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Newmark's Method and Discrete Energy Applied to Resistive MHD Equation

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Abstract. For the second order time evolution equation with a general dissipation term, we introduce a recurrence relation of Newmark's method. Deriving an energy inequality from this relation, we obtain the stability and the convergence theories of Newmark's method. Next we take up resistive MHD equation as an application of Newmark's method. We introduce a discrete energy of the solution derived from the above energy inequality. We investigate this quantity using numerical experiments.

1. Introduction of Newmark's Method

We recall the basic ideas of Newmark's method [7, 10] for the second order time evolution equation in the d-dimensional Euclidean space $\mathbb{R}^d$. Let M, C and K be $d \times d$ real symmetric matrices on $\mathbb{R}^d$ which are constant in time, and let $f(t)$ be a given $\mathbb{R}^d$-valued function on $[0, \infty)$. We use the usual Euclidean scalar product $(\cdot, \cdot)$ and the corresponding norm $\| \cdot \|$ in $\mathbb{R}^d$. Throughout this paper, we assume that M is positive definite:

$$(Mx, x) > 0, \quad \forall x \neq 0, \tag{1}$$

which is equivalent to the condition:

$$(Mx, x) \geq m(x, x), \quad \forall x, \tag{2}$$

with a positive constant m. The identity matrix on $\mathbb{R}^d$ is denoted by I. We consider the approximation method for the following initial value problem of the second order time evolution equation for an $\mathbb{R}^d$-valued function $u(t)$:

$$M\frac{d^2u}{dt^2}(t) + C\frac{du}{dt}(t) + Ku(t) = f(t), \quad u(0) = u_0, \quad \frac{du}{dt}(0) = v_0. \tag{3}$$

Let a_n, v_n and u_n be approximations of $(d^2/dt^2)u(t)$, $(d/dt)u(t)$ and $u(t)$ respectively at $t = \tau n$ with a positive time step τ and an integer n, and let $f_n = f(\tau n)$. Then Newmark's method for (3) is defined through the following relations:

$$\begin{cases} Ma_n + Cv_n + Ku_n = f_n, \\ u_{n+1} = u_n + \tau v_n + \frac{1}{2}\tau^2 a_n + \beta\tau^2(a_{n+1} - a_n), \\ v_{n+1} = v_n + \tau a_n + \gamma\tau(a_{n+1} - a_n). \end{cases} \tag{4}$$

The first relation corresponds to the equation (3), and the second and the third relations correspond to the Taylor expansions of $u(t+\tau)$ and $v(t+\tau)$ at $t = \tau n$ respectively. Here, β and γ are positive tuning parameters of the method.

An interpretation of the parameter β related to the acceleration $(d^2/dt^2)u(t)$ can be seen in [7]: The case $\beta = 1/6$ corresponds to the approximation where the acceleration is a linear function of t in each time interval; the case $\beta = 1/4$ corresponds to the approximation of the acceleration to be a constant function during the time interval which is equal to the mean value of the initial and final values of acceleration; the case $\beta = 1/8$ corresponds to the case where the acceleration is a step function with the initial value for the first half of each time interval and the final value for the second half of the interval. It is often the case that γ is equal to $1/2$.

2. Iteration Scheme of Newmark's Method

Based on the formulas in (4), Newmark's method generates the approximation sequence $u_n, n = 0, 1, 2, \ldots, N$ by the following iteration scheme:

Step 1. For $n = 0$, compute a_0 from the initial data u_0 and v_0:

$$a_0 = M^{-1}(f_0 - Cv_0 - Ku_0).$$

Step 2. Compute a_{n+1} from f_{n+1}, u_n, v_n and a_n solving a linear equation:

$$\begin{aligned} a_{n+1} = {} & (M + \gamma\tau C + \beta\tau^2 K)^{-1} \\ & \times \Big[- Ku_n - (C + \tau K)v_n + \{(\gamma - 1)\tau C + (\beta - \frac{1}{2})\tau^2 K\}a_n + f_{n+1}\Big] \end{aligned}$$

Step 3. Compute u_{n+1} from u_n, v_n, a_n and a_{n+1}:

$$u_{n+1} = u_n + \tau v_n + (\frac{1}{2} - \beta)\tau^2 a_n + \beta\tau^2 a_{n+1}.$$

Step 4. Compute v_{n+1} from v_n, a_n and a_{n+1}:

$$v_{n+1} = v_n + (1 - \gamma)\tau a_n + \gamma\tau a_{n+1}.$$

Step 5. Replace n by $n + 1$, and return to Step 2.

Here, Step 1 is nothing but the first relation of (4) with $n = 0$. The expression of a_{n+1} in Step 2 is obtained by eliminating u_{n+1} and v_{n+1} from the second and the third equalities in (4) together with the first equality in (4) with n replaced by $n+1$:

$$Ma_{n+1} + Cv_{n+1} + Ku_{n+1} = f_{n+1}.$$

Step 3 and Step 4 are from the second and the third relations of (4).

3. Recurrence Relation of Newmark's Method

Newmark's method (4) for the second order equation (3) is reformulated as follows in the recurrence relation [1, 11]. Eliminating v_n and a_n from (4) by a series of lengthy computations, we have

$$\begin{aligned}(M + \beta\tau^2 K)D_{\tau\bar{\tau}}u_n + \gamma C D_\tau u_n + \{(1-\gamma)C + \tau(\gamma - \frac{1}{2})K\}D_{\bar{\tau}}u_n + Ku_n \\ = \{I + \tau(\gamma - \frac{1}{2})D_{\bar{\tau}} + \beta\tau^2 D_{\tau\bar{\tau}}\}f_n,\end{aligned} \tag{5}$$

where

$$\begin{cases} D_\tau u_n = & (u_{n+1} - u_n)/\tau, \\ D_{\bar{\tau}} u_n = & (u_n - u_{n-1})/\tau, \\ D_{\tau\bar{\tau}} u_n = & (D_\tau u_n - D_{\bar{\tau}} u_n)/\tau. \end{cases}$$

We confirmed this result by a formula manipulation software NCAlgebra [4] on Mathematica [12]. Especially, in the case $\gamma = 1/2$, we have:

$$(M + \beta\tau^2 K)D_{\tau\bar{\tau}}u_n + \frac{1}{2}C(D_\tau + D_{\bar{\tau}})u_n + Ku_n = (I + \beta\tau^2 D_{\tau\bar{\tau}})f_n. \tag{6}$$

These recurrence relations are useful for the stability and error analysis of Newmark's method. See [6, 9] for the case with $C \equiv 0$.

4. Derivation of an Energy Inequality

Taking a scalar product of (5) and $(D_\tau + D_{\bar{\tau}})u_n$, we can derive an energy inequality for Newmark's method. From this inequality, we obtain the stability conditions for Newmark's method.

From now on, let T be a fixed positive number and N be a positive integer, and define $\tau := T/N$ and $n = 0, 1, 2, \ldots, N$. Let C be nonnegative: $(Cx, x) \geq 0$, K be positive definite: $(Kx, x) \geq k(x, x)$ with $k > 0$, and $m > 0$ be the smallest eigenvalue of M. We also assume that $\gamma \geq 1/2$. We have the following theorem for the energy inequality.

Theorem 1. *Let $\{u_n\}_{n=0}^N$ be a sequence generated by the scheme in Sec.* 2. *Then for a positive constant τ_0 determined as below, there exists a positive constant C_0 such that the inequality*

$$\|M^{1/2}D_\tau u_n\|^2 + \tau^2\Big\{\beta - \frac{1}{2}\Big(\gamma - \frac{1}{2}\Big) - \frac{1}{2}\alpha\Big\}\|K^{1/2}D_\tau u_n\|^2 + \Big(1 - \frac{1}{2\alpha}\Big)\|K^{1/2}u_n\|^2 \leq C_0 \tag{7}$$

holds for all $\tau \le \tau_0$ and an arbitrary positive constant α, where τ_0 is determined either as

$$\tau_0 > 0 \quad \textit{when} \quad \beta \ge \frac{\gamma}{2}, \tag{8}$$

or as

$$0 < \tau_0 < \sqrt{\frac{m}{(\frac{1}{2}\gamma - \beta)\|K\|}} \quad \textit{when} \quad 0 \le \beta < \frac{\gamma}{2}. \tag{9}$$

Remark 1. The constant C_0 is defined as follows:

$$\begin{aligned} C_0 := & \left(1 - \frac{1}{2}\delta\tau_0\right)^{-1} \left[w_0 + \frac{T}{\delta^2} \left\{ (4\beta + 2\gamma) \sup_{0 \le t \le T} \|f(t)\| \right\}^2 \right] \\ & \times \ \exp\left(\delta \left(1 - \frac{1}{2}\delta\tau_0\right)^{-1} T \right), \end{aligned} \tag{10}$$

where

$$\begin{aligned} w_0 = & \|M^{1/2} D_\tau u_0\|^2 + \tau^2 \left\{ \beta - \frac{1}{2}\left(\gamma - \frac{1}{2}\right) \right\} \|K^{1/2} D_\tau u_0\|^2 \\ & + \tau(K D_\tau u_0, u_0) + \|K^{1/2} u_0\|^2 + \tau\Big(\gamma - \frac{1}{2}\Big)\|C^{1/2} D_\tau u_0\|^2, \end{aligned} \tag{11}$$

and δ in (11) is defined as:

$$0 < \delta \le m, \quad \text{in the case (8)}, \tag{12}$$

or

$$0 < \delta \le m - \tau_0^2 \left(\frac{1}{2}\gamma - \beta \right) \|K\|, \quad \text{in the case (9)}. \tag{13}$$

Proof of Theorem 1. Using (5), we derive an energy inequality as follows (see [1] for the case $\gamma = 1/2$). Rearranging (5), we have

$$\begin{aligned} & (M + \beta\tau^2 K) D_{\tau\bar\tau} u_n + \tfrac{1}{2} C (D_\tau + D_{\bar\tau}) u_n \\ & \quad + (\gamma - \frac{1}{2}) C (D_\tau - D_{\bar\tau}) u_n + \tau(\gamma - \frac{1}{2}) K D_{\bar\tau} u_n + K u_n = g_n, \end{aligned} \tag{14}$$

where

$$g_n := \Big\{ I + \tau\Big(\gamma - \frac{1}{2}\Big) D_{\bar\tau} + \beta\tau^2 D_{\tau\bar\tau} \Big\} f_n. \tag{15}$$

We take a scalar product of (14) and $(D_\tau + D_{\bar\tau})u_n$. Since C is nonnegative, we have

$$\begin{aligned} & ((M + \beta\tau^2 K) D_{\tau\bar\tau} u_n, (D_\tau + D_{\bar\tau}) u_n) + \Big(\gamma - \frac{1}{2}\Big)(C(D_\tau - D_{\bar\tau})u_n, (D_\tau + D_{\bar\tau})u_n) \\ & + \tau\Big(\gamma - \frac{1}{2}\Big)(K D_{\bar\tau} u_n, (D_\tau + D_{\bar\tau}) u_n) + (K u_n, (D_\tau + D_{\bar\tau}) u_n) - (g_n, (D_\tau + D_{\bar\tau}) u_n) \\ & = -\Big(\frac{1}{2} C (D_\tau + D_{\bar\tau}) u_n, (D_\tau + D_{\bar\tau}) u_n\Big) \le 0. \end{aligned} \tag{16}$$

Using the assumption for M, C and K, we obtain the following lemma:

Lemma 1. *When we put*

$$w_n := ((M + \beta\tau^2 K)D_\tau u_n, D_\tau u_n) + (Ku_{n+1}, u_n)$$
$$+ \tau\Big(\gamma - \frac{1}{2}\Big)(CD_\tau u_n, D_\tau u_n) - \frac{1}{2}\tau^2\Big(\gamma - \frac{1}{2}\Big)(KD_\tau u_n, D_\tau u_n),$$

we have

$$w_n \leq w_{n-1} + \tau(g_n, D_\tau u_n + D_\tau u_{n-1}). \tag{17}$$

Proof. Multiplying τ to both side of (16) and using the relation:

$$\tau D_{\tau\bar{\tau}} = D_\tau - D_{\bar{\tau}},$$

we have

$$\begin{aligned}
&((M+\beta\tau^2 K)(D_\tau - D_{\bar{\tau}})u_n, (D_\tau + D_{\bar{\tau}})u_n)\\
&\quad +\tau\Big(\gamma - \frac{1}{2}\Big)(C(D_\tau - D_{\bar{\tau}})u_n, (D_\tau + D_{\bar{\tau}})u_n)\\
&\quad +\frac{1}{2}\tau^2\Big(\gamma - \frac{1}{2}\Big)(K(D_\tau + D_{\bar{\tau}})u_n, (D_\tau + D_{\bar{\tau}})u_n)\\
&\quad -\frac{1}{2}\tau^2\Big(\gamma - \frac{1}{2}\Big)(K(D_\tau - D_{\bar{\tau}})u_n, (D_\tau + D_{\bar{\tau}})u_n)\\
&\quad +\tau(Ku_n, (D_\tau + D_{\bar{\tau}})u_n) - \tau(g_n, (D_\tau + D_{\bar{\tau}})u_n) \leq 0.
\end{aligned} \tag{18}$$

Using the identities:

$$((D_\tau - D_{\bar{\tau}})u_n, (D_\tau + D_{\bar{\tau}})u_n) = (D_\tau u_n, D_\tau u_n) - (D_\tau u_{n-1}, D_\tau u_{n-1}),$$

and

$$D_{\bar{\tau}} u_n = \frac{1}{2}(D_\tau + D_{\bar{\tau}})u_n - \frac{1}{2}(D_\tau - D_{\bar{\tau}})u_n,$$

we have

$$\begin{aligned}
&((M + \beta\tau^2 K)D_\tau u_n, D_\tau u_n) - ((M + \beta\tau^2 K)D_\tau u_{n-1}, D_\tau u_{n-1})\\
&+ \tau\Big(\gamma - \frac{1}{2}\Big)(CD_\tau u_n, u_n) - \tau\Big(\gamma - \frac{1}{2}\Big)(CD_\tau u_{n-1}, u_{n-1})\\
&+ \frac{1}{2}\tau^2(\gamma - \frac{1}{2})(K(D_\tau + D_{\bar{\tau}})u_n, (D_\tau + D_{\bar{\tau}})u_n)\\
&+ \frac{1}{2}\tau^2\Big(\gamma - \frac{1}{2}\Big)(KD_\tau u_n, D_\tau u_n) - \frac{1}{2}\tau^2\Big(\gamma - \frac{1}{2}\Big)(KD_\tau u_{n-1}, D_\tau u_{n-1})\\
&+ \tau(Ku_n, (D_\tau + D_{\bar{\tau}})u_n) - \tau(g_n, (D_\tau + D_{\bar{\tau}})u_n) \leq 0.
\end{aligned} \tag{19}$$

Furthermore, using the relation:

$$\tau(Ku_n, (D_\tau + D_{\bar{\tau}})u_n) = (Ku_n, u_{n+1} - u_{n-1}) = (Ku_{n+1}, u_n) - (Ku_n, u_{n-1}),$$

we have

$$\begin{aligned}
&((M+\beta\tau^2K)D_\tau u_n, D_\tau u_n) - ((M+\beta\tau^2K)D_\tau u_{n-1}, D_\tau u_{n-1})\\
&+\tau\Big(\gamma-\frac{1}{2}\Big)(CD_\tau u_n, u_n) - \tau\Big(\gamma-\frac{1}{2}\Big)(CD_\tau u_{n-1}, u_{n-1})\\
&+\frac{1}{2}\tau^2\Big(\gamma-\frac{1}{2}\Big)(KD_\tau u_n, D_\tau u_n) - \frac{1}{2}\tau^2\Big(\gamma-\frac{1}{2}\Big)(KD_\tau u_{n-1}, D_\tau u_{n-1})\\
&+(Ku_{n+1}, u_n) - (Ku_n, u_{n-1}) + \frac{1}{2}\tau^2\Big(\gamma-\frac{1}{2}\Big)\\
&\times (K(D_\tau + D_{\bar\tau})u_n, (D_\tau + D_{\bar\tau})u_n) \le \tau(g_n, (D_\tau + D_{\bar\tau})u_n).
\end{aligned} \tag{20}$$

We neglect the nonnegative term $(1/2)\tau^2(\gamma-1/2)(K(D_\tau+D_{\bar\tau})u_n, (D_\tau+D_{\bar\tau})u_n)$ in the left-hand side and obtain (17). ■

Proof of Theorem 1 (continued). Using the equality:

$$(Ku_{n+1}, u_n) = \tau(KD_\tau u_n, u_n) + (Ku_n, u_n),$$

we can modify the expression of w_n as

$$\begin{aligned}
w_n = \|M^{1/2}D_\tau u_n\|^2 + \ \tau^2\Big\{\beta - \frac{1}{2}(\gamma-\frac{1}{2})\Big\}\|K^{1/2}D_\tau u_n\|^2\\
+\tau(KD_\tau u_n, u_n) + \|K^{1/2}u_n\|^2 + \tau\Big(\gamma-\frac{1}{2}\Big)\|C^{1/2}D_\tau u_n\|^2.
\end{aligned} \tag{21}$$

Summing up (17) from $n = 1$ to n, we obtain

$$w_n \le w_0 + \sum_{i=1}^{n} \tau(g_i, D_\tau u_i + D_\tau u_{i-1}). \tag{22}$$

To modify (22), we show the next lemma.

Lemma 2. *Under either the condition*

$$(8) \text{ and } 0 < \delta \le m, \tag{23}$$

or

$$(9) \text{ and } \tau \le \tau_0 \text{ and } 0 < \delta \le m - \tau_0^2\Big(\frac{1}{2}\gamma - \beta\Big)\|K^{1/2}\|^2, \tag{24}$$

we have

$$\delta\|D_\tau u_i\|^2 \le w_i. \tag{25}$$

Proof. Let δ be a positive number. Using the assumption that M and K are positive definite, C is nonnegative and $\gamma \ge 1/2$, we have

$$\begin{aligned}
w_i - \delta\|D_\tau u_i\|^2 &\ge \|M^{1/2}D_\tau u_i\|^2 + \tau^2\Big(\beta - \frac{1}{2}\gamma\Big)\|K^{1/2}D_\tau u_i\|^2\\
&\quad + \frac{1}{4}\|K^{1/2}(\tau D_\tau u_i + 2u_i)\|^2 - \delta\|D_\tau u_i\|^2\\
&\ge (m-\delta)\|D_\tau u_i\|^2 + \tau^2(\beta - \frac{1}{2}\gamma)\|K^{1/2}D_\tau u_i\|^2\\
&\quad + \frac{1}{4}\|K^{1/2}(\tau D_\tau u_i + 2u_i)\|^2.
\end{aligned}$$

If β and γ satisfy the condition (8) and $\delta \le m$, then $w_i - \delta\|D_\tau u_i\|^2$ becomes nonnegative for any i.

On the other hand, if $\beta - \gamma/2 < 0$, then we have

$$\begin{aligned} w_i - \delta\|D_\tau u_i\|^2 &\ge m\|D_\tau u_i\|^2 - \tau^2\Big(\frac{1}{2}\gamma - \beta\Big)\|K^{1/2}\|^2\|D_\tau u_i\|^2 - \delta\|D_\tau u_i\|^2 \\ &= \{m - \tau^2(\frac{1}{2}\gamma - \beta)\|K^{1/2}\|^2 - \delta\}\|D_\tau u_i\|^2. \end{aligned}$$

If τ_0 satisfies the condition (9) and $\tau \le \tau_0$ then

$$m - \tau^2\Big(\frac{1}{2}\gamma - \beta\Big)\|K^{1/2}\|^2 - \delta > m - \tau_0^2(\frac{1}{2}\gamma - \beta)\|K^{1/2}\|^2 - \delta.$$

Hence, if $\delta \le m - \tau_0^2(\frac{1}{2}\gamma - \beta)\|K^{1/2}\|^2$, then $w_i - \delta\|D_\tau u_i\|^2 \ge 0$ for any i. Thus we obtain (25). ■

Using Lemma 2, we have the next lemma.

Lemma 3. *Under either the condition*

$$(8)\ \ \textit{and}\ 0 < \delta \le \min\{m, 2/\tau_0\}, \tag{26}$$

or

$$(9)\ \ \textit{and}\ \ \tau \le \tau_0\ \textit{and}\ 0 < \delta \le \min\Big\{m - \tau_0^2\Big(\frac{1}{2}\gamma - \beta\Big)\|K^{1/2}\|^2, 2/\tau_0\Big\}, \tag{27}$$

we have, for $\tau \le \tau_0$,

$$w_n \le C_0, \quad n = 0, 1, 2, \ldots, N-1. \tag{28}$$

Proof. Using (25), we modify (22) as follows:

$$\begin{aligned} w_n &\le w_0 + \tau\sum_{i=1}^{n}(g_i, D_\tau u_i + D_\tau u_{i-1}) \\ &\le w_0 + \tau\sum_{i=1}^{n}\|g_i\|(\|D_\tau u_i\| + \|D_\tau u_{i-1}\|) \\ &\le w_0 + \sum_{i=1}^{n}\{\frac{\tau}{\delta^2}\|g_i\|^2 + (\frac{\tau\delta^2}{2}\|D_\tau u_i\|^2 + \frac{\tau\delta^2}{2}\|D_\tau u_{i-1}\|^2)\} \\ &\le w_0 + \frac{\delta\tau}{2}\sum_{i=1}^{n}(w_i + w_{i-1}) + \frac{\tau}{\delta^2}\sum_{i=1}^{n}\|g_i\|^2 \\ &\le w_0 + \frac{\delta\tau}{2}w_n + \delta\tau\sum_{i=0}^{n-1}w_i + \frac{\tau}{\delta^2}\sum_{i=1}^{N-1}\|g_i\|^2. \end{aligned}$$

If $f(t)$ is bounded on $[0, T]$, then, for $i \ge 1$, we have

$$\|g_i\| = \|\{I + \tau(\gamma - \frac{1}{2})D_{\bar{\tau}} + \beta\tau^2 D_{\tau\bar{\tau}}\}f_i\|$$
$$= \|f_i + (\gamma - \frac{1}{2})(f_i - f_{i-1}) + \beta(f_{i+1} - 2f_i + f_{i-1})\| \leq (4\beta + 2\gamma) \sup_{0\leq t\leq T} \|f(t)\|.$$

Hence we have

$$\begin{aligned} w_n &\leq w_0 + \frac{\delta\tau}{2}w_n + \delta\tau\sum_{i=0}^{n-1} w_i + \frac{\tau}{\delta^2}\sum_{i=1}^{N-1}\{(4\beta+2\gamma)\sup_{0\leq t\leq T}\|f(t)\|\}^2 \\ &\leq w_0 + \frac{\delta\tau_0}{2}w_n + \delta\tau\sum_{i=0}^{n-1} w_i + \frac{T}{\delta^2}\{(4\beta+2\gamma)\sup_{0\leq t\leq T}\|f(t)\|\}^2. \end{aligned}$$

Since $0 < 1 - \delta\tau_0/2$ from (26) or (27), we have

$$w_n \leq \Big(1 - \frac{1}{2}\delta\tau_0\Big)^{-1}\Big[w_0 + \delta\tau\sum_{i=0}^{n-1} w_i + \frac{T}{\delta^2}\{(4\beta+2\gamma)\sup_{0\leq t\leq T}\|f(t)\|\}^2\Big].$$

Using the Gronwall inequality, we then obtain

$$w_n \leq \Big(1 - \frac{1}{2}\delta\tau_0\Big)^{-1}\Big[w_0 + \frac{T}{\delta^2}\{(4\beta+2\gamma)\sup_{0\leq t\leq T}\|f(t)\|\}^2\Big]\exp\Big(\delta\Big(1 - \frac{1}{2}\delta\tau_0\Big)^{-1}T\Big),$$

where we use $N\tau = T$. Thus, we can define C_0 as follows

$$C_0 := \Big(1 - \frac{1}{2}\delta\tau_0\Big)^{-1}\Big[w_0 + \frac{T}{\delta^2}\{(4\beta+2\gamma)\sup_{0\leq t\leq T}\|f(t)\|\}^2\Big]\exp\Big(\delta\Big(1 - \frac{1}{2}\delta\tau_0\Big)^{-1}T\Big), \tag{29}$$

where

$$\begin{aligned} w_0 = \|M^{1/2}D_\tau u_0\|^2 &+ \tau^2\Big\{\beta - \frac{1}{2}\Big(\gamma - \frac{1}{2}\Big)\Big\}\|K^{1/2}D_\tau u_0\|^2 \\ &+ \tau(KD_\tau u_0, u_0) + \|K^{1/2}u_0\|^2 + \tau\Big(\gamma - \frac{1}{2}\Big)\|C^{1/2}D_\tau u_0\|^2. \end{aligned} \tag{30}$$

■

Proof of Theorem 1 (continued). Lastly, under either the condition (8) or (9), using the following inequality with an arbitrary α:

$$\begin{aligned} \tau(K^{1/2}D_\tau u_n, K^{1/2}u_n) &\geq -\tau\,\|K^{1/2}D_\tau u_n\| \times \sqrt{\alpha} \times \frac{1}{\sqrt{\alpha}} \times \|K^{1/2}u_n\| \\ &\geq -\frac{1}{2}\Big\{\alpha\tau^2\|K^{1/2}D_\tau u_n\|^2 + \frac{1}{\alpha}\|K^{1/2}u_n\|^2\Big\}, \end{aligned}$$

we obtain the energy inequality (7) from (28) in Lemma 3. From (22) we have

$$\text{If } f_n = g_n = 0, \quad \text{then} \quad C_0 = w_0.$$

Remark 2. When $C = 0$, $\gamma = 1/2$ and $f(t) \equiv 0$, we obtain

$$((M+\beta\tau^2 K)D_\tau u_n, D_\tau u_n)+(Ku_{n+1}, u_n) = ((M+\beta\tau^2 K)D_\tau u_0, D_\tau u_0)+(Ku_1, u_0).$$

Namely, in this case, Newmark's method conserves 'energy'.

Remark 3. The constant C_0 in the theorem depends primarily on the coefficient matrices M, C and K, and secondarily on the tuning parameters β and γ and it also depends on the initial values u_0 and v_0 and the inhomogeneous term f, and finally on τ_0 in the way described above.

Remark 4. In the case C is negative or K is not positive, we consider $v(t) := e^{-\lambda t}u(t)$ instead of $u(t)$. By choice of appropriate $\lambda > 0$, we may obtain new matrices satisfying the condition in Theorem 1 as coefficients of equation of $v(t)$.

5. Stability and Convergence Conditions for Newmark's Method

In this section, using the energy inequality (7) we derive stability conditions for Newmark's method. With respect to a parameter β, we divide the stability condition into two cases and lead to the next theorem.

5.1. Stability Theorem

Theorem 2. *Let M and K be positive definite and C be nonnegative. Let m and k be the smallest eigenvalues of M and K. Assume $\gamma \geq 1/2$. Then, Newmark's method for* (3) *in a time interval* $[0, T]$ *is stable in the following two cases with respect to β:*

Case 1: *If $(1/2)\gamma < \beta$, then with $\tau_0 > 0$ and δ given in* (12) *and* (13) *we have, for $\tau < \tau_0$ and $N = T/\tau$,*

$$\|u_n\| \leq \sqrt{\frac{(4\beta - 2\gamma + 1)}{(4\beta - 2\gamma)k} C_0}, \quad n = 0, 1, 2, \dots, N. \tag{31}$$

Case 2: *If $0 \leq \beta \leq (1/2)\gamma$, then for τ_0 satisfying*

$$0 < \tau_0 < \sqrt{\frac{m}{(\frac{1}{2}\gamma - \beta)\|K\|}}, \tag{32}$$

we have for $\tau \leq \tau_0$ and $N = T/\tau$

$$\|u_n\| \leq \|u_0\| + \sqrt{\frac{C_0}{m - \tau_0^2\left(\frac{1}{2}\gamma - \beta\right)\|K\|}}\,\tau n, \quad n = 0, 1, 2, \dots, N. \tag{33}$$

Here, in both cases, C_0 is given by(10) *together with* (11).

Proof. First, we treat Case 1. We consider the following condition for the coefficients in (7):

$$\beta - \frac{1}{2}(\gamma - \frac{1}{2}) - \frac{1}{2}\alpha = 0 \quad \text{and} \quad 1 - \frac{1}{2\alpha} > 0.$$

This implies that

$$\beta - \frac{1}{2}\Big(\gamma - \frac{1}{2}\Big) = \frac{1}{2}\alpha > \frac{1}{4},$$

and hence we have

$$\beta > \frac{1}{2}\gamma \geq \frac{1}{4}.$$

Conversely, if we assume that $\beta > \gamma/2 \geq 1/4$ and put $\alpha = 2\beta - (\gamma - 1/2)$, then we have $1 - 1/2\alpha > 0$. Using (7), we have

$$k\|u_n\|^2 \leq \|K^{1/2}u_n\|^2 \leq (1 - \frac{1}{2\alpha})^{-1}C_0.$$

Hence we have the stability estimate (31).

Next, we consider Case 2. Putting $\alpha = 1/2$ in (7), we have

$$\begin{aligned} C_0 &\geq \|M^{1/2}D_\tau u_n\|^2 + \tau^2\Big(\beta - \frac{1}{2}\gamma\Big)\|K^{1/2}D_\tau u_n\|^2 \\ &\geq m\|D_\tau u_n\|^2 + \tau^2(\beta - \frac{1}{2}\gamma)\|K^{1/2}D_\tau u_n\|^2. \end{aligned}$$

Noticing the condition $\gamma/2 - \beta > 0$, we have

$$\{m - \tau^2\Big(\frac{1}{2}\gamma - \beta\Big)\|K^{1/2}\|^2\}\|D_\tau u_n\|^2 \leq C_0.$$

Under the condition:

$$\tau \leq \tau_0 < \sqrt{\frac{m}{\Big(\frac{1}{2}\gamma - \beta\Big)\|K^{1/2}\|^2}},$$

the coefficient in the left-hand side becomes positive and we havevs

$$\|D_\tau u_n\|^2 \leq \frac{C_0}{m - \tau^2\Big(\frac{1}{2}\gamma - \beta\Big)\|K^{1/2}\|^2} \leq \frac{C_0}{m - \tau_0^2\Big(\frac{1}{2}\gamma - \beta\Big)\|K^{1/2}\|^2}.$$

Thus we have (33) in the same way as in Case 1. ■

Remark 5. Fujii investigated in [2, 3] the stability condition for the Rayleigh damping case with $C = aK + bM,\ a, b \in \mathbb{R}$.

5.2. Convergence Theorem

Using the recurrence relation (5) and the stability theorem, we can show in this section the convergence of Newmark's method together with its convergence order.

Let $u(t)$ be the solution of (3) and u_n $(n = 0, 1, 2, \ldots, N)$ be the solution of Newmark's method for (3). We assume that M and K are positive definite and C is nonnegative.

The discretization error e_n is defined as $e_n := u(\tau n) - u_n$. Then we have the following theorem.

Theorem 3. *We assume that $f \in C^2([0,T])$, then we have the estimate $\|e_n\| = O(\tau^l)$, where*

$$\begin{cases} l = 2 \quad ;\text{for} \quad \gamma = \frac{1}{2}, \\ l = 1 \quad \text{for} \quad \gamma > \frac{1}{2}. \end{cases}$$

Proof. To prove this theorem, we first show the following two lemmas.

Lemma 4. *At the mesh points $t = \tau n$ with $n = 0, 1$, we have*

$$e_0 = 0,\ e_1 = O(\tau^3).$$

Proof. Since $u_0 = u(0)$, we have $e_0 = u(0) - u_0 = 0$. Next we estimate e_1. From Step 3 in the iteration scheme in Sec. 2, we have

$$u_1 = u_0 + \tau v_0 + (\frac{1}{2} - \beta)\tau^2 a_0 + \beta\tau^2 a_1,$$

where

$$v_0 = \frac{\mathrm{d}u}{\mathrm{d}t}(0),\ a_0 = \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(0)$$

and

$$a_1 = (M+\gamma\tau C+\beta\tau^2 K)^{-1}\times[f_1-(C+\tau K)v_0-Ku_0+\{(\gamma-1)\tau C+(\beta-\frac{1}{2})\tau^2 K\}a_0].$$

On the other hand, by Taylor's theorem, we have

$$u(\tau) = u_0 + \tau v_0 + \frac{1}{2}\tau^2 a_0 + O(\tau^3) = u_0 + \tau v_0 + (\frac{1}{2} - \beta)\tau^2 a_0 + \beta\tau^2 a_0 + O(\tau^3).$$

So, we obtain

$$e_1 = u(\tau) - u_1 = \beta\tau^2 a_0 + O(\tau^3) - \beta\tau^2 a_1 = \beta\tau^2(a_0 - a_1) + O(\tau^3),$$

and from the relation

$$a_0 = M^{-1}(f_0 - Cv_0 - Ku_0),$$

and the above expression of a_1, we have

$$\begin{aligned} a_1 &= (M^{-1} + O(\tau)) \times [(f_0 + O(\tau)) - (Cv_0 + O(\tau)) - Ku_0 + O(\tau)] \\ &= M^{-1}(f_0 - Cv_0 - Ku_0) + O(\tau). \end{aligned}$$

Thus we obtain $e_1 = O(\tau^3)$. ■

At the mesh points $t = \tau n,\ n = 0, 1, 2, \dots, N$, we have $u_n = u(\tau n) - e_n$ and $f_n = f(\tau n)$. Using the recurrence relation (5), we can prove the next technical lemma.

Lemma 5. *Define* p_n *as*

$$p_n := (M + \beta\tau^2 K)D_{\tau\bar{\tau}}e_n + [\{\gamma C D_\tau + (1-\gamma)C D_{\bar{\tau}}\} + \tau\Big(\gamma - \frac{1}{2}\Big)K D_{\bar{\tau}}]e_n + K e_n. \tag{34}$$

Then it is expressed as follows:

$$p_n = -\tau\Big(\gamma - \frac{1}{2}\Big)M\frac{\mathrm{d}^3 u}{\mathrm{d}t^3}(\tau n) + O(\tau^2). \tag{35}$$

Proof. Substituting $u(\tau n) - e_n$ for u_n in the recurrence relation (5), we have

$$\begin{aligned} p_n &= (M + \beta\tau^2 K)D_{\tau\bar{\tau}}e_n + [\{\gamma C D_\tau + (1-\gamma)C D_{\bar{\tau}}\} + \tau\Big(\gamma - \frac{1}{2}\Big)K D_{\bar{\tau}}]e_n + K e_n \\ &= (M + \beta\tau^2 K)D_{\tau\bar{\tau}}u(\tau n) + [\{\gamma C D_\tau + (1-\gamma)C D_{\bar{\tau}}\} + \tau\Big(\gamma - \frac{1}{2}\Big)K D_{\bar{\tau}}]u(\tau n) \\ &\quad + K u(\tau n) - \{I + \tau(\gamma - \frac{1}{2})D_{\bar{\tau}} + \beta\tau^2 D_{\tau\bar{\tau}}\}f(\tau n). \end{aligned}$$

Using the expressions

$$\begin{aligned} D_\tau u(\tau n) &= \{u(\tau(n+1)) - u(\tau n)\}/\tau, \quad D_{\bar{\tau}} f(\tau n) = \{f(\tau n) - f(\tau(n-1))\}/\tau, \\ D_{\tau\bar{\tau}}u(\tau n) &= \{u(\tau(n+1)) - 2u(\tau n) + u(\tau(n-1))\}/\tau^2, \text{ etc.,} \end{aligned}$$

we rewrite the right hand side of the above formula. Applying Taylor's theorem to $u(\tau(n+1))$, $f(\tau(n-1))$, etc. at $t = \tau n$, we have

$$\begin{aligned} p_n &= M\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(\tau n) + C\frac{\mathrm{d}u}{\mathrm{d}t}(\tau n) + K u(\tau n) - f(\tau n) \\ &\quad + \tau\Big(\gamma - \frac{1}{2}\Big)\Big(C\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(\tau n) + K\frac{\mathrm{d}u}{\mathrm{d}t}(\tau n) - \frac{\mathrm{d}f}{\mathrm{d}t}(\tau n)\Big) + O(\tau^2). \end{aligned}$$

Using the equalities:

$$M\frac{\mathrm{d}^2 u}{\mathrm{d}^2 t}(\tau n) + C\frac{\mathrm{d}u}{\mathrm{d}t}(\tau n) + K u(\tau n) - f(\tau n) = 0,$$

and

$$C\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(\tau n) + K\frac{\mathrm{d}u}{\mathrm{d}t}(\tau n) - \frac{\mathrm{d}f}{\mathrm{d}t}(\tau n) = -M\frac{\mathrm{d}^3 u}{\mathrm{d}t^3}(\tau n),$$

we obtain (35). ∎

Proof of Theorem 3 (*continued*). Using Lemmas 4, 5, and the stability theorem we can obtain the estimate of $\|e_n\|$. To apply Theorem 2 to (34) we consider a modification of (10). If we look the proof of Theorem 1 again, we can replace $(4\beta + 2\gamma)\sup_{0\le t\le T}\|f(t)\|^2$ with $\sup_{1\le n\le N-1}\|p_n\|^2$. Then we have in this case

$$C_0 = \Big(1 - \frac{1}{2}\delta\tau_0\Big)^{-1}\{w_0 + \frac{T}{\delta^2}\sup_{1\le n\le N-1}\|p_n\|^2\}\ \exp\Big(\delta\Big(1-\frac{1}{2}\delta\tau_0\Big)^{-1}T\Big).$$

Hence we have from (31) or (33)

$$\|e_n\| \le C_1\sqrt{C_0} + \|e_0\| = C_1\sqrt{C_0},$$

where C_1 is independent of u_0, v_0 and p_n. By Lemma 4 and (11), where u_0 is replaced by e_0, we have

$$\begin{aligned} w_0 &= \|M^{1/2}D_\tau e_0\|^2 + \tau^2\{\beta - \frac{1}{2}\Big(\gamma - \frac{1}{2}\Big)\}\|K^{1/2}D_\tau e_0\|^2 \\ &\quad + \tau(KD_\tau e_0, e_0) + \|K^{1/2}e_0\|^2 + \tau\Big(\gamma-\frac{1}{2}\Big)\|C^{1/2}D_\tau e_0\|^2 \\ &= \|M^{1/2}\frac{1}{\tau}e_1\|^2 + \tau^2\{\beta - \frac{1}{2}\Big(\gamma-\frac{1}{2}\Big)\}\|K^{1/2}\frac{1}{\tau}e_1\|^2 + \tau\Big(\gamma-\frac{1}{2}\Big)\|C^{1/2}\frac{1}{\tau}e_1\|^2 \\ &= O(\tau^4). \end{aligned}$$

From this we obtain with another constant C_2

$$C_0 \le C_2\big\{O(\tau^4) + \sup_{1\le n\le N-1}\|p_n\|^2\big\}.$$

On the other hand, from Lemma 5, we have

$$\begin{cases} \sup_{1\le n\le N-1}\|p_n\|^2 = O(\tau^4) & \text{for} \quad \gamma = \frac{1}{2}, \\ \sup_{1\le n\le N-1}\|p_n\|^2 = O(\tau^2) & \text{for} \quad \gamma > \frac{1}{2}. \end{cases}$$

Thus we obtain the results. ■

5.3. Energy Inequality and Discrete Energy

Related to an energy inequality, we define the following quantity:

$$\begin{aligned} w_n &:= (MD_\tau u_n, D_\tau u_n) + \tau\left(\gamma - \frac{1}{2}\right)(CD_\tau u_n, D_\tau u_n) \\ &\quad + \tau^2\left\{\beta - \frac{1}{2}\left(\gamma-\frac{1}{2}\right)\right\}(KD_\tau u_n, D_\tau u_n) + (Ku_{n+1}, u_n) \\ &= \Big(\Big(M + \tau\Big(\gamma-\frac{1}{2}\Big)C + \tau^2\Big(\beta - \frac{1}{2}\Big(\gamma-\frac{1}{2}\Big)\Big)K\Big)D_\tau u_n, D_\tau u_n\Big) \\ &\quad + (Ku_{n+1}, u_n) = (\mathcal{M}D_\tau u_n, D_\tau u_n) + (Ku_{n+1}, u_n), \end{aligned} \tag{36}$$

where

$$\mathcal{M} := M + \tau\left(\gamma - \frac{1}{2}\right)C + \tau^2\left(\beta - \frac{1}{2}\left(\gamma - \frac{1}{2}\right)\right)K \simeq M.$$

We call w_n a discrete energy. We also define other quantities w'_n and w''_n:

$$w'_n := (MD_\tau u_n, D_\tau u_n) + (Ku_n, u_n), \tag{37}$$

and

$$w''_n := (Mv_n, v_n) + (Ku_n, u_n), \tag{38}$$

where v_n is a velocity given in Newmark's method together with u_n. These are discrete analogues of the continuous energy:

$$\mathcal{E}(t) := \left(M\frac{d}{dt}u(t), \frac{d}{dt}u(t)\right) + (Ku(t), u(t)). \tag{39}$$

The difference quotient $D_\tau u_n$ approximates the derivative of u at $t = \tau n + \tau/2$ with the second order in τ. So the average of $(\mathcal{M}D_\tau u_n, D_\tau u_n)$ and $(\mathcal{M}D_\tau u_{n-1}, D_\tau u_{n-1})$ is a second order approximation of $(M(d/dt)u, (d/dt)u)$ at $t = \tau n$. The average of (Ku_{n+1}, u_n) and (Ku_n, u_{n-1}) is represented as follows:

$$\frac{1}{2}\{(Ku_{n+1}, u_n) + (Ku_n, u_{n-1})\} = \left(Ku_n, \frac{1}{2}\{u_{n+1} + u_{n-1}\}\right),$$

where we use the assumption that K is symmetric. Since $(1/2)\{u_{n+1} + u_{n-1}\}$ is a second order approximation in τ of $u(t)$ at $t = \tau n$, $(1/2)\{(Ku_{n+1}, u_n) + (Ku_n, u_{n-1})\}$ is a second order approximation of (Ku, u) at $t = \tau n$. Now we define the following quantity W_n as a second order approximation in τ of (39) at $t = \tau n$:

$$W_n := \begin{cases} w_0 & \text{for } n = 0, \\ (1/2)(w_n + w_{n-1}) & \text{for } n \geq 1. \end{cases}$$

5.4. Convergence Theorem of W_n

Theorem 4. *For a fixed $T > 0$, the convergence order of $\mathcal{E}(T) - W_N$ with $T = \tau N$ is estimated as follows:*

$$|\mathcal{E}(T) - W_N| = \begin{cases} O(\tau^2) & \text{for } \gamma = \frac{1}{2}, \\ O(\tau) & \text{for } \gamma > \frac{1}{2}. \end{cases}$$

Proof. Using $e_n = u(\tau n) - u_n$ in Theorem 3, we replace $u_n = u(\tau n) - e_n$ in $\mathcal{E}(\tau n) - W_n$ and using the Taylor expansion of $u(t)$ and the fact $e_n = O(\tau^l), l = 1$ or 2, we obtain

$$\begin{aligned} \mathcal{E}(\tau n) - W_n = & -\frac{1}{2\tau^2}\{(M(e_{n+1} - e_n), e_{n+1} - e_n) + (M(e_n - e_{n-1}), e_n - e_{n-1})\} \\ & + \frac{1}{\tau}\left(M(e_{n+1} - e_{n-1}), \frac{du}{dt}(\tau n)\right) + O(\tau^l). \end{aligned}$$

So, we consider the estimation of $e_{n+1} - e_n$ and $e_{n+1} - e_{n-1}$. We define the quantities

$$E_n = \frac{1}{\tau}(e_{n+1} - e_{n-1}) \quad \text{for} \quad n \geq 1,$$

and, using p_n defined in Lemma 5, we put

$$\begin{aligned} P_n &= (M + \beta\tau^2 K)D_{\tau\bar{\tau}}E_n + [\{\gamma C D_\tau + (1-\gamma)CD_{\bar{\tau}}\} \\ &\quad + \tau(\gamma - \frac{1}{2})KD_{\bar{\tau}}]E_n + KE_n = \frac{1}{\tau}(p_{n+1} - p_{n-1}) \quad \text{for} \quad n \geq 1. \end{aligned} \tag{40}$$

By the similar calculation in the proof of Theorem 3, we have

$$E_1 = O(\tau^2) \quad \text{and} \quad E_2 = O(\tau^2),$$

and also we get

$$P_n = \tau(2\gamma - 1)\left(\frac{d^2u}{dt^2}(T) + C\frac{d^3u}{dt^3}(T)\right) + O(\tau^2).$$

Hence we have $P_n = O(\tau^3)$ when $\gamma = 1/2$ and $P_n = O(\tau^2)$ when $\gamma > 1/2$. In the same way as the proof of Theorem 3, we have the estimate of E_n

$$\|E_n\| \leq \sqrt{O(\tau^4) + O\big(\sup_{0\leq n\leq N-1} \|P_n\|^2\big)}.$$

In the same way as for E_n, we have the estimate of $e_{n+1} - e_n$

$$\|e_{n+1} - e_n\| = \begin{cases} O(\tau^3) & \text{for} \quad \gamma = 1/2, \\ O(\tau^2) & \text{for} \quad \gamma > 1/2. \end{cases}$$

Thus we obtain the following estimation:

$$|\mathcal{E}(T) - W_N| \leq O(E_N) = \begin{cases} O(\tau^2) & \text{for} \ \gamma = 1/2, \\ O(\tau) & \text{for} \ \gamma > 1/2. \end{cases}$$

■

6. Application to Resistive MHD Equation

6.1. Simplified Resistive MHD Equation

We investigate the behavior of solution to resistive MHD equation(see [8, 5] for details). Let $\psi(t,x) := B_x(t,x)$ be the x-component of magnetic field at time t which we assume depends only on $x-$variable in space direction and periodic with period one, and hence the function on $\mathbb{R} \times S$, where $S = \mathbb{R}/\mathbb{N}$ (the unit circle). Then ψ satisfies the following second order equation in t if we neglect a weak interaction term along magnetic lines:

$$\begin{cases} \frac{\partial^2}{\partial t^2}\psi - \eta\Delta\frac{\partial}{\partial t}\psi + H(x)^2\psi = 0, \\ \Delta = \frac{\partial^2}{\partial x^2}, \end{cases} \tag{41}$$

where η is a nonnegative constant called resistivity, and $H(x)$ is a smooth real periodic function which represents the Alfvén frequency at x. We call the problem (41) a model equation of magnetic line for the solution and the resistive linearized MHD. Based on Newmark's scheme, we compute the time evolution of the discrete approximations of energy defined in the formulas in (36) - (38).

For general discussion, we formulate the model problem to the abstract equation for $u(t)$:

$$\frac{\partial^2}{\partial t^2}u + \eta L \frac{\partial}{\partial t}u + Hu = 0, \tag{42}$$

and we consider this equation in a Hilbert space $\mathcal{H}$. We assume that L is a nonnegative self-adjoint operator and the inverse operator $(L+1)^{-1}$ is compact in $\mathcal{H}$ and H is a bounded operator. We refomulate the equation (42) to the first order system:

$$\frac{d}{dt}\begin{pmatrix} u \\ u' \end{pmatrix} = F_\eta \begin{pmatrix} u \\ u' \end{pmatrix}, \quad F_\eta := \begin{pmatrix} 0 & I \\ -H & -\eta L \end{pmatrix}. \tag{43}$$

We define the domain of F_η as follows:

$$\mathcal{D}(F_\eta) := \mathcal{H} \times \mathcal{D}(L).$$

The operator F_η is the generator of a C_0-semigroup and its resolvent is given as follows:

$$(F_\eta - \lambda)^{-1} = \begin{pmatrix} -\frac{1}{\lambda} + \frac{1}{\lambda}D_\eta^{-1}H & -D_\eta^{-1} \\ D_\eta^{-1}H & -\lambda D_\eta^{-1} \end{pmatrix}, \tag{44}$$

where $D_\eta(\lambda)(= D_\eta$ for short) is defined as follows:

$$D_\eta(\lambda) := \lambda^2 + \eta\lambda L + H, \quad \mathcal{D}(D_\eta) = \mathcal{D}(L). \tag{45}$$

Since H is bounded and $(L+1)^{-1}$ is compact, D_η has a compact inverse for sufficient large positive λ. Hence, (44) is well defined and has its essential spectrum $\{0,\ \infty\}$.

We consider (43) on $\mathcal{H}\times\mathcal{H}$. In the weak form, (43) becomes to find $\{v, w\} \in \mathcal{H} \times \mathcal{H}$ such that

$$\frac{\partial}{\partial t}\{(v,\psi) + (w,\zeta)\}(w,\psi) - (Hv,\zeta) - \eta(Lw,\zeta), \quad \text{for all} \quad \psi,\zeta \in \mathcal{H}. \tag{46}$$

We factorize L in the third term of the left-hand side and adopt the representation:

$$(Lv,\zeta) = (L^{1/2}v, L^{1/2}\zeta).$$

Then we can finally formulate the equation in $\mathcal{V}\times\mathcal{V}$ with $\mathcal{V} = \mathcal{D}\left(L^{1/2}\right)$ to find $\{v, w\} \in \mathcal{V}\times\mathcal{V}$ such that

$$\frac{\partial}{\partial t}\{(v,\psi) + (w,\zeta)\} = (w,\psi) - (Hv,\zeta) - \eta(L^{1/2}w, L^{1/2}\zeta), \quad \text{for all} \quad \psi,\zeta \in \mathcal{V}. \tag{47}$$

6.2. Numerical Method for the Model Problem

The weak formulation (47) introduces the following discrete approximation. We takes as an approximate space $\mathcal{V}_h \times \mathcal{V}_h$, where $\mathcal{V}_h$ is a finite dimensional approximate subspace of $\mathcal{V}$. Then the finite element approximation problem is to restrict the problem on $\mathcal{V}_h$ to find $\{v_h, w_h\} \in \mathcal{V}_h \times \mathcal{V}_h$ such that

$$\frac{\partial}{\partial t}\{(v_h, \psi_h) + (w_h, \zeta_h)\} = (w_h, \psi_h) - (Hv_h, \zeta_h) - \eta(L^{1/2}w_h, L^{1/2}\zeta_h), \quad \text{for all } \psi_h, \zeta_h \in \mathcal{V}_h. \tag{48}$$

Since $L = -\Delta$ with $\mathcal{D}(L) = H^2(S)$, we have

$$\mathcal{D}(L^{1/2}) = \mathcal{V} = H^1(S).$$

We define a finite dimensional subspace $\mathcal{V}_h$ of $\mathcal{V}$ as a set of piecewise linear continuous functions for the equipartition of $S = [0, 1)$ with N mesh points. Namely, we take $h = 1/N$ and define $\mathcal{V}_h$ as

$$\mathcal{V}_h := \{u_h : u_h \text{ is periodic with period 1 and continuous, and linear on } [k\,h,\ (k+1)h],\ k = 0, 1, \dots, N-1\}. \tag{49}$$

6.3. Numerical Tests of Newmark's Method Applied to the Model MHD Equation

We investigate numerically the convergence order of Newmark's method. We calculate the numerical solution at $t = T$ by Newmark's method with a time step $\tau_p = T/2^p, p = 1, 2, 3, \dots$. We put $H(x) = 2 + \sin 2\pi x$ and $h = 1/256$. Let us denote by $u_{n,\tau}$ the numerical solution with a time step τ at $t = \tau n$ of (41), we assume the convergence order as follows:

$$\|u_{2^{16},\tau_{16}} - u_{2^p,\tau_p}\|_{L^2(0,1)} = C_1 \tau_p^l, \tag{50}$$

where we use $u_{2^{16},\tau_{16}}$ as the fine enough approximation to the exact solution at $t = T(= 10)$. We compute numerically the convergence order l based on the above assumption (50). In these calculations, we set $\eta = 0.001, 0.0001, 0$, $\beta = 1/6$, $\gamma = 1/2,\ 0.51,\ 2/3,\ 3/4$ and $p = 6, 7, \dots, 16$. We use the least square method for estimating l. Table 1 shows the estimation of the convergence order l for each γ. The results are consistent with the theoretical orders in Theorem 3.

Table 1. Convergence order l for various γ

γ	1/2	0.51	2/3	3/4
$\eta = 0.001$	2.0371	1.2517	1.0149	0.9841
$\eta = 0.0001$	2.0331	1.2019	1.0158	0.9873
$\eta = 0$	2.0302	1.1995	1.0165	0.9884

6.4. Numerical Tests of Discrete Energy

We estimate the convergence order of the quantities w_n, w_n', w_n'' and W_n. We calculate these quantities at $t = T$ with a time step $\tau_p = T/2^p, p = 1, 2, 3, \ldots$. We put $\beta = 1/6$, $\gamma = 1/2$ and $H(x) = 2 + \sin 2\pi x$. In the case of w_n, let us denote by $w_{n,\tau}$ the numerical solution with a time step τ at $t = \tau n$, and we assume the relation of convergence order as follows:

$$|w_{2^{16},\tau_{16}} - w_{2^p,\tau_p}| = C_2 \tau_p^l, \tag{51}$$

where we use $w_{2^{16},\tau_{16}}$ on behalf of the exact solution at $t = T(= 10)$ as its fine enough approximation to the exact solution. We estimate the convergence order l based on the above assumption (51). In these calculations, we put $\eta = 0.001, 0.0001, 0$, $h = 1/256$ and $p = 6, 7, \ldots, 16$. We use the least square method for estimating l. In other cases of w_n', w_n'' and W_n, we estimate l similarly. Table 2 shows the estimation of l in each case. It shows that W_n and w_n have very good performance.

Fig. 1. The case $\tau = 0.5$: Frame 1=w_n, Frame 2=w_n', Frame 3 =w_n'' and Frame 4=W_n

Table 2. Estimates of l at $h = 1/256$

	w_n	w_n'	w_n''	W_n
$\eta = 0.001$	1.0774	1.1038	2.0311	2.1457
$\eta = 0.0001$	1.3514	1.0940	2.0422	2.0265
$\eta = 0$	2.0242	1.0480	2.0182	2.0242

Fig. 1, 2, 3 and 4 show the time evolution behaviors of w_n, w_n', w_n'' and W_n.

Fig. 2. The case $\tau = 0.5$: the lower one is W_n, and another one is w''_n

Fig. 3. The case $\tau = 0.125$: Frame 1=w_n, Frame 2=w'_n, Frame 3 =w''_n and Frame 4=W_n

Fig. 4. The case $\tau = 0.125$: the lower one is W_n, and another one is w''_n

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