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Convergence Rates of Representation Formulas for M- Parameter Semigroups*

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Abstract. Some representation formulas with rates for strongly continuous m-parameter semigroups of operators are deduced by applying a vector-valued version of Shisha and Mond's quantitative approximation theorem to some m-dimensional approximation operators.

1. Introduction

For $t = (t_1, t_2, ..., t_m) \in \mathbb{R}^m$, denote $\overline{t} := t_1 + t_2 + \cdots + t_m$. Let $\mathbb{T} := \{t = (t_1, t_2, ..., t_m); 0 \le t_i \le 1, i = 1, ..., m\}$ and $\tilde{\mathbb{T}} := \{t \in \mathbb{T}; 0 \le \overline{t} \le 1\}$. Let X be a Banach space. As usual, $\mathcal{C}(\mathbb{T}, X)$ and $\mathcal{C}(\tilde{\mathbb{T}}, X)$ denote the Banach spaces of X-valued continuous functions on $\mathbb{T}$ and $\tilde{\mathbb{T}}$, respectively.

For $n \in \mathbb{N}$, $k \in (\{0\} \cup \mathbb{N})^m$ with $0 \le k_i \le n,\ i = 1, ..., m$, let the function $p_{n,k} \in \mathcal{C}(\mathbb{T}, \mathbb{R})$ be defined as:

$$p_{n,k}(t) := \prod_{i=1}^{m} \binom{n}{k_i} (t_i)^{k_i} (1 - t_i)^{n-k_i} \text{ for } t \in \mathbb{T}. \tag{1}$$

For k with $0 \le \overline{k} \le n$, let the function $\tilde{p}_{n,k} \in \mathcal{C}(\tilde{\mathbb{T}}, \mathbb{R})$ be defined as:

$$\tilde{p}_{n,k}(t) := \binom{n}{k} \left(\prod_{i=1}^{m} (t_i)^{k_i} \right) (1 - \overline{t})^{n-\overline{k}} \text{ for } t \in \tilde{\mathbb{T}} \tag{2}$$

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where $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n(n-1)\cdots(n-\overline{k}+1)}{k_1!k_2!\cdots k_m!}$.

Using the above two functions one can define two kinds of *m-dimensional Bernstein operators*. The first kind is the operator $B_n : \mathcal{C}(\mathbb{T}, X) \to \mathcal{C}(\mathbb{T}, X)$ defined (cf. [4]) as:

$$(B_n f)(t) := \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} f(\frac{k_1}{n}, ..., \frac{k_m}{n}) p_{n,k}(t),\ f \in \mathcal{C}(\mathbb{T}, X),\, t \in \mathbb{T}. \tag{3}$$

The second kind of m-dimensional Bernstein operator $\tilde{B}_n : \mathcal{C}(\tilde{\mathbb{T}}, X) \to \mathcal{C}(\tilde{\mathbb{T}}, X)$ is defined (cf. [6]) as:

$$(\tilde{B}_n f)(t) := \sum_{\overline{k}=0}^{n} f(\frac{k_1}{n}, ..., \frac{k_m}{n}) \tilde{p}_{n,k}(t),\ f \in \mathcal{C}(\tilde{\mathbb{T}}, X),\, t \in \tilde{\mathbb{T}}. \tag{4}$$

As is shown in [1, 6 - 9], a Korovkin-type approximation theorem [6, Theorem 2.1] can be used to derive many representation formulas of strongly continuous semigroups of operators and cosine families of operators. In particular, by applying it to the above two m-dimensional Bernstein operators, one can deduce two representation formulas for a strongly continuous m-parameter semigroup of operators on X (see Theorem 4.1 of [6]).

In order to estimate convergence rates of representation formulas, we shall first prove, in Sec. 2, a vector-valued version of the quantitative approximation theorem of Shisha and Mond [10]. In Secs. 3 and 4, we shall use (3) and (4), respectively, as a model to define two kinds of m-dimensional generalizations of each of the approximation operators of Chung, Kantorovitch, and Durrmeyer, and then obtain estimates of their convergence rates by applying Theorem 2.1.

Finally, we employ the estimates obtained in Secs. 3 and 4 to deduce the following theorems about representation formulas with rates for strongly continuous m-parameter semigroups of operators.

Let $T(\cdot) = \{T(t); t \in \mathbb{R}^m_{0+}\}$ be an m-parameter semigroup of bounded linear operators on a Banach space X. Let us assume that $T(\cdot)$ is strongly continuous on $\mathbb{R}^m_{0+} = \{t \in \mathbb{R}^m : t_i \geq 0,\ i = 1, 2, \cdots, m\}$. If $T_i(\cdot) = \{T_i(t_i); 0 \leq t_i < \infty\}$ denotes the restriction of $T(t_1, t_2, \cdots, t_m)$ to the half line $\{(0, \cdots, 0, t_i, 0, \cdots, 0) : 0 \leq t_i < \infty\}$, then $T_i(\cdot)$ is an one-parameter semigroup which is strongly continuous on $[0, \infty)$. $T(\cdot)$ is the product of T_i; $T(t_1, t_2, \cdots, t_m) = \prod_{i=1}^{m} T_i(t_i)$. Let A_i be the infinitesimal generator of $T_i(\cdot)$.

We can easily deduce from Theorems 3.2 and 4.2 the two formulas in the following theorem.

Theorem 1.1. *Let $T(\cdot)$ be a strongly continuous m-parameter semigroup. Then we have for every $x \in X$*

$$\Big\| \prod_{i=1}^{m} \Big(I + t_i \Big(T_i(\frac{1}{n}) - I \Big) \Big)^n x - T(t)x \Big\| \leq 2\omega \left(T(\cdot)x, \mathbb{T}, \sqrt{\overline{t}/n} \right), \quad t \in \mathbb{T}; \tag{5}$$

$$\Big\| \Big(I + \sum_{i=1}^{m} t_i \Big(T_i(\frac{1}{n}) - I \Big) \Big)^n x - T(t)x \Big\| \leq 2\omega \Big(T(\cdot)x, \mathbb{T}, \sqrt{\overline{t}/n} \Big), \quad t \in \tilde{\mathbb{T}}. \tag{6}$$

Next, we can deduce from Theorems 3.3, 4.3, 3.5, and 4.5 the respective four formulas in the following two theorems.

Theorem 1.2. *Let $T(\cdot)$ be a strongly continuous m-parameter semigroup, and A_i be the infinitesimal generator of $T_i(t_i)$, $i = 1, 2, \cdots, m$. If $x \in X$ is such that $T(\cdot)x$ is uniformly continuous on $\mathbb{R}^m_{0+}$, then*

$$\begin{aligned}&\Big\|\prod_{i=1}^{m}\Big(I + t_i\Big(\Big(I - \frac{A_i}{n}\Big)^{-1} - I\Big)\Big)^n x - T(t)x\Big\| \\ &\le 2\omega\Big(T(\cdot)x, \mathbb{R}^m_{0+}, \sqrt{2\overline{t}/n}\Big),\ t \in \mathbb{T};\end{aligned} \tag{7}$$

$$\begin{aligned}&\Big\|\Big(I + \sum_{i=1}^{m} t_i\Big(\Big(I - \frac{A_i}{n}\Big)^{-1} - I\Big)\Big)^n x - T(t)x\Big\| \\ &\le 2\omega\Big(T(\cdot)x, \mathbb{R}^m_{0+}, \sqrt{2\overline{t}/n}\Big),\ t \in \tilde{\mathbb{T}}.\end{aligned} \tag{8}$$

Theorem 1.3. *Let $T(\cdot)$ be a strongly continuous m-parameter semigroup, and A_i be the infinitesimal generator of $T_i(t_i)$, $i = 1, 2, \cdots, m$. If A_i are invertible, then we have for every $x \in X$*

$$\begin{aligned}&\Big\|\prod_{i=1}^{m}\Big\{(n+1)A_i^{-1}\Big(T_i(\frac{1}{n+1}) - I\Big) \\ &\cdot \Big(I + t_i\Big(T_i(\frac{1}{n+1} - I)\Big)\Big)^n\Big\}x - T(t)x\Big\| \\ &\le 2\omega\Big(T(\cdot)x, \mathbb{T}, \sqrt{\frac{(3n-2)m}{3(n+1)^2}}\Big),\ t \in \mathbb{T};\end{aligned} \tag{9}$$

$$\begin{aligned}&\Big\|\Big[\prod_{i=1}^{m}\Big((n+1)A_i^{-1}\Big(T_i(\frac{1}{n+1}) - I\Big)\Big)\Big] \\ &\times \Big(I + \sum_{i=1}^{m} t_i\Big(T_i(\frac{1}{n+1}) - I\Big)\Big)^n x - T(t)x\Big\| \\ &\le 2\omega\Big(T(\cdot)x, \tilde{\mathbb{T}}, \sqrt{\frac{(3n-2)m}{3(n+1)^2}}\Big),\ t \in \tilde{\mathbb{T}}.\end{aligned} \tag{10}$$

2. The Shisha-Mond Theorem for Vector-Valued Functions

Let E, F be subsets of a normed linear space such that $F \subset E$. Let $\mathcal{B}(F, X)$ denote the set of all bounded X-valued functions on F, and $\mathcal{C}(E, X)$ (resp. $\mathcal{UC}(E, X)$) the set of all continuous (resp. uniformly continuous) functions on E. Let L be a linear operator from $\mathcal{C}(E, X)$ to $\mathcal{B}(F, X)$, and P be a positive linear operator from $\mathcal{C}(E, \mathbb{R})$ to $\mathcal{B}(F, \mathbb{R})$. L is said to be *dominated* by P on F if $\|(Lf)(t)\| \le (P\|f\|)(t)$ for all $f \in \mathcal{C}(E, X)$ and $t \in F$ (cf. [5]). L is said to *preserve constants* if $L(1_E x)(t) = x$ for all $x \in X$ and $t \in F$. Here 1_E denotes the constant function on E with value 1.

For $f \in \mathcal{UC}(E, X)$, $\omega(f, E, \delta)$ denotes the modulus of continuity of f,

$$\omega(f, E, \delta) := \sup\{\|f(u) - f(t)\|; u, t \in E, \|u - t\| \leq \delta\} \quad (\delta > 0). \tag{11}$$

The quantitative approximation theorem of Shisha and Mond [10] can be modified in the following theorem for approximation of vector-valued functions.

Theorem 2.1. *Suppose a linear operator $L : \mathcal{C}(E, X) \to \mathcal{B}(F, X)$ is dominated by a positive linear operator $P : \mathcal{C}(E, \mathbb{R}) \to \mathcal{B}(F, \mathbb{R})$. Then for every $f \in \mathcal{UC}(E, X)$ and $t \in F$ we have*

$$\|(Lf)(t) - f(t)\| \leq \|L(f(t)1_E)(t) - f(t)\| + \omega(f, E, \gamma(t))[(P1_E)(t) + 1] \tag{12}$$

where $\gamma^2(t) = P(\| \cdot - t\|^2)(t)$. Moreover, if L and P preserve constants, then

$$\|(Lf)(t) - f(t)\| \leq 2\omega(f, E, \gamma(t)), \ t \in F. \tag{13}$$

Proof. Fix $f \in \mathcal{UC}(E, X)$. For any given $\delta > 0$, if $\delta \leq \|u - t\|$, then

$$\begin{aligned} \omega(f, E, \|u - t\|) &\leq \omega\Big(f, E, \Big(\Big[\frac{\|u - t\|}{\delta}\Big] + 1\Big)\delta\Big) \leq \Big(\Big[\frac{\|u - t\|}{\delta}\Big] + 1\Big)\omega(f, E, \delta) \\ &\leq (1 + \delta^{-2}\|u - t\|^2)\omega(f, E, \delta). \end{aligned}$$

If $\|u - t\| < \delta$, we also have

$$\omega(f, E, \|u - t\|) \leq \omega(f, E, \delta) \leq (1 + \delta^{-2}\|u - t\|^2)\omega(f, E, \delta).$$

Hence we have

$$\|f(u) - f(t)\| \leq \omega(f, E, \|u - t\|) \leq (1 + \delta^{-2}\|u - t\|^2)\omega(f, E, \delta)$$

for all $u \in E, \ t \in F$. Applying the positive operator P we have for $t \in F$

$$P(\|f - 1_E f(t)\|)(t) \leq \omega(f, E, \delta)\big(P(1_E)(t) + \delta^{-2}\gamma^2(t)\big).$$

As L is dominated by P, we have

$$\begin{aligned} \|(Lf)(t) - f(t)\| &\leq \|L\big(f - 1_E f(t)\big)(t)\| + \|L\big(1_E f(t)\big)(t) - f(t)\| \\ &\leq P\big(\|f - 1_E f(t)\|\big)(t) + \|L\big(1_E f(t)\big)(t) - f(t)\| \\ &\leq \|L(1_E f(t))(t) - f(t)\| + \omega(f, E, \delta)\big(P(1_E)(t) + \delta^{-2}\gamma^2(t)\big) \end{aligned}$$

for all $t \in F$. Now, (12) follows by taking $\delta = \gamma(t)$. If L and P preserve constants, then $L(1_E f(t))(t) = f(t)$ and $P(1_E)(t) = 1$, we have

$$\|(Lf)(t) - f(t)\| \leq 2\omega\big(f, E, \gamma(t)\big).$$

The proof is complete. ∎

3. Approximation of Functions on $\mathbb{T}$

In this section we define some m-dimensional approximation operators for approximation of functions on $\mathbb{T}$ and then apply Theorem 2.1 to them.

Let $p^{(i)}_{n,k_i}(t_i) := \binom{n}{k_i}(t_i)^{k_i}(1-t_i)^{n-k_i}$ for $t \in [0,1]$. Then $p_{n,k}(t) = \prod_{i=1}^{m} p^{(i)}_{n,k_i}(t_i)$ for $t \in \mathbb{T}$. We shall need the lemma.

Lemma 3.1. *$\sum_{k_i=0}^{n} p^{(i)}_{n,k_i}(t_i) = 1$, $\sum_{k_i=1}^{n} k_i p^{(i)}_{n,k_i}(t_i) = nt_i$, and $\sum_{k_i=2}^{n} k_i(k_i-1)$ $p^{(i)}_{n,k_i}(t_i) = n(n-1)t_i^2$ for all $t_i \in [0,1]$, $i = 1, 2, \cdots, m$.*

For the m-dimensional Bernstein operator $B_n : \mathcal{C}(\mathbb{T}, X) \to \mathcal{C}(\mathbb{T}, X)$ as defined in (3), we have the next theorem.

Theorem 3.2. *For every $f \in \mathcal{C}(\mathbb{T}, X)$, we have*

$$\|(B_n f)(t) - f(t)\|_X \le 2\omega(f, \mathbb{T}, \sqrt{\frac{\bar{t}}{n}}),\ t \in \mathbb{T}. \tag{14}$$

Proof. Using Lemma 3.1 we have $(B_n 1_{\mathbb{T}})(t) = 1$, $(B_n e_j)(t) = t_j$, $(B_n e_j^2)(t) = t_j^2 + \frac{1}{n}(t_j - t_j^2)$ for all $t \in \mathbb{T}$, $j = 1, 2, ..., m$, $n \in \mathbb{N}$. Hence, B_n preserves constants, and

$$\gamma_n^2(t) = B_n(\| \cdot - t\|_2^2)(t) = \sum_{j=1}^{m} \frac{t_j - t_j^2}{n} \le \frac{\bar{t}}{n}.$$

The conclusion now follows from Theorem 2.1. ■

If we replace $f(\frac{k}{n}) = f(\frac{k_1}{n}, ..., \frac{k_m}{n})$ in (3) with

$$c(f, n, k) := \int_0^\infty \cdots \int_0^\infty \Big(\prod_{i=1}^{m} \frac{n^{k_i}}{(k_i - 1)!} e^{-nu_i} u_i^{k_i - 1}\Big) f(u_1, ..., u_m) du_1 \cdots du_m, \tag{15}$$

then we can define the *m-dimensional Chung operator* $C_n : \mathcal{BC}(\mathbb{R}^m_{0+}, X) \to \mathcal{C}(\mathcal{T}, X)$ as follows:

$$(C_n f)(t) := \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} c(f, n, k) p_{n,k}(t) \text{ for } f \in \mathcal{BC}(\mathbb{R}^m_{0+}, X),\ t \in \mathbb{T}. \tag{16}$$

Here $\mathbb{R}^m_{0+} = \{t \in \mathbb{R}^m : t_i \ge 0,\ i = 1, 2, \cdots, m\}$

This is a generalization of the one-dimensional Chung operator considered in [6].

Theorem 3.3. *For every $f \in \mathcal{BUC}(\mathbb{R}^m_{0+}, X)$, we have*

$$\|(C_n f)(t) - f(t)\|_X \le 2\omega(f, \mathbb{R}^m_{0+}, \sqrt{\frac{2\bar{t}}{n}}),\ t \in \mathbb{T}. \tag{17}$$

Proof. Using Lemma 3.1 we have $(C_n 1)(t) = 1$,

$$(C_n e_j)(t) = \sum_{k_j=1}^{n} p_{n,k_j}^{(j)}(t_j)(\frac{k_j}{n}) = \frac{1}{n}(nt_j) = t_j,$$

$$(C_n e_j^2)(t) = \sum_{k_j=1}^{n} p_{n,k_j}^{(j)}(t_j)(\frac{k_j(k_j+1)}{n^2}) = \frac{1}{n^2}\{2nt_j + n(n-1)t_j^2\}$$

$$= t_j^2 + \frac{1}{n}(2t_j - t_j^2)$$

for all $t \in \mathbb{T}$, $j = 1, 2, ..., m$, $n \in \mathbb{N}$. Hence, C_n preserves constants, and

$$\gamma_n^2(t) = C_n(\| \cdot - t\|_2^2)(t) = \sum_{j=1}^{m} \frac{2t_j - t_j^2}{n} \leq \frac{2\bar{t}}{n},$$

from which we conclude the proof. ■

Next, if we replace $f(\frac{k}{n})$ in (3) with

$$\kappa(f, n, k) := (n+1)^m \Big(\int_{\frac{k_m}{n+1}}^{\frac{k_m+1}{n+1}} \cdots \int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}} f(u_1, ..., u_m) du_1 \cdots du_m \Big), \tag{18}$$

we obtain the *m-dimensional Kantorovitch operator* $K_n : \mathcal{C}(\mathbb{T}, X) \to \mathcal{C}(\mathbb{T}, X)$ as follows:

$$(K_n f)(t) := \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} \kappa(f, n, k) p_{n,k}(t) \text{ for } f \in \mathcal{C}(\mathbb{T}, X), \ t \in \mathbb{T}. \tag{19}$$

For this operator, we have the following theorem.

Theorem 3.4. *For every $f \in \mathcal{C}(\mathbb{T}, X)$ and $t \in \mathbb{T}$,*

$$\|(K_n f)(t) - f(t)\|_X \leq 2\omega\Big(f, \mathbb{T}, \sqrt{\frac{(3n-2)m}{3(n+1)^2}}\Big). \tag{20}$$

Proof. Using Lemma 3.1 we have $(K_n 1_{\mathbb{T}})(t) = 1$,

$$(K_n e_j)(t) = \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} \prod_{i=1}^{m} p_{n,k_i}^{(i)}(t_i)\Big((n+1)^m \int_{\frac{k_m}{n+1}}^{\frac{k_m+1}{n+1}} \cdots \int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}} u_j du_1 \cdots du_m\Big)$$

$$= \sum_{k_j=0}^{n} p_{n,k_j}^{(j)}(t_i)(\frac{1}{2}(n+1)\frac{2k_j+1}{(n+1)^2}) = t_j + \frac{1}{2(n+1)}(1 - 2t_j),$$

$$(K_n e_j^2)(t)$$

$$= \sum_{k_j=0}^{n} p_{n,k_j}^{(j)}(t_i)\Big((n+1) \int_{\frac{k_j}{n+1}}^{\frac{k_j+1}{n+1}} u_j^2 \, du_j\Big) = t_j^2 + \frac{1}{(n+1)^2}(\frac{1}{3} + 2nt_j - (3n+1)t_j^2)$$

for all $t \in \mathbb{T},\ j = 1, 2, ..., m,\ n \in \mathbb{N}$. Hence, K_n preserves constants, and

$$\gamma_n^2(t) = K_n(\|\cdot - t\|_2^2)(t) = \sum_{j=1}^{m} \frac{3(n-1)(t-t^2)+1}{3(n+1)^2} \leq m\frac{(3n-2)}{3(n+1)^2}.$$

The estimate now follows from Theorem 2.1. ∎

If we replace $f(\frac{k}{n}) = f(\frac{k_1}{n}, ..., \frac{k_m}{n})$ in (3) with

$$d(f,n,k) := (n+1)^m \Big(\int_0^1 \cdots \int_0^1 p_{n,k}(u_1,..,u_m) f(u_1,..,u_m) du_1 \cdots du_m \Big), \tag{21}$$

then we obtain the *m-dimensional Durrmeyer operator* $D_n : \mathcal{C}(\mathbb{T}, X) \to \mathcal{C}(\mathbb{T}, X)$ defined as follows:

$$(D_n f)(t) := \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} d(f,n,k) p_{n,k}(t) \quad \text{for} \quad f \in \mathcal{C}(\mathbb{T}, X),\ \ t \in \mathbb{T}. \tag{22}$$

The next theorem gives an estimate for this operator.

Theorem 3.5. *For every $f \in \mathcal{C}(\mathbb{T}, X)$ and $t \in \mathbb{T}$,*

$$\|(D_n f)(t) - f(t)\|_X \leq 2\omega\big(f, \mathbb{T}, \sqrt{\frac{2(n-2)m}{(n+2)(n+3)}}\big). \tag{23}$$

Proof. Using Lemma 3.1 we have for all $t \in \mathbb{T},\ j = 1, 2, ..., m,\ n \in \mathbb{N}$

$$\begin{aligned}
(D_n 1_{\mathbb{T}})(t) &= \prod_{i=1}^{m} \Big\{ \sum_{k_i=0}^{n} p_{n,k_i}^{(i)}(t_i)(n+1) \int_0^1 \binom{n}{k_i} u_i^{k_i}(1-u_i)^{n-k_i} du_i \Big\} \\
&= \prod_{i=1}^{m} \Big\{ \sum_{k_i=0}^{n} p_{n,k_i}^{(i)}(t_i)(n+1) \binom{n}{k_i} B(k_i+1, n-k_i+1)\, du_i \Big\} \\
&= \prod_{i=1}^{m} \Big(\sum_{k_i=0}^{n} p_{n,k_i}^{(i)}(t_i) \Big) = 1, \\
(D_n e_j)(t) &= \sum_{k_j=0}^{n} p_{n,k_j}^{(j)}(t_j) \Big((n+1) \binom{n}{k_j} \int_0^1 u_j^{k_j+1}(1-u_j)^{n-k_j}\, du_j \Big) \\
&= \sum_{k_j=0}^{n} p_{n,k_j}^{(j)}(t_j) \Big((n+1) \binom{n}{k_j} B(k_j+2, n-k_j+1) \Big) \\
&= \sum_{k_j=0}^{n} p_{n,k_j}^{(j)}(t_j) (\frac{k_j+1}{n+2}) = t_j + \frac{1}{n+2}(1-2t_j),
\end{aligned}$$

$$
\begin{aligned}
(D_n e_j^2)(t) &= \sum_{k_j=0}^{n} p_{n,k_j}^{(j)}(t_j)\big((n+1)\binom{n}{k_j}B(k_j+3, n-k_j+1)\big)\\
&= \sum_{k_j=0}^{n} p_{n,k_j}^{(j)}(t_j)\Big(a\frac{(k_j+2)(k_j+1)}{(n+3)(n+2)}\Big)\\
&= t_j^2 + \frac{1}{n^2+5n+6}\big(2+4nt_j-6n(n+1)t_j^2\big),
\end{aligned}
$$

where $B(\cdot,\cdot)$ is the Beta function, $B(k+1, n-k+1) = \frac{k!(n-k)!}{(n+1)!}$. Hence D_n preserves constants and

$$
\gamma_n^2(t) = D_n(\|\cdot - t\|_2^2)(t) = \sum_{j=1}^{m} \frac{2(n-3)(t-t^2)+2}{(n+2)(n+3)} \le m\frac{2(n-2)m}{(n+2)(n+3)}.
$$

From Theorem 2.1 we conclude the proof. ∎

4. Approximation of Functions on $\tilde{\mathbb{T}}$

In this section we define some m-dimensional approximation operators for approximation of functions on $\tilde{\mathbb{T}}$ and then apply Theorem 2.1 to them. We need the next Lemma.

Lemma 4.1. $\sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t) = 1$, $\sum_{\bar{k}=1}^{n} k_i\tilde{p}_{n,k}(t) = nt_i$, *and* $\sum_{\bar{k}=2}^{n} k_i(k_i-1)\tilde{p}_{n,k}(t) = n(n-1)t_i^2$ *for all* $t \in \tilde{\mathbb{T}}$, $i = 1, 2, ..., m$.

For the m-dimensional Bernstein operator $\tilde{B}_n : \mathcal{C}(\tilde{\mathbb{T}}, X) \to \mathcal{C}(\tilde{\mathbb{T}}, X)$ as defined in (4), we have the next theorem.

Theorem 4.2. *For every* $f \in \mathcal{C}(\tilde{\mathbb{T}}, X)$, *we have*

$$
\|(\tilde{B}_n f)(t) - f(t)\|_X \le 2\omega(f, \tilde{\mathbb{T}}, \sqrt{\frac{\bar{t}}{n}}),\ t \in \tilde{\mathbb{T}}. \tag{24}
$$

Proof. Using Lemma 4.1 we have $(\tilde{B}_n 1_{\tilde{\mathbb{T}}})(t) = \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t) = 1$, $(\tilde{B}_n e_j)(t) = t_j^2 + (1/n)(t_j - t_j^2)$ for all $t \in \tilde{\mathbb{T}}$, $j = 1, 2, ..., m$, $n \in \mathbb{N}$. Hence, $\tilde{B}_n$ preserves constants, and

$$
\gamma_n^2(t) = \tilde{B}_n(\|\cdot - t\|_2^2)(t) = \sum_{j=1}^{m} \frac{t_j - t_j^2}{n} \le \frac{\bar{t}}{n}.
$$

The conclusion now follows from Theorem 2.1. ∎

If we replace $f(k/n) = f(k_1/n, ..., k_m/n)$ in (4) with (15) then we define the *m-dimensional Chung operator* $\tilde{C}_n : \mathcal{BC}(\mathbb{R}_{0^+}^m, X) \to \mathcal{C}(\tilde{\mathbb{T}}, X)$ as follows:

$$(\tilde{C}_n f)(t) := \sum_{\bar{k}=0}^{n} c(f,n,k)\tilde{p}_{n,k}(t) \text{ for } f \in \mathcal{BC}(\mathbb{R}^m_{0+}, X),\ t \in \tilde{\mathbb{T}}. \tag{25}$$

This is another generalization of the one-dimensional Chung operator considered in [6].

Theorem 4.3. *For every* $f \in \mathcal{BUC}(\mathbb{R}^m_{0+}, X)$*, we have*

$$\|(\tilde{C}_n f)(t) - f(t)\|_X \le 2\omega(f, \mathbb{R}^m_{0+}, \sqrt{\frac{2\bar{t}}{n}}),\ t \in \tilde{\mathbb{T}}. \tag{26}$$

Proof. Using Lemma 4.1 we have $(\tilde{C}_n 1)(t) = 1$,

$$\begin{aligned}
&(\tilde{C}_n e_j)(t)\\
&= \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\Big(\int_0^\infty \cdots \int_0^\infty \big(\prod_{i=1}^{m} \frac{n^{k_i}}{(k_i-1)!} e^{-nu_i} u_i^{k_i-1}\big) u_j \, du_1 \cdots du_m\Big)\\
&= \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\Big(\frac{n^{k_j}}{(k_j-1)!} \int_0^\infty e^{-nu_j} u_j^{k_j} \, du_j\Big) = \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\big(\frac{k_j}{n}\big) = t_j,\\
&(\tilde{C}_n e_j^2)(t)\\
&= \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\Big(\frac{n^{k_j}}{(k_j-1)!} \int_0^\infty e^{-nu_j} u_j^{k_j+1} \, du_j\Big) = \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\big(\frac{k_j(k_j+1)}{n^2}\big)\\
&= \frac{1}{n^2}\{(2nt_j) + \big(n(n-1)t_j^2\big)\} = t_j^2 + \frac{1}{n}(2t_j - t_j^2)
\end{aligned}$$

for all $t \in \tilde{\mathbb{T}},\ j = 1, 2, ..., m,\ n \in \mathbb{N}$. Hence $\tilde{C}_n$ preserves constants and

$$\gamma_n^2(t) = \tilde{C}_n(\|\cdot - t\|_2^2)(t) = \sum_{j=1}^{m} \frac{2t_j - t_j^2}{n} \le \frac{2\bar{t}}{n}.$$

Thus Theorem 2.1 implies the estimate in (26). ∎

Next, if we replace $f(\frac{k}{n})$ in (4) with (18), we obtain the *m-dimensional Kantorovitch operator* $\tilde{K}_n : \mathcal{C}(\mathbb{T}, X) \to \mathcal{C}(\tilde{\mathbb{T}}, X)$ as follows:

$$(\tilde{K}_n f)(t) := \sum_{\bar{k}=0}^{n} \kappa(f,n,k)\tilde{p}_{n,k}(t) \text{ for } f \in \mathcal{C}(\mathbb{T}, X),\ t \in \tilde{\mathbb{T}}. \tag{27}$$

This is another generalization of the original one-dimensional Kantorovitch operator. For this operator, we have the following theorem.

Theorem 4.4. *For every* $f \in \mathcal{C}(\tilde{\mathbb{T}}, X)$ *and* $t \in \tilde{\mathbb{T}}$,

$$\|(\tilde{K}_n f)(t) - f(t)\|_X \le 2\omega(f, \tilde{\mathbb{T}}, \sqrt{\frac{(3n-2)m}{3(n+1)^2}}). \tag{28}$$

Proof. Using Lemma 4.1 we have $(\tilde{K}_n 1_{\tilde{\mathbb{T}}})(t) = \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t) = 1$,

$$(\tilde{K}_n e_j)(t) = \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\big((n+1)^m(\int_{\frac{k_m}{n+1}}^{\frac{k_m+1}{n+1}} \cdots \int_{\frac{k}{n+1}}^{\frac{k_1+1}{n+1}} u_j \, du_1 \cdots du_m)\big)$$

$$= \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\big(\frac{1}{2}(n+1)\frac{2k_j+1}{(n+1)^2}\big) = t_j + \frac{1}{2(n+1)}(1-2t_j),$$

$$(\tilde{K}_n e_j^2)(t) = \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\big((n+1)(\int_{\frac{k_j}{n+1}}^{\frac{k_j+1}{n+1}} u_j^2 \, du_j\big)$$

$$= \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\big(\frac{1}{3}(n+1)\frac{3k_j^2+3k_j+1}{(n+1)^3}\big)$$

$$= t_j^2 + \frac{1}{(n+1)^2}\big(\frac{1}{3} + 2nt_j - (3n+1)t_j^2\big)$$

for all $t \in \tilde{\mathbb{T}}$, $j = 1, 2, ..., m$, $n \in \mathbb{N}$.

Hence $\tilde{K}_n$ preserves constants, and

$$\gamma_n^2(t) = \tilde{K}_n(\| \cdot - t\|_2^2)(t) = \sum_{j=1}^{m} \frac{3(n-1)(t-t^2)+1}{3(n+1)^2} \leq m\frac{(3n-2)}{3(n+1)^2}.$$

Then the estimate (28) follows from Theorem 2.1. ∎

If we replace $f(\frac{k}{n}) = f(\frac{k_1}{n}, ..., \frac{k_m}{n})$ in (4) with (21), then we obtain the *m-dimensional Durrmeyer operator* $\tilde{D}_n : \mathcal{C}(\mathbb{T}, X) \to \mathcal{C}(\tilde{\mathbb{T}}, X)$ defined as follows:

$$(\tilde{D}_n f)(t) := \sum_{\bar{k}=0}^{n} d(f, n, k)\tilde{p}_{n,k}(t) \text{ for } f \in \mathcal{C}(\mathbb{T}, X),\ t \in \tilde{\mathbb{T}}. \tag{29}$$

Theorem 4.5. *For every $f \in \mathcal{C}(\mathbb{T}, X)$ and $t \in \tilde{\mathbb{T}}$,*

$$\|(\tilde{D}_n f)(t) - f(t)\|_X \leq 2\omega(f, \mathbb{T}, \sqrt{\frac{2(n-2)m}{(n+2)(n+3)}}). \tag{30}$$

Proof. Using Lemma 4.1 we have $(\tilde{D}_n 1_{\tilde{\mathbb{T}}})(t) = 1$,

$$(\tilde{D}_n e_j)(t)$$
$$= \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\big((n+1)^m \int_0^1 \cdots \int_0^1 \prod_{i=1}^{m} \binom{n}{k_i} u_i^{k_i}(1-u_i)^{n-k_i} u_j \, du_1 \cdots du_m\big)$$

$$= \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\big((n+1)\binom{n}{k_j}\int_0^1 u_j^{k_j+1}(1-u_j)^{n-k_j}\,du_j\big)$$

$$= \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)(\frac{k_j+1}{n+2}) = t_j + \frac{1}{n+2}(1-2t_j),$$

$$(\tilde{D}_n e_j^2)(t) = \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\big((n+1)\binom{n}{k_j}\int_0^1 u_j^{k_j+2}(1-u_j)^{n-k_j}\,du_j\big)$$

$$= \sum_{\bar{k}=0}^{n} \tilde{p}_{n,k}(t)\big(\frac{(k_j+2)(k_j+1)}{(n+3)(n+2)}\big) = t_j^2 + \frac{1}{n^2+5n+6}\big(2+4nt_j-6(n+1)t_j^2\big)$$

for all $t \in \tilde{\mathbb{T}},\ j = 1, 2, ..., m,\ n \in \mathbb{N}$. Hence $\tilde{D}_n$ preserves constants and

$$\gamma_n^2(t) = \tilde{D}_n(\| \cdot - t\|_2^2)(t) = \sum_{j=1}^{m} \frac{3(n-1)(2(n-3)(t-t_2)+2}{(n+2)(n+3)}$$

$$\le m\frac{2(n-2)}{(n+2)(n+3)}.$$

Hence Theorem 2.1 implies the estismate in (30). ■

5. Derivation of Representation Formulas for m-Parameter Semigroups

The following lemmas present some representation formulas for an m-parameter semigroup $T(\cdot)$. Then the estimates (5) - (10) in Theorems 1.1 - 1.3 follow immediately from the theorems in Secs. 3 and 4.

First, because of the nice semigroup property: $T_i(k_i t_i) = \big(T_i t_i\big)^{k_i}$, we can write for $t \in \mathbb{T}$

$$B_n\big(T(\cdot)x\big)(t) = \sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\Big(\prod_{i=1}^{m}\binom{n}{k_i}(t_i)^{k_i}(1-t_i)^{n-k_i}\Big)T(\frac{k_1}{n},\cdots,\frac{k_m}{n})x$$

$$= \sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\prod_{i=1}^{m}\Big(\binom{n}{k_i}(t_i)^{k_i}(1-t_i)^{n-k_i}\Big(T_i\Big(\frac{1}{n}\Big)\Big)^{k_i}\Big)x$$

$$= \prod_{i=1}^{m}\Big\{\sum_{k_i=0}^{n}\Big(\binom{n}{k_i}\Big(t_i T_i\Big(\frac{1}{n}\Big)\Big)^{k_i}(1-t_i)^{n-k_i}\Big)x\Big\}$$

$$= \prod_{i=1}^{m}\Big(1 + t_i\Big(T_i(\frac{1}{n}) - 1\Big)\Big)^n x.$$

This and Theorem 3.2 yield formula (5).

Next, let $t \in \tilde{\mathbb{T}}$. Since

$$\begin{aligned}\tilde{B}_n\big(T(\cdot)x\big)(t) &= \sum_{\bar{k}=0}^{n}\binom{n}{k}\Big(\prod_{i=1}^{m}(t_i)^{k_i}(1-\bar{t})^{n-\bar{k}}\Big)T(\frac{k_1}{n},\cdots,\frac{k_m}{n})x\\ &= \sum_{\bar{k}=0}^{n}\binom{n}{k}\prod_{i=1}^{m}\Big((t_i)^{k_i}(1-\bar{t})^{n-\bar{k}}\big(T_i(\frac{1}{n})\big)^{k_i}\Big)x\\ &= \Big(1-\bar{t}+\sum_{i=1}^{m}t_iT_i(\frac{1}{n})\Big)^n x = \Big(1+\sum_{i=1}^{m}t_i\big(T_i(\frac{1}{n})-1\big)\Big)^n x,\end{aligned}$$

(6) follows from this and Theorem 4.2.

Next, using the property that

$$A_i\int_0^t T(s)xds = T_i(t)x - x \text{ and } (\lambda - A_i)^{-1}x = \int_0^\infty e^{-\lambda}T_i(t)xdt$$

for all $x \in X,\ t \geq 0$ and large λ, we have for $t \in \mathbb{T}$

$$\begin{aligned}&C_n\big(T(\cdot)x\big)(t)\\ &= \sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\Big(\prod_{i=1}^{m}\binom{n}{k_i}(t_i)^{k_i}(1-t_i)^{n-k_i}\Big)\\ &\times\int_0^\infty\cdots\int_0^\infty\big(\prod_{i=1}^{m}\frac{n^{k_i}}{(k_i-1)!}e^{-nu_i}u_i^{k_i-1}\big)T(u_1,\cdots,u_m)x\,du_1\cdots du_m\\ &= \sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\prod_{i=1}^{m}\Big(\binom{n}{k_i}(t_i)^{k_i}(1-t_i)^{n-k_i}\frac{n^{k_i}}{(k_i-1)!}\\ &\times\Big(\int_0^\infty e^{-nu_i}u_i^{k_i-1}T_i(u_i)\,du_i\Big)\Big)x\\ &= \sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\prod_{i=1}^{m}\Big(\binom{n}{k_i}(t_i)^{k_i}(1-t_i)^{n-k_i}n^{k_i}(n-A_i)^{-k_i}\Big)x\\ &= \prod_{i=1}^{m}\Big(1-t_i+nt_i(n-A_i)^{-1}\Big)^n x = \prod_{i=1}^{m}\Big(1+t_i\big((1-\frac{A_i}{n})^{-1}-1\big)\Big)^n x.\end{aligned}$$

Thus (7) follows from this and Theorem 3.3.

Next, since for $t \in \tilde{\mathbb{T}}$

$$\begin{aligned}&\tilde{C}_n\big(T(\cdot)x\big)(t)\\ &= \sum_{\bar{k}=0}^{n}\binom{n}{k}\big(\prod_{i=1}^{m}(t_i)^{k_i}(1-\bar{t})^{n-\bar{k}}\big)\\ &\quad\times\int_0^\infty\cdots\int_0^\infty\big(\prod_{i=1}^{m}\frac{n^{k_i}}{(k_i-1)!}e^{-nu_i}u_i^{k_i-1}\big)T(u_1,\cdots,u_m)x\,du_1\cdots du_m\end{aligned}$$

$$= \sum_{\bar{k}=0}^{n} \binom{n}{k} (1-\bar{t})^{n-\bar{k}} \prod_{i=1}^{m} \Big(t_i^{k_i} \frac{n^{k_i}}{(k_i-1)!} (\int_0^\infty e^{-nu_i} u_i^{k_i-1} T_i(u_i)\, du_i) \Big) x$$

$$= \sum_{\bar{k}=0}^{n} \binom{n}{k} (1-\bar{t})^{n-\bar{k}} \prod_{i=1}^{m} \Big((nt_i)^{k_i} (n-A_i)^{-k_i} \Big) x$$

$$= \Big(1 - \bar{t} + \sum_{i=1}^{m} t_i (1 - \frac{A_i}{n})^{-1} \Big)^n x = \Big(1 + \sum_{i=1}^{m} t_i \big((1-\frac{A_i}{n})^{-1} - 1 \big) \Big)^n x,$$

(8) follows from this and Theorem 4.3.

Next, since for $t \in \mathbb{T}$

$$K_n\big(T(\cdot)x\big)(t)$$

$$= (n+1)^m \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} \prod_{i=1}^{m} \Big(\binom{n}{k_i} (t_i)^{k_i} (1-t_i)^{n-k_i} \Big(\int_{\frac{k_i}{n+1}}^{\frac{k_i+1}{n+1}} T_i(u_i)\, du_i \Big) \Big) x$$

$$= (n+1)^m \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} \prod_{i=1}^{m} \Big(\binom{n}{k_i} (t_i)^{k_i} (1-t_i)^{n-k_i}$$

$$\times A_i^{-1} \big(T_i(\frac{k_i+1}{n+1}) - T_i(\frac{k_i}{n+1}) \big) \Big) x$$

$$= (n+1)^m \sum_{k_1=0}^{n} \cdots \sum_{k_m=0}^{n} \prod_{i=1}^{m} \Big(\binom{n}{k_i} \big(t_i T_i(\frac{1}{n+1}) \big)^{k_i} (1-t_i)^{n-k_i}$$

$$\times A_i^{-1} \big(T_i(\frac{1}{n+1}) - I \big) \Big) x$$

$$= \prod_{i=1}^{m} \Big((n+1) A_i^{-1} \big(T_i(\frac{1}{n+1}) - I \big) \big(1 - t_i + t_i T_i(\frac{1}{n+1}) \big)^n \Big) x,$$

(9) follows from this and Theorem 3.4. Next, let $t \in \tilde{\mathbb{T}}$. Since

$$\tilde{K}_n\big(T(\cdot)x\big)(t) = (n+1)^m \sum_{\bar{k}=0}^{n} \binom{n}{k} (1-\bar{t})^{n-\bar{k}} \left[\prod_{i=1}^{m} \Big(t_i^{k_i} \int_{\frac{k_i}{n+1}}^{\frac{k_i+1}{n+1}} T_i(u_i)\, du_i \Big) \right] x$$

$$= (n+1)^m \sum_{\bar{k}=0}^{n} \binom{n}{k} (1-\bar{t})^{n-\bar{k}} \left[\prod_{i=1}^{m} \Big(t_i^{k_i} A_i^{-1} \big(T_i(\frac{k_i+1}{n+1}) - T_i(\frac{k_i}{n+1}) \big) \Big) \right] x$$

$$= (n+1)^m \sum_{\bar{k}=0}^{n} \binom{n}{k} (1-\bar{t})^{n-\bar{k}} \prod_{i=1}^{m} \Big(t_i^{k_i} A_i^{-1} \big(T_i(\frac{1}{n+1}) - I \big) \big(T_i(\frac{1}{n+1}) \big)^{k_i} \Big) x$$

$$= \left[\prod_{i=1}^{m} \Big((n+1) A_i^{-1} \big(T_i(\frac{1}{n+1}) - I \big) \Big) \right] \Big(1 - \bar{t} + \sum_{i=1}^{m} t_i T_i(\frac{1}{n+1}) \Big)^n x,$$

(10) follows from this and Theorem 4.4.

References

1. P. L. Butzer and H. Berens, *Semi-group of Operators and Approximation,* Springer-Verlag, New York, 1967.
2. M. M. Derriennic, Sur l'approximation de functions intégrables sur $[0, 1]$ par des polynômes de Bernstein modifies, *J. Approx. Theory* **31** (1981) 325–343.
3. P. P. Korovkin, *Linear Operators and Approximation Theory,* Hindustan Publishing Corp., Delhi, 1960.
4. G. G. Lorentz, *Bernstein Polynomials,* Univ. of Toronto Press, Toronto, 1953.
5. J. B. Prolla, *Approximation by Vector-valued Functions,* North-Holland, 1977.
6. S.-Y. Shaw, Approximation of unbounded functions and applications to representations of semigroups, *J. Approx. Theory* **28** (1980) 238–259.
7. S.-Y. Shaw, Some exponential formulas for m-parameter semigroups, *Bull. Inst. Math. Acad. Sinica* **9** (1981) 221–228.
8. S.-Y. Shaw, C.-S. Lee, and W.-L. Chiou, Representation formulas for cosine and sine functions of operators II, *Aequationes Mathematicae* **31** (1986) 64–75.
9. S.-Y. Shaw and Y.-C. Li, Representation formulas for C-semigroups, *Semigroup Forum* **46** (1993) 123–125.
10. O. Shisha and B. Mond, The degree of convergence of sequences of linear positive operators, *Proc. Nat. Acad. Sci. U.S.A.* **60** (1968) 1196–1200.