \xi; f) \in W^{cu}(V)$ on $[0, b]$,*

(iv) *if u is a solution of Eq. (5) on an interval J with $u_t \in W^{cu}(V)$ on J, then the function $y(t)$ determined by $\Phi y(t) = \Pi^{C,U} u_t$ satisfies the ordinary differential equation*

$$\dot{y}(t) = Gy(t) + \langle x^*, f(\Phi y(t) + H(\Phi y(t)))\rangle, \quad t \in J. \tag{9}$$

Conversely, if $y(t)$ satisfies the equation (9) on an interval J with $\Phi y(t) \in V$ on J, then there is a solution u of (5) on J such that $u_t \in W^{cu}(V)$ and $\Pi^{C,U} u_t = \Phi y(t)$ on J.

Proof. The theorem can be established by almost the same argument employed in the proof of [7, Theorem 10.2.1, pp.314–316] (cf. [2]). In what follows, we will give the proof of the theorem for completeness.

Let $\chi : \mathbb{R}^d \mapsto [0, 1]$ be a C^∞ function with $\chi(y) = 1$ if $|y| \le 1$, and $\chi(y) = 0$ if $|y| > 2$. For any $\delta > 0$ we set

$$\tilde{f}_\delta(y, \phi^S) = f\Big(\chi(\frac{y}{\delta})\Phi y + \phi^S\Big)$$

for all $(y, \phi^S) \in \mathbb{R}^d \times S$, where $\mathbb{R}^d \times S$ is equipped with the norm $\|(y, \phi^S)\| = |y| + \|\phi^S\|_{\mathcal{B}}$. There is a positive constant C_0 such that $\|\Phi y\|_{\mathcal{B}} \le C_0|y|$ for $y \in \mathbb{R}^d$. Define constants $\bar{C}$ and C_2 by

$$\bar{C} = 2C_0 + 1, \quad C_2 = 1 + C_0 \sup_{|y|\le 2} |(d/dy)(y\chi(y))|.$$

It is not difficult to check the following inequalities:

$$\|\tilde{f}_\delta(y,\phi^S)\|_{\mathbb{X}} \le \zeta(\bar{C}\delta)\min\{\bar{C}\delta, C_0|y| + \|\phi^S\|_{\mathcal{B}}\}, \tag{10}$$

$$\|\tilde{f}_\delta(y,\phi^S) - \tilde{f}_\delta(z,\psi^S)\|_{\mathbb{X}} \le C_2\zeta(\bar{C}\delta)\|(y,\phi^S) - (z,\psi^S)\|_{\mathcal{B}}. \tag{11}$$

In the above, (y,ϕ^S) and (z,ψ^S) are any elements in $\mathbb{R}^d \times (S \cap \mathcal{B}_\delta)$, and ζ is the one appeared in the proof of Theorem 3.1.

Now we take a positive constant δ such that

$$\zeta(\bar{C}\delta) < \min\{1,\ \alpha/(CC_1\bar{C}K_0),\ (\alpha-\epsilon)/(2CC_2(\|x^*\| + CC_1K_0)),$$
$$\alpha/(4CC_1C_2K_0),\ \epsilon(\alpha-\epsilon)/(2CC_2\|x^*\|(\epsilon + 4CC_1C_2K_0))\},$$

where $K_0 := K(1)$. Let us consider a closed subset $\mathcal{M}$ of the Banach space $BC(\mathbb{R}^d, S)$ (equipped with the supremum norm $\|\cdot\|$) defined by

$$\mathcal{M} = \{h \in BC(\mathbb{R}^d, S) : h(0) = 0, \|h\| \le \delta,$$
$$\|h(y) - h(\bar{y})\|_{\mathcal{B}} \le |y - \bar{y}| \quad (\forall y,\ \bar{y} \in \mathbb{R}^d)\}.$$

Let $h \in \mathcal{M}$ be given. For any $y_0 \in \mathbb{R}^d$ the ordinary differential equation

$$\dot{y} = Gy + \langle x^*, \tilde{f}_\delta(y, h(y))\rangle \tag{12}$$

has a unique solution on $\mathbb{R}$ such that $y(0) = y_0$ (which will be denoted by $y(\cdot, y_0; h)$). Set

$$(\mathcal{G}h)(y_0) = \lim_{n\to\infty} \int_{-\infty}^{0} V(-\tau)\Pi^S\Gamma^n \tilde{f}_\delta(y(\tau, y_0; h), h(y(\tau, y_0; h)))d\tau, \qquad y_0 \in \mathbb{R}^d.$$

Then $\mathcal{G}h$ is a function mapping $\mathbb{R}^d$ into S. We assert that $\mathcal{G}\mathcal{M} \subset \mathcal{M}$. Since $y(\cdot, 0; h) \equiv 0$, it follows that $(\mathcal{G}h)(0) = 0$. Also, we get $\|\tilde{f}_\delta(y(\tau, y_0; h), h(y(\tau, y_0; h)))\|_{\mathbb{X}} \le \zeta(\bar{C}\delta)\bar{C}\delta$ by (10), and hence $\|(\mathcal{G}h)(y_0)\|_{\mathcal{B}} \le (CC_1\bar{C}K_0\delta/\alpha)\zeta(\bar{C}\delta) < \delta$. Furthermore, since $\|e^{G\tau}\| \le Ce^{-\epsilon\tau}$ for $\tau \le 0$, using the inequality (11) and applying Gronwall's inequality one can see that

$$|y(\tau, y_0; h) - y(\tau, z_0; h)| \le C|y_0 - z_0|e^{-\tau(\epsilon + 2CC_2\|x^*\|\zeta(\bar{C}\delta))}, \quad \forall \tau \le 0, \tag{13}$$

from which it follows that

$$\begin{aligned}
&\|(\mathcal{G}h)(y_0) - (\mathcal{G}h)(z_0)|_{\mathcal{B}} \\
&\le 2CC_1C_2K_0\zeta(\bar{C}\delta) \int_{-\infty}^{0} e^{\alpha\tau}|y(\tau, y_0; h) - y(\tau, z_0; h)|d\tau \\
&\le \Big(2C^2C_1C_2K_0\zeta(\bar{C}\delta)/(\alpha - \epsilon - 2CC_2\|x^*\|\zeta(\bar{C}\delta))\Big)|y_0 - z_0| \\
&\le |y_0 - z_0|
\end{aligned}$$

for any $y_0,\ z_0 \in \mathbb{R}^d$. Thus $\mathcal{G}\mathcal{M} \subset \mathcal{M}$, as required.

We next assert that $\mathcal{G}$ is a contraction on $\mathcal{M}$. Indeed, using the inequality (11) and applying Gronwall's inequality again, one can see that

$$\begin{aligned}&|y(\tau, y_0; h_1) - y(\tau, y_0; h_2)| \\ &\quad \le \|h_1 - h_2\|(CC_2\|x^*\|\zeta(\bar{C}\delta)/\epsilon)e^{-\tau(\epsilon+2CC_2\|x^*\|\zeta(\bar{C}\delta))}, \quad \forall \tau \le 0,\end{aligned}$$

and

$$\begin{aligned}&\|(\mathcal{G}h_1)(y_0) - (\mathcal{G}h_2)(y_0)\|_{\mathcal{B}} \\ &\le CC_1C_2K_0\zeta(\bar{C}\delta)\int_{-\infty}^{0} e^{\alpha\tau}\{2|y(\tau, y_0; h_1) - y(\tau, y_0; h_2)| + \|h_1 - h_2\|\}d\tau \\ &\le (2C^2C_1C_2^2\|x^*\|K_0\zeta(\bar{C}\delta)/\epsilon)\|h_1 - h_2\| \int_{-\infty}^{0} e^{(\alpha-\epsilon-2CC_2\|x^*\|\zeta(\bar{C}\delta))\tau}d\tau \\ &\quad + (CC_1C_2K_0\zeta(\bar{C}\delta)/\alpha)\|h_1 - h_2\| \\ &\le (1/4)\|h_1 - h_2\| + (1/4)\|h_1 - h_2\| = (1/2)\|h_1 - h_2\|\end{aligned}$$

for any $y_0 \in \mathbb{R}^d$ and $h_1,\ h_2 \in \mathcal{M}$. Consequently, $\mathcal{G}$ is a contraction on $\mathcal{M}$, as required. By virtue of the contraction mapping theorem, the mapping $\mathcal{G}$ has a unique fixed point in $\mathcal{M}$ which we denote by h^*. Then

$$h^*(y_0) = \lim_{n\to\infty} \int_{-\infty}^{0} V(-\tau)\Pi^S\Gamma^n \tilde{f}_\delta(y(\tau, y_0; h^*), h^*(y(\tau, y_0; h^*)))d\tau, \quad \forall y_0 \in \mathbb{R}^d.$$

Noting that $y(\tau, y(t, y_0; h^*); h^*) = y(t + \tau, y_0; h^*)$, we get

$$\begin{aligned}h^*(y(t, y_0; h^*)) &= \lim_{n\to\infty} \int_{-\infty}^{0} V(-\tau)\Pi^S\Gamma^n \tilde{f}_\delta(y(t + \tau, y_0; h^*), h^*(y(t + \tau, y_0; h^*)))d\tau \\ &= \lim_{n\to\infty} \int_{-\infty}^{t} V(t - \tau)\Pi^S\Gamma^n \tilde{f}_\delta(y(\tau, y_0; h^*), h^*(y(\tau, y_0; h^*)))d\tau.\end{aligned}$$

Hence it follows that

$$\begin{aligned}&\Phi y(t, y_0; h^*) + h^*(y(t, y_0; h^*)) \\ &= \Phi\Big(e^{Gt}y_0 + \int_0^t e^{G(t-s)}\langle x^*, \tilde{f}_\delta(y(s, y_0; h^*), h^*(y(s, y_0; h^*)))\rangle ds\Big) \\ &\qquad + h^*(y(t, y_0; h^*))\end{aligned}$$

$$= V(t)(\Phi y_0) + \int_0^t V(t-s)\Phi\langle x^*, \tilde{f}_\delta(y(s,y_0;h^*), h^*(y(s,y_0;h^*)))\rangle ds$$
$$+ h^*(y(t,y_0;h^*))$$
$$= V(t)(\Phi y_0) + \lim_{n\to\infty}\int_0^t V(t-s)\Phi\langle \Psi, \Gamma^n\tilde{f}_\delta(y(s,y_0;h^*), h^*(y(s,y_0;h^*)))\rangle ds$$
$$+ h^*(y(t,y_0;h^*))$$
$$= V(t)\Pi^{C,U}(\Phi y_0) + \lim_{n\to\infty}\int_0^t V(t-s)\Pi^{C,U}\Gamma^n\tilde{f}_\delta(y(s,y_0;h^*), h^*(y(s,y_0;h^*)))ds$$
$$+ \lim_{n\to\infty}\int_{-\infty}^t V(t-s)\Pi^S\Gamma^n\tilde{f}_\delta(y(s,y_0;h^*), h^*(y(s,y_0;h^*)))ds.$$

Therefore, if we define a $\bar{f}_\delta \in C(\mathcal{B};\mathbb{X})$ by

$$\bar{f}_\delta(\Phi y + \phi^S) = \tilde{f}_\delta(y, \phi^S), \quad (y, \phi^S) \in \mathbb{R}^d \times S,$$

then the $\mathcal{B}$-valued function $\nu(t) := \Phi y(t,y_0;h^*) + h^*(y(t,y_0;h^*))$ satisfies the equation

$$\nu(t) = V(t)\Pi^{C,U}\nu(0) + \lim_{n\to\infty}\int_0^t V(t-s)\Pi^{C,U}\Gamma^n\bar{f}_\delta(\nu(s))ds$$
$$+ \lim_{n\to\infty}\int_{-\infty}^t V(t-s)\Pi^S\Gamma^n\bar{f}_\delta(\nu(s))ds, \quad t \in \mathbb{R},$$

and by the same reasoning as in the latter part of the Lemma 3.2. we get

$$\nu(t) = V(t-\sigma)\nu(\sigma) + \lim_{n\to\infty}\int_\sigma^t V(t-s)\Gamma^n\bar{f}_\delta(\nu(s))ds, \quad \forall t \geq \sigma. \tag{14}$$

Then, by virtue of [12, Theorem 4.2], it follows that $u(t) := [\nu(t)](0)$ is a solution of Eq. (3) with $h(t) = \bar{f}(\nu(t))$, and it satisfies $u_t = \nu(t)$ on $\mathbb{R}$. We now consider an open neighborhood of 0 in $\mathcal{B}$ and a Lipschitz continuous mapping $H : CN \oplus U \to S$ defined by

$$V = \{\xi \in \mathcal{B} : |\langle \Psi, \xi\rangle| < \delta\}$$

and

$$H(\Phi y) = h^*(y), \qquad y \in \mathbb{R}^d.$$

In what follows, we will certify the properties (i)–(iv) in the theorem. Observe that $W^{cu}(V) = \{\xi = \Phi y + h^*(y) : y \in \mathbb{R}^d, |y| < \delta\}$ and $(CN \oplus U) \cap V = \{\Phi y : y \in \mathbb{R}^d, |y| < \delta\}$. Then Property (i) holds true.

In oreder to check Property (iii)-(a), let $\xi \in W^{cu}(V)$ be given. Then $\xi = \Phi y_0 + h^*(y_0)$ for some $y_0 \in \mathbb{R}^d$ with $|y_0| < \delta$. There exists a $t_0 > 0$ such that $|y(t, y_0 : h^*)| < \delta$ on $|t| < t_0$. Since $\chi(y(t, y_0; h)/\delta) = 1$ on $|t| < t_0$, we see that the $\mathcal{B}$-valued function $\nu(t) := \Phi y(t, y_0; h^*) + h^*(y(t, y_0; h^*))$ satisfies $\bar{f}_\delta(\nu(t)) = f(\nu(t))$ on $|t| < t_0$. Then Property (iii)-(a) follows from this observation.

Next, let us assume that $\xi \in W^{cu}(V)$ and $u_t(0, \xi; f) \in V$ on $[0, b]$. It follows that $u_t(0, \xi; f) = \Phi z(t) + \Pi^S u_t(0, \xi; f)$ for some $z(t)$. Observe that $|z(t)| = |\langle \Psi, u_t(0, \xi; f)\rangle| < \delta$ or $\bar{f}_\delta(u_t(0, \xi; f)) = f(u_t(0, \xi; f))$, and hence

$$
\begin{aligned}
u_t(0, \xi; f) &= V(t)\xi + \lim_{n\to\infty} \int_0^t V(t-s)\Gamma^n f(u_s(0, \xi; f))ds \\
&= V(t)\xi + \lim_{n\to\infty} \int_0^t V(t-s)\Gamma^n \bar{f}_\delta(u_s(0, \xi; f))ds
\end{aligned}
$$

for all $t \in [0, b]$. Since the solution of Eq. (5) with $f = \bar{f}_\delta$ is unique for the initial value problem, we get

$$u_t(0, \xi; f) \equiv u_t(0, \xi; \bar{f}_\delta), \quad \forall t \in [0, b].$$

By the same reasoning as above, it follows from the relation (14) that $u_t(0, \xi; \bar{f}_\delta) = \nu(t) := \Phi y(t, y_0; h^*) + h^*(y(t, y_0; h^*))$ for all $t \geq 0$. Hence we get $z(t) = y(t, y_0; h^*)$, and consequently $u_t(0, \xi; f) = \Phi z(t) + h^*(z(t)) \in W^{cu}(V)$ for all $t \in [0, b]$, which proves (iii)-(b).

Next we will certify Property (iv). If u is a solution of Eq. (5) with $u_t \in W^{cu}(V)$ on J, we get $u_t = \Phi y(t) + \Pi^S u_t$ with $|y(t)| < \delta$ and $\Pi^S u_t = h^*(y(t)) = H(\Phi y(t))$ on J. Then it follows that $f(\Phi y(t) + \Pi^S u_t) = f(\Phi y(t) + H(\Phi y(t)))$ on J, which proves the first part of Property (iv) because y satisfies Eq. (8).

Conversely, let y be a solution of Eq. ; (9) on J with $\Phi y(t) \in V$ on J. y can uniquely be extended on $\mathbb{R}$ as a solution of (12) with $h = h^*$ on $\mathbb{R}$, which will be denoted by y again. Set $z(t) = y(t, y(0); h^*)$ for $t \in \mathbb{R}$. By virtue of the uniqueness of solutions for the initial value problems of $\dot{z} = Gz + \langle x^*, \tilde{f}_\delta(z, h^*(z))\rangle$, we get $y(t) \equiv z(t)$ on J. Define a $\mathcal{B}$-valued function ξ by $\xi(t) = \Phi z(t) + h^*(z(t))$. By the same reasoning as for (14), one can see that ξ satisfies

$$\xi(t) = V(t-\sigma)\xi(\sigma) + \lim_{n\to\infty} \int_\sigma^t V(t-s)\Gamma^n \tilde{f}_\delta(z(s), h^*(z(s))ds, \quad \forall t \geq \sigma,$$

and consequently it follows from [12, Theorem 4.2] that $u(t) := [\xi(t)](0)$ is a solution of Eq. (3) with $h(t) = \tilde{f}_\delta(z(t), h^*(z(t))$ and satisfies $u_t = \xi(t)$. Since $\Phi y(t) \in V$ on J, we get $|z(t)| = |y(t)| < \delta$ on J. Moreover, we get $\Pi^S u_t = \Pi^S \xi(t) = h^*(z(t))$, and

$$
\begin{aligned}
\tilde{f}_\delta(z(t), h^*(z(t)) &= f(\chi(z(t)/\delta)\Phi z(t) + h^* z(t)) \\
&= f(\Phi z(t) + h^*(z(t))) \\
&= f(\Pi^{C,U} u_t + \Pi^S u_t) = f(u_t)
\end{aligned}
$$

on J. Therefore, u is a solution of Eq. (5) on J which satisfies $\Pi^{C,U}u_t = \Phi y(t)$ on J.

Finally, we will certify Property (ii). To do this, it suffices to establish that

$$\limsup_{y_0\in\mathbb{R}^d,\ y_0\to 0} \frac{\|h^*(y_0)\|_{\mathcal{B}}}{|y_0|} = 0.$$

Note that $y(\tau,0;h)\equiv 0$, and consequently

$$|y(\tau,y_0;h)| \le C|y_0|e^{-\tau(\epsilon+2CC_2\|x^*\|\zeta(\bar{C}\delta))}, \quad \forall\tau\le 0,$$

by (13). Let us write $y(t)=y(t,y_0;h^*)$. By using (10), we get

$$\begin{aligned}\|h^*(y_0)\|_{\mathcal{B}} \le CC_1K_0\Big(&\int_{-\infty}^{-L} e^{\alpha\tau}\zeta(\bar{C}\delta)(C_0|y(\tau)|+|h^*(y(\tau))|_{\mathcal{B}})d\tau\\ &+\int_{-L}^{0} e^{\alpha\tau}|f(\chi(\frac{y(\tau)}{\delta})\Phi y(\tau)+h^*(y(\tau)))|_{\mathbb{X}}d\tau\Big)\\ &\le \frac{C^2C_1(C_0+1)K_0\zeta(\bar{C}\delta)}{\alpha-\epsilon-2CC_2\|x^*\|\zeta(\bar{C}\delta)}|y_0|e^{-(\alpha-\epsilon-2CC_2\|x^*\|\zeta(\bar{C}\delta))L}\\ &\quad+\frac{C_1CK_0}{\alpha}\sup_{-L\le\tau\le 0}|f(\chi(\frac{y(\tau)}{\delta})\Phi y(\tau)+h^*(y(\tau)))|_{\mathbb{X}}\end{aligned}$$

for any $L>0$.

Since $\|\chi(\frac{y(\tau)}{\delta})\Phi y(\tau)+h^*(y(\tau))\|_{\mathcal{B}} \le C(C_0+1)|y_0|e^{L(\epsilon+2CC_2\|x^*\|\zeta(\bar{C}\delta))}$ for $\tau\in[-L,0]$, it follows that

$$\begin{aligned}&\sup_{-L\le\tau\le 0}|f(\chi(\frac{y(\tau)}{\delta})\Phi y(\tau)+h^*(y(\tau))|_{\mathbb{X}}\\ &\le C(C_0+1)|y_0|e^{L(\epsilon+2CC_2\|x^*\|\zeta(\bar{C}\delta))}\times\zeta(C(C_0+1)|y_0|e^{L(\epsilon+2CC_2\|x^*\|\zeta(\bar{C}\delta))}).\end{aligned}$$

Note that ζ is continuous with $\zeta(0)=0$. Thus we get

$$\limsup_{y_0\in\mathbb{R}^d,\ y_0\to 0}\frac{\|h^*(y_0)\|_{\mathcal{B}}}{|y_0|} \le \frac{C^2C_1(C_0+1)K_0\zeta(\bar{C}\delta)}{\alpha-\epsilon-2CC_2\|x^*\|\zeta(\bar{C}\delta)}e^{-(\alpha-\epsilon-2CC_2\|x^*\|\zeta(\bar{C}\delta))L}.$$

Since $L>0$ is an arbitrary positive number, the required one follows from the above inequality. This completes the proof of Theorem 3.5. ∎

4. Linearized Stabilities and Instabilities

As an application of the results in the preceding section, we will give some results on stabilities and instabilities of the zero solution of Eq. (5) which is sometimes called the principle of linearization in the theory of ordinary differential equations.

Theorem 4.1. *Let $f \in C^1(\mathcal{B};\mathbb{X})$ satisfy $f(0) = f'(0) = 0$, and suppose that the operator $\lambda I - A - L(e^{\lambda \cdot} I)$ is invertible in $\mathcal{L}(\mathbb{X})$ for any λ with $\Re\lambda \geq 0$. Then the zero solution of* (5) *is exponentially asymptotically stable.*

Proof. Since $\Sigma = \emptyset$ by the assumption of the theorem, we get $U = \{0\}$ and $W^s(\delta) = \mathcal{B}_{\delta/(2C)}$ in Theorem 3.1. Hence it follows from Property (iii) in Theorem 3.1 that

$$\begin{aligned}\|u_t(\sigma,\phi;f)\|_{\mathcal{B}} &= \|u_{t-\sigma}(0,\phi;f)\|_{\mathcal{B}}\\ &\leq Me^{-\beta(t-\sigma)}\|\phi\|_{\mathcal{B}}, \qquad t \geq \sigma,\ \phi \in \mathcal{B}_{\delta/(2C)},\end{aligned}$$

which shows that the zero solution of (5) is exponentially asymptotically stable. ∎

Theorem 4.2. *Let $f \in C^1(\mathcal{B};\mathbb{X})$ satisfy $f(0) = f'(0) = 0$, and suppose that the operator $\lambda I - A - L(e^{\lambda \cdot} I)$ is not invertible in $\mathcal{L}(\mathbb{X})$ for some λ with $\Re\lambda > 0$. Then the zero solution of* (5) *is not stable.*

Proof. By virtue of Theorem 3.5, there exists the local center-unstable manifold $W^{cu}(V)$ of Eq. (5). Observe that the function $p(z) := \langle x^*, f(\Phi z + H(\Phi z))\rangle$ satisfies $h(z) = o(z)$ as $z \to 0$ because of the inequality $|f(\Phi z + H(\Phi z))| \leq \zeta(\|\Phi z + H(\Phi z)\|_{\mathcal{B}})(C_0+1)|z|$. Therefore, by [3, Theorem 13.1.2, p.317], the zero solution of Eq. (14) is unstable, because $\sigma(G) = \Sigma$ contains a number λ with $\Re\lambda > 0$ by the assumption. Then this observation and Property (iv) of Theorem 3.5 yield that the the zero solution of (5) is unstable. This completes the proof of the theorem. ∎

Example 4.3. We consider the following (scalar) partial-integrodifferential equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + u(1 - au - b\int_{-\infty}^{0} e^{\theta}u(t+\theta)d\theta), \quad t > 0,\ 0 < x < \pi \quad (15)$$

with the boundary condition $\frac{\partial u}{\partial t}(t,0) = \frac{\partial u}{\partial t}(t,\pi) = 0$ for $t > 0$, where a and b are positive constants. Take $\mathbb{X} = C([0,\pi];\mathbb{R})$ and set $A\xi = \xi''$ for $\xi \in D(A) = \{\xi \in C^2([0,\pi];\mathbb{R}) : \xi'(0) = \xi'(\pi) = 0\}$. Then Eq. (15) can be set up as a functional differential equation with the phase space $\mathcal{B} = C_{-1/2}(\mathbb{X})$. The linearized equation of (15) for the zero solution is given by $\dot{u} = Au + u$ with the characteristic equations

$$\lambda - 1 = -n^2, \qquad n = 0,\ 1,\ 2, \cdots.$$

Then $1 \in \Sigma$, and the zero solution of (15) is unstable by Theorem 4.2. On the other hand, the linearized equation for the solution $u \equiv 1/(a+b)$ is given by

$\dot{v} = Av - (av + b \int_{-\infty}^{0} e^s v(t+s)ds)/(a+b)^2$ with the characteristic equations

$$\lambda + \frac{1}{(a+b)^2}(a + \frac{b}{\lambda+1}) = -n^2, \quad n = 0,\ 1,\ 2, \cdots .$$

Observe that if $\Re\lambda \geq 0$, then $\Re\Big[\lambda + \frac{1}{(a+b)^2}(a + \frac{b}{\lambda+1})\Big] \geq \frac{a}{(a+b)^2} > 0$. Hence $\Sigma = \emptyset$, and the the solution $u \equiv 1/(a+b)$ of (15) is exponentially asymptotically stable by Theorem 4.1.

Remark 4.4. Throughout this paper, $\|\cdot\|_{\mathcal{B}}$ is assumed to be a norm. In fact, by introducing some cumbersome ones such as the quotient space, the quotient operator and so on as in [11], one can establish all the theorems in this paper under a weaker situation that $\|\cdot\|_{\mathcal{B}}$ is a complete *seminorm*; that is, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a complete seminormed linear space.

Remark 4.5. We can derive the smoothness of the manifolds established in Theorems 3.1, 3.3 and 3.5. Indeed, replacing the number δ in the proofs of the theorems by a smaller one if necessary, one can see that the manifolds are continuously differentiable by applying [7, Lemma 10.1.2]; the details are omitted.

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