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Stabilization of Dynamic Systems by Bounded Feedbacks*

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Abstract. The paper deals with an approach to the solution of stabilization problems for dynamic systems based on positional solutions of auxiliary optimal control problems. The specific character of the approach consists in using bounded controls, nonfixed beforehand structures of feedbacks. In addition, by choosing parameters of control constraints and cost functions of optimal control problems in an appropriate way, one can provide high characteristics of transients. Stabilization of linear and nonlinear dynamic systems is under discussion. Results of computer experiments on the stabilization of an inverted pendulum and the damping of string oscillations are given.

1. Introduction

The stabilization problem of dynamic systems is one of central problems of control theory and in a certain way it gave birth to the latter. Stabilization theory can be considered as antipode to stability theory since it deals with the synthesis of dynamic systems whereas subjects of stability theory is the analysis of motions. The close relation between two problems (stability and stabilization) allowed during a long period of time to use results of stability theory to create dynamic systems which possessed stable behavior [1 - 3].

As is known, the classical statement of the problem consists in the following. Let a control system

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^n,\ f(0,0) = 0. \tag{1}$$

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be given. We assume that the equilibrium state $x = 0$ of the system without controls ($u \equiv 0$) is not asymptotically stable. In addition to (1), a set of functions $\{u = u(x),\ x \in \mathbb{R}^n\ (u(0) = 0)\}$ called feedbacks is given. If a control system is linear

$$\dot{x} = Ax + Bu, x \in \mathbb{R}^n, u \in \mathbb{R}^r, \tag{2}$$

then a linear feedback $u = Kx$ is usually used.

The problem consists in constructing a feedback $u^* = u^*(x), x \in \mathbb{R}^n$, such that it provides asymptotic stability of (1) or (2) [1, 2].

From the modern point of view the solution of the classical stabilization problem has at least tree drawbacks:

(1) it does not take into account that control functions can be bounded;
(2) the domain of attraction of the zero solution of (1) or (2) can be not large;
(3) nothing is known (or demanded) about transients except for property of asymptotic stability.

To overcome shortages of the classical approach one can use the optimal control methods. The first result on stabilization of linear dynamic systems (2) by optimal control methods was obtained by Kalman and Lyotov [3, 4]. Their classical results on analytical design of optimal regulators had been at once applied to stabilization problems. The prevalence and the success of their results can partly be explained by absence of constraints on control functions due to which the problem in question becomes a problem of calculus of variations. Later on to solve stabilization problems linear-quadratic optimal control problems with a finite horizon [5, 6] were used. The construction of stabilizing feedbacks in [3 - 6] was based on possibility of explicit positional solutions of the problem and the absence of any constraints on control functions as well. The constructing of bounded stabilizing feedbacks in an explicit form represents an extremely complex mathematical problem. Below one approach is suggested to solve the stabilization problem which is based on constructing in real-time optimal feedbacks for auxiliary optimal control problems [7, 8] and the principle of receding horizon. At first consider linear control system (2).

2. Statement of the Problem

Let G be a neighborhood of the equilibrium state $x = 0$ of (2), $u \equiv 0,\ L \in (0, +\infty)$ be a given number, b a vector ($u(t)$ is a scalar control).

Definition 1. *A function*

$$u = u(x),\ x \in G,$$

is called a bounded stabilizing feedback for system (2) *if*

(1) $u(0) = 0$;
(2) $|u(x)| \leq L,\ x \in G$;
(3) *equation* (2) *closed by* $u(x)$

$$\dot{x} = Ax + bu(x) \tag{3}$$

has a solution $x(t),\ t \geq 0$, *for all* $x_0 \in G$.

(4) *system* (3) *is asymptotically stable in* G.

Introduce an auxiliary (accompanying) optimal control problem

$$\dot{x} = Ax + bu,\ x(0) = z, \quad \operatorname{rank}(b, Ab, \ldots, A^{n-1}b) = n. \tag{4}$$

For a fixed number Θ (a parameter of the method) put

$$V(z) = \min_u \int_0^{\Theta} |u(t)| dt, x(\Theta) = 0, |u(t)| \leq L,\ t \in T = [0, \Theta]. \tag{5}$$

Denote by $G(\Theta)$ the set of initial states z for which problem (4), (5) has optimal open-loop solutions $u^0(t|z),\ t \in T$. It can be proved that for any $\varepsilon > 0$ there exists such $\Theta < +\infty$ that an ε-vicinity $G(\Theta)$ of G contains all initial states that can be transferred to $x = 0$ by bounded controls.

Definition 2. *A function* $u^0(z) = u^0(0|z),\ z \in G(\Theta)$, *is called a start optimal positional control (a start optimal feedback control).*

It can be proved that the function $u(x) = u^0(x),\ x \in G(\Theta)$, is a bounded stabilizing feedback for (2). As the construction of $u(x),\ x \in G(\Theta)$, is a rather complex problem, one can use the mentioned approach to calculate a realization $u^*(t) = u^0(x^*(t)),\ t \geq 0$, in real-time mode [7]. The essence of the approach consists in introducing a concrete process and investigating a part of the feedback which corresponds to real situations of control. Suppose the bounded stabilizing feedback $u(x),\ x \in G$, is constructed. Consider the behavior of the closed control system in a real situation

$$\dot{x}^*(t) = Ax^*(t) + bu(x^*(t)),\ x^*(0) = x_0^*,\ x_0^* \in G.$$

In any concrete process the feedback $u(x),\ x \in G$, is not used as the whole. All one needs is values of the feedback along an isolated curve $x^*(t),\ t \geq 0$, and these values are not supposed to be calculated beforehand but only at every current moment $\tau \geq 0$, when system (3) turns out to be at the state $x^*(\tau)$.

Definition 3. *A function* $u^*(t) = u(x^*(t)),\ t \geq 0$, *is called a realization of the stabilizing feedback.*

Thus, the stabilization problem is reduced to constructing an algorithm of functioning Optimal Controller for (4), (5). Two ways of realization of the stabilizing feedbacks can be suggested: "continuous" and "discrete". The continuous way deals with defining elements (switching points of optimal open-loop controls and the Lagrange multipliers) which satisfy algebraic equations originating from the maximum principle ($\psi'(t_k|\tau, x(\tau))b = 0$) and terminal constraints ($x(\Theta) = 0$). Under rather general conditions the Jacobi matrix of the equations is nonsingular that makes possible to use the Newton method to solve the defining equations in real time [8]. Another approach is based on using discrete (piecewise-constant) controls with a constant period of quantization that reduces stabilization problems to linear programming problems solved by a dual methods [9 - 11]. In the paper we shall restrict ourselves by the class of discrete controls and give new results on stabilization problems obtained by optimal control methods in which constructive optimization methods combine with the modern computer tools.

3. Stabilization of Time-varying Dynamic Systems

Consider a time-varying dynamic system

$$\dot{x} = A(t)x, \quad T = \{t \in \mathbb{R} : t \geq t_*\}, \tag{6}$$

where $A(t)$, $t \in T$, is a piecewise continuous $(n \times n)$- matrix function.

Suppose that system (6) is not asymptotically stable. For the purpose of the solution of the stabilization problem we shall consider the dynamic system

$$\dot{x} = A(t)x + b(t)u, \tag{7}$$

with a constrained piecewise continuous n-vector function $b(t)$, $t \in T$.

A function

$$u = u(t, x), \quad x \in X_t \subset \mathbb{R}^n, \quad t \in T_h = \{t_*, t_* + h, \dots\}; \tag{8}$$

is called a bounded stabilizing discrete control of feedback type with a quantization period $h > 0$ if
(1) $u(t, 0) = 0, t \in T_h$;
(2) $|u(t)| \leq 1, x \in X_t, t \in T_h$;
(3) a trajectory $x(t)$, $t \in T_h$, of closed system

$$\dot{x} = A(t)x + b(t)u(t, x), \quad x(t_*) = x_0 \in X_{t_*}, \tag{9}$$

is a trajectory of linear system (7) with the condition $x(t_*) = x_0$ and the control

$$u = u(t) = u(t_* + kh, x(t_* + kh)), \quad t \in [t_* + kh, t + (k+1)h[, \quad k = 1, 2, \dots ;$$

(4)$x(t) \in X_t, t \in T_h$;
(5) the zero solution $x(t) = 0, \quad t \in T$, of closed system (9) is asymptotically stable in X_{t_*}.

From practical point of view it is important that feedback (8) would possess additional properties such as
(6) the domain of attraction X_{t_*} of the zero solution is sufficiently large;
(7) transients $(x(t) \to 0, \quad t \to \infty)$ possess definite qualities with respect to a cost function.

Let the functions $A(t), \quad b(t), t \in T$, satisfy

$$\lambda_1 = \inf_{\tau \in T_h} \min_{\|z\|=1} \|F(\tau + \theta, \tau)z\| > 0,$$

$$\lambda_2 = \inf_{\tau \in T_h} \min_{\int_{\tau}^{\tau+\theta} |u(t)|dt=1} \| \int_{\tau}^{\tau+\theta} F(\tau + \theta, t)b(t)u(t)dt\| > 0.$$

Here $F(t,\tau) = F(t)F^{-1}(\tau)$, $t \in T$, is the fundamental matrix of the solution of (6), $\dot{F} = A(t)F$, $F(t_*) = E$, $\theta = Nh > 0$ is a parameter of the method. For $\tau \in T_h$, $z \in \mathbb{R}^n$ in the class of discrete functions $u(t)$, $t \in T(\tau) = [\tau, \tau + \theta[$:

$$u(t) = u(t_* + kh), t \in [t_* + kh, t_* + (k+1)h[, k = 0, ..., N-1,$$

consider an accompanying problem of optimal control

$$B(\tau, z) = \min \int_{\tau}^{\tau+\theta} |u(t)|dt, \quad \dot{x} = A(t)x + b(t)u(t), \tag{10}$$
$$x(\tau) = z, \quad x(\tau + \theta) = 0, \quad |u(t)| \le 1, \quad t \in T(\tau).$$

Let $u^0(t|\tau, z)$, $t \in T(\tau)$, be an optimal open-loop control to (10), X_τ be a set of vectors $z \in \mathbb{R}^n$ for which problem (10) has a solution at the fixed τ.

Definition 3. *A function*

$$u^0(\tau, z) = u^0(t|\tau, z), \quad z \in X_\tau, \quad \tau \in T_h, \tag{11}$$

is called a start optimal control of feedback type.

At first we prove that the feedback

$$u(t, x) = u^0(t, x), \quad x \in X_t, \quad t \in T_h, \tag{12}$$

is a solution to the stabilization problem. The properties (1) - (3) are evident due to (9), (11) and (12). Let $x(\tau) \in X_\tau$ i.e. there exists an admissible control $u(t|\tau, x(\tau))$, $t \in T(\tau)$, to (9), at which $x(\tau + \theta) = 0$. Under $u(t) = u^0(\tau, x(\tau))$, $t \in [\tau, \tau + h[$, system (7) gets at $x(\tau + h)$ at the moment $\tau + h$. For this state the control $u(t|\tau, x(\tau))$, $t \in [\tau + h, \tau + \theta[$; $u(t) = 0$, $t \in [\tau + \theta, \tau + h + \theta[$, is admissible, i.e. $x(\tau + h) \in X_{\tau+h}$ (property (4)).

To prove (5) let us investigate the function $B(t, z)$, $z \in X_\tau$, $\tau \in T_h$. It is clear that $B(\tau, 0) = 0$, $\tau \in T_h$. According to the Cauchy formula

$$F(\tau + \theta, \tau)z + \int_{\tau}^{\tau+\theta} F(\tau + \theta, t)b(t)u(t|\tau, z)dt = 0, \tag{13}$$

if $u(t|\tau, z)$, $t \in T(\tau)$, is admissible.

Hence

$$\lambda_1 \|z\| \le \|F(\tau + \theta, \tau)z\| \le \max_{t \in T(\tau)} \|F(\tau + \theta, \tau)b(t)\| \int_{\tau}^{\tau+\theta} |u(t|\tau, z)|dt$$

for all admissible controls. In particular, for $u^0(t|\tau, z)$, $t \in T(\tau)$,

$$B(\tau, z) = \int_{\tau}^{\tau+\theta} |u^0(t|\tau, z)|dt \ge \mu_1 \|z\|, \quad \mu_1 = \text{const}, \tag{14}$$

that proves the definite positiveness of $B(\tau, z)$, $z \in X_\tau$, $t \in T_h$.

The function $B(\tau, z)$, $z \in X_\tau$, $t \in T_h$, decreases along any trajectory $x(t)$, $t \in T$, of (9). Really, if $x(\tau)$, $x(\tau + h)$ are arbitrary neighboring states of (9), then

$$B(\tau, z) = \int_{\tau}^{\tau+\theta} |u^0(t|\tau, z)|dt = \int_{\tau}^{\tau+h} |u^0(t|\tau, z)|dt + \int_{\tau+h}^{\tau+\theta} |u^0(t|\tau, z)|dt \geq$$
$$\geq \int_{\tau+h}^{\tau+\theta+h} |u^0(t|\tau + h, x(\tau + h))|dt = B(\tau + h, x(\tau + h)). \tag{15}$$

Let $\tau + \nu(\tau)$, $\nu(\tau) > 0$ be the first after τ moment when $u^0(\tau + \nu(\tau), x(\tau + \nu(\tau))) \neq 0$. Then from (15) it follows that $B(\tau + \nu(\tau), x(\tau + \nu(\tau))) < B(\tau, x(\tau))$. Thus the function $B(\tau, z)$ $z \in X_\tau$, $t \in t_h$, does not increase along the trajectory $x(t)$, $t \in T$ of (9) and there exists such an increasing sequence of moments $t^k \leq T_h$, $k = 1, 2 \dots$, that $B(t^{k+1}, x(t^{k+1})) < B(t^k, x(t^k))$, $k = 1, 2 \dots$.

The property takes place

$$B(t^k, x(t^k)) \to 0, \quad k \to \infty, \tag{16}$$

as the function $B(\tau, z)$, $z \in X_\tau$, $t \in t_h$, possesses a superior limit. In fact, from (13) it follows that

$$B(\tau, z) = \int_{\tau}^{\tau+\theta} |u^0(t|\tau, z)|dt \leq \frac{1}{\lambda_2} \| \int_{\tau}^{\tau+\theta} F(\tau + \theta, t)b(t)u^0(t|\tau, z)\|dt$$
$$\leq \frac{1}{\lambda_2} \|F(\tau + \theta, \tau)\| \|z\| \leq \mu_2 \|z\|.$$

Due to (14) from (16) one gets

$$x(t^k) \to 0, \quad k \to \infty. \tag{17}$$

Let $t \in T$ be a moment from $[t^k, t^{k+1}[$. Then

$$\max_{t \in [t^k, t^{k+1}]} \|x(t)\| \leq \max_{t \in [t^k, t^{k+1}]} \|F(t, t^k)x(t)\|$$
$$+ \max_{s \in [t^k, t[, t \in [t^k, t^{k+1}]]} \|F(t, s)b(s)\| \int_{t^k}^{t^k+1} |u(t)|dt.$$

As (16), (17) hold, then $x(t) \to 0$, $t \to \infty$, that proves the property 5).

The set X_{t_*} is widened if θ increases. Denote by X_* the set of controllability to the zero of (7) with the help of discrete bounded control. Then for any $\varepsilon > 0$ there exists such $\theta = \theta(\varepsilon)$ that all elements of X_* will be in the ε-vicinity $[X_*]_\varepsilon$ of X_{t_*}. So by selection of θ the domain X_{t_*} can be made as close to a maximal one as possible.

An extremal property of transients generated by (9) with (8) satisfies the inequality

$$\int_{\tau^*}^{\infty} |u^*(t)|dt \leq \int_{\tau}^{\tau+\theta} |u^0(t)|dt, \quad \tau \in T_h. \tag{18}$$

Here $u^*(t)$, $t \in T$, is values of the feedback along the trajectory $x^*(t)$, $t \in T$, of system (9) with the initial state $x(\tau) \in X_\tau$, $u^0(t)$, $t \in T(\tau)$, is an optimal open-loop control damping at $\tau + \theta$ system (9) with $x(\tau) \in X_\tau$.

If one interprets $|u(t)|$ as fuel consumption per second at t, then property (18) means that fuel consumption spent to stabilize $x(\tau)$ on $[\tau, \infty[$ does not exceed the minimal value which is necessary to damp system (7) with the initial state $x(\tau)$ on $[\tau, \tau + \theta[$.

Property (18) follows from inequalities

$$B(\tau, x(\tau)) = \int_{\tau}^{\tau+\theta} |u^0(t|\tau, x(\tau))|dt \geq \int_{\tau}^{\tau+h} |u^*(t)|dt$$
$$+ \int_{\tau+h}^{\tau+h+\theta} |u^0(t|\tau+h, x(\tau+h))|dt \geq \ldots \geq \int_{\tau}^{\infty} |u^*(t)|dt.$$

The value $\int_{\tau}^{\tau+\theta} |u^0(t)|dt$, $\tau \in T_h$, does not increase if θ increases. Therefore, by choosing θ the fuel consumption spent on stabilization can be reduced as close as desired to minimal possible.

Realization of the method. Before the stabilization process we take a vicinity of equilibrium state $x = 0$ that consists of perturbed initial states $x(t_*)$ and cover it with a dense network with nodes x^k, $k \in K$. Steps for the realization are as follows:
1. Optimal support K^k_{sup} is constructed for every node [11];
2. At the initial state of stabilization t_* using the realizing $x^*(t_*)$ we construct the nearest node x^m, and the support K^m_{sup} is corrected by the duel method up to the optimal support $K^0_{sup}(t_*)$ [11]. A number of nodes $|K|$ of the network is chosen in such a way that the time of correcting for any $x \in X_{t_*}$ does not exceed h;
3. The function $u^*(t) = u^*(t_*) = u^0(t_*|t_*, x(t_*))$, $t \in [t_*, t_* + h[$ is calculated by $K^0_{sup}(t_*)$. Under influence of $u^*(t)$, $t \in [t_*, t_* + h[$, and a disturbance system (9) gets at the state $x^*(t_* + h)$ at $t_* + h$. Under the condition $x^*(t_* + h) \notin X_{t_*+h}$, the stabilization of (6) is impossible.
4. Let $x^*(t_* + h) \in X_{t_*+h}$. For this state we construct the optimal support $K^0_{sup}(t_* + h)$ correcting $K^0_{sup}(t_*)$ by the mentioned duel method.
5. Repeat the procedure from the step 4 for $t_* + 2h$ and so on.

The process of stabilization can be realized in real time mode with computer tools which is used in the procedure of correction. For that the time of correcting at any current moment $\tau \in T_h$, of the stabilization process has not to exceed h.

4. Stabilization of Nonlinear Dynamic Systems

Consider the problem of stabilization of a nonlinear system in the class of piecewise constant controls with a constant quantization period $h > 0$:

$$\dot{x} = f(x),\ (x \in \mathbb{R}^n, \quad f(0) = 0),\ \ x \in X, \tag{19}$$

$$U_h = \{u(x(kh)), \quad t \in [kh, (k+1)h[,\ k = 0, 1, 2, \dots\}. \tag{20}$$

Suppose that the equilibrium state $x = 0$ of (19) is not asymptotically stable. Construct a feedback which possesses the stabilizing property. Assume that

$$\dot{x} = a(x) + b(x, u),\ (a(x) + b(x, 0) = f(x),\ a(0) + b(0, 0) = 0) \tag{21}$$

and
1) equation (21) has a unique solution $x(t),\ t \geq 0$, for all $x(0) \in X$ and controls from class (20);
2) for any $x(0) \in X$ there exist a finite t^* and a control $u(t),\ |u(t)| \leq L,\ t \geq 0,\ u(t) \in U_h$, such that the corresponding trajectory $x(t),\ t \geq 0$, of (21) satisfies the condition $x(t^*) = 0$.

Definition 4. *A function*

$$u = u(x),\ x \in X, \tag{22}$$

is called a bounded discrete stabilizing feedback if
(1) $u(0) = 0,\ |u(x)| \leq L$;
(2) *a trajectory of the closed-loop system*

$$\dot{x} = a(x) + b(x, u(x)),\ x(0) = x_0 \in X, \tag{23}$$

coincides with a solution of the equation

$$\dot{x} = a(x) + b(x, u(t)),\ x(0) = x_0, \tag{24}$$
$$u(t) = u(x(kh)),\ t \in [kh, (k+1)h[,\ k = 0, 1, 2, \dots;$$

(3) *the zero solution* $x(t) = 0,\ t \geq 0,$ *of* (23) *is asymptotically stable in* X.

To construct function (22) choose a parameter $\Theta,\ 0 < \Theta < +\infty$, and consider an auxiliary (accompanying) optimal control problem

$$V_\Theta(z) = \min_u \int_0^\Theta c(x(t), u(t))dt, \dot{x} = a(x) + b(x, u),\ x(0) = z,\ x(\Theta) = 0, \tag{25}$$
$$|u(t)| \leq L,\ t \in T = [0, \Theta = Nh], (c(x, u) > 0,\ x \neq 0,\ u \neq 0;\ c(0, 0) = 0)$$

taking $u(t),\ t \geq 0$, from U_h.

Let $u^0_\Theta(t|z),\ t \in T$, be an optimal open-loop control of (25), X_Θ be a set of z for which problem (25) has a solution.

Definition 5. *A function*

$$u^0_\Theta(z) = u^0_\Theta(0|z),\ z \in X_\Theta, \tag{26}$$

is called a start optimal control of feedback type.

Theorem 1. *The function*

$$u(x) = u^0_\Theta(z),\ z \in X_\Theta, \tag{27}$$

is a bounded discrete stabilizing feedback in X_Θ.

Proof. Property 1) for function (27) follows from (25), (26) and the inequality $|u^0(t|z)| \le L,\ t \in T,\ z \in X_\Theta$. Property 2) originates from (20), (25). Consider $V_\Theta(z),\ z \in X_\Theta$, to verify that it is the Lyapunov function providing asymptotic stability of the zero solution of (21) with feedback (26) in X_Θ.

Really, $V_\Theta(0) = 0,\ V_\Theta(z) > 0,\ z \in X_\Theta,\ z \neq 0$. Let $x^*_0 \in X_\Theta,\ x^*_0 \neq 0$, be an initial state of a process. At the moment h using the control $u^*_\Theta(0) = u^0(x^*_0) = u^0_\Theta(0|x^*_0)$ system (21) gets at the position $x^*(h)$. Consider the control $u_\Theta(0|x^*(h)) = u^0_\Theta(h|x^*_0),\ u_\Theta(h|x^*(h)) = u^0_\Theta(2h|x^*_0),\ \ldots,\ u_\Theta(\Theta - 2h|x^*(h)) = u^0_\Theta(\Theta - h|x^*_0),\ u_\Theta(\Theta - h|x^*_0(0)) = 0$ which is admissible for the state $x^*(h)$ and

$$\int_0^\Theta c(x_\Theta(t), u_\Theta(t|x^*(h))dt = V_\Theta(x^*_0).$$

Due to the inequality $c(x,u) > 0$ at $x \neq 0,\ u \neq 0$, the function $V_\Theta(x),\ x \in X_\Theta$ does not increase at passing $x^*_0 \to x^*(h)$: $V_\Theta(x^*(h)) \le V_\Theta(x^*_0)$. Moving along trajectory (21) with control (26) one can find $l,\ 0 \le l \le N$, such that $V_\Theta(x(lh)) < V_\Theta(x^*_0)$. So we conclude that the positive definite function $V_\Theta(z),\ z \in X_\Theta$, decreases strongly along any trajectory of (21) and its limit value is equal to zero. According to the proof of the Lyapunov theorem on asymptotic stability [1, 2] the zero solution of system (21) is asymptotically stable in X_Θ. ■

Choosing Θ it is possible to make the domain X_Θ close to the maximum domain of the stabilizability X^* of system (22), i.e., for any $\varepsilon > 0$ there exists such $\Theta < +\infty$ that the ε-vicinity of X_Θ contains X^*.

Stabilizing feedback (27) satisfies the inequality

$$\int_0^\infty c(x^*_\Theta(t), u^*_\Theta(t))dt \le \int_0^\Theta c(x^0_\Theta(t), u^0_\Theta(t|x^*_\Theta))dt. \tag{28}$$

Let the value

$$\int_0^\tau c(x^*_\Theta(t), u^*_\Theta(t))dt$$

be “the cost” of the damping of system (21) on the segment $[0,\tau]$. The value

$$\int_0^\infty c(x^*_\Theta(t), u^*_\Theta(t))dt$$

represents “the cost” of all process of stabilization of (21). Inequality (28) means that “the cost” of all process of stabilization of (21) does not exceed the minimal cost of the system damping for the time Θ.

The value

$$\alpha^0 = \lim_{\Theta\to\infty} \int_0^{\Theta} c(x^0_{\Theta}(t), u^0_{\Theta}(t))dt$$

is called the minimal cost damping system (21).

From (28) it follows that for any $\varepsilon > 0$ there exists such a parameter $\Theta(\varepsilon)$ of accompanying problem (25) that the inequality

$$\int_0^{\infty} c(x^*_{\Theta(\varepsilon)}(t), u^*_{\Theta(\varepsilon)}(t))dt \le \alpha^0 + \varepsilon,$$

holds, i.e. "the cost" of stabilization of the system can be made as close as desired to the minimal cost of its damping.

Remarks. By introducing special constraints in problem (25) one can get required properties of transients for closed-loop system (21). Transients with given degree of asymptotic stability, given degree of oscillations and monotonicity are more spread among them. More details are given in [9, 10].

Scheme of constructing stabilizing feedbacks.
Now let us pass to constructive realization of feedbacks. Divide the domain X into subdomains $X_i,\ i = 1,...,q$. At each $X_i, i = 1,...,q$ substitute the function $a(x),\ x \in X$, by a linear approximations $a_q(x),\ x \in X$, and the function $b(x,u),\ x \in X,\ u \in U = \{u \in \mathbb{R} : |u| \le 1\}$ by the constant functions $b_q(x,u),\ x \in X,\ u \in U$. Choose p points $u_1, \dots, u_p$ in U and construct the vector $b_{qj}(x) = b_q(x, u_j),\ j = 1,...,p$. The system

$$\dot{x} = a_q(x) + \sum_{j=1}^{p} b_{qj}(x)\nu_j,\ 0 \le \nu_j \le 1,\ j = 1,...,p,\ \sum_{j=1}^{p} \nu_j = 1, \qquad (29)$$

is called a piecewise-linear approximation of (21).

Two types of approximation of the criterion can be used to construct the realization of stabilizing feedbacks:
1. Linear approximation of the criterion is made by substitution of an integral criterion by a linear terminal one on account of increasing the dimension of the state vector with the consequent linearization;
2. A quadratic approximation is made when the function $c(x,u),\ x \in X,\ u \in U$, changes by a piecewise-quadratic function.

For both cases fast algorithms of open-loop and closed-loop optimization can be justified [11]. The scheme is completed by using asymptotic improvement of results of the approximation procedure [12].

Example. An inverted pendulum
Consider the mathematical model of an inverted pendulum which is controlled by horizontal movements of the pin [13]

$$\ddot{x} - \sin x + u \cos x = 0. \qquad (30)$$

Here x is an angle between the vertical and the pendulum. Let $x = x_1,\ \dot{x} = x_2$. The state $(\pi, 0)$ is the lower stable equilibrium of (30), $(0,0)$ is the upper unstable one. Denote

$$X = \{(x_1, x_2) : \ -\pi \le x_1 \le \pi\} \tag{31}$$

and divide domain (31) into subdomains $X_1 = \{(x_1, x_2) : \ -\pi \le x_1 \le -3\pi/4\}$, $X_2 = \{(x_1, x_2) : \ -3\pi/4 \le x_1 \le -\pi/2\}$, $X_3 = \{(x_1, x_2) : \ -\pi/2 \le x_1 \le -\pi/4\}$, $X_4 = \{(x_1, x_2) : \ -\pi/4 \le x_1 \le \pi/4\}$, $X_5 = \{(x_1, x_2) : \ \pi/4 \le x_1 \le \pi/2\}$, $X_6 = \{(x_1, x_2) : \ \pi/2 \le x_1 \le 3\pi/4\}$, $X_7 = \{(x_1, x_2) : \ 3\pi/4 \le x_1 \le \pi\}$. In computer experiments a piecewise linear approximation for $\sin x$ and a piecewise-constant approximation for $\cos x$ were used (see (29)).

Let the initial state $x_0 = (\pi, 0)$, values L, Θ, and h be given. The problem is to transfer the pendulum to the upper unstable state $(0, 0)$ and stabilize it at this position. Consider the linearized optimal control problem

$$\int_0^\Theta |u(t)|dt \longrightarrow \min_{u, \Theta_1, \Theta_2, \Theta_3}, \tag{32}$$

$$\begin{aligned}
&\ddot{x}^1 = \, -x^1 + \pi - u, \ x^1(0) = z_1, \dot{x}^1(0) = z_2, \quad t \in [0, \Theta_1];\\
&\ddot{x}^2 = (1 - \pi/4)x^2 + 3/2 - \pi/2 - u(1 - 4/\pi), \quad t \in [\Theta_1, \Theta_2];\\
&\ddot{x}^3 = (\pi/4 - 1)x^3 + \pi/2 - 1 + u(4/\pi - 1), \quad t \in [\Theta_2, \Theta_3];\\
&\ddot{x}^4 = x^4 - u, \ t \in [\Theta_3, \Theta];\\
&x^1(\Theta_1) = \pi/4, \ x^2(\Theta_2) = \pi/2, \ x^3(\Theta_3) = 3\pi/4,\\
&x^4(\Theta) = 0, \ \dot{x}^4(\Theta) = 0; \ |u(t)| \le L, \ t \in T.
\end{aligned}$$

Let the optimal trajectory of (30) on the phase plane go through four domains I: X_4, II: X_5, III: X_6, IV: X_7 and

$$\begin{aligned}
&\Theta_1^0 = \Theta_1^0(0) = \Theta_1^0(x(0), \dot{x}(0)), \ \ \Theta_2^0 = \Theta_2^0(0) = \Theta_2^0(x(0), \dot{x}(0)),\\
&\Theta_3^0 = \Theta_3^0(0) = \Theta_3^0(x(0), \dot{x}(0))
\end{aligned}$$

are optimal moments of crossing boundaries of the corresponding domains: I $\to$ II, II $\to$ III, III $\to$ IV.

At testing the algorithm the influence of parameters L, Θ was studied. The results are given in Table 1. The values of minimal cost functions are given in the first line (corresponds to the problem of damping the pendulum to (0, 0)) and in the second line (corresponds to the solution of problem (32) which gives a realization of the stabilizing feedback on the interval $(0, 20)$).

On Fig. 1 the trajectories of the closed-loop system for different L, Θ are pictured. The curve number 1 corresponds to numbers given in Table 1. The stabilizing feedback realized for the piecewise linear approximation of (30) by solving problem (32) manages to damp out oscillations for nonlinear model (30). The corresponding phase trajectories are presented on Fig. 2 for $\Theta = 3, \ L = 4$. Both the piecewise-linear system and the original nonlinear system were closed by the optimal feedback for (32). The curve 1 and the curve 2 correspond to them. It can be seen from Table 1 that inequality (28) holds.

Table 1. Influence of Θ, L

$\Theta(h)$	$L = 4$	$L = 2$	$L = 1$
3 (0.1)	40.54041		
	21.77021_1		
4 (0.1)	25.44218		
	21.64750_1		
5 (0.2)	16.34493	17.79221	
	15.41292_1	16.57428_1	
6 (0.2)	15.72967	16.70155	
	14.41151_1	16.41495_1	
8 (0.2)	15.45285	16.44656	16.90072
	15.41129_1	16.10531_2	16.45942_2
10 (0.4)	12.24193	13.22911	13.61861
	12.23775_1	13.10527_2	13.45920_2
12 (0.4)	12.23832	12.62847	12.92553
	12.23775_1	12.59836_2	12.89541_2
14 (0.4)	12.23457	12.57533	12.75824
	12.23331_2	12.56470_2	12.74239_2
16 (0.4)	12.23309	12.56614	12.74453
	12.22891_3	12.56647_3	12.74238_3
18 (0.4)	12.22947	12.46490	12.74237
	12.22891_3	12.46470_3	12.74238_3

Fig. 1 *Fig.2*

5. Damping Oscillations of a String

Consider one generalization of the approach. Given a string the right end of which is fixed but the left end can be removed to damp oscillations generated by initial disturbances of its equilibrium state. The mathematical model of the string behavior is as follows

$$\frac{\partial^2 y(t,x)}{\partial t^2} = a^2 \frac{\partial^2 y(t,x)}{\partial x^2}, \tag{33}$$

$$y(t,0) = u(t),\ y(t,l) = 0,\ t \geq 0;$$

$$y(0,x) = \varphi(x),\ \ y_t(0,x) = \psi(x),\ x \in \Omega = \{x:\ x \in [0,l]\}.$$

A piecewise-continuous function $u(t),\ t \geq 0$, is called an open-loop control, the corresponding solution $y(t,x),\ ;x \in \Omega,\ t \geq 0$, of equation (33) is called a trajectory of dynamic system (33) with distributed parameters.

Let $z = (z_1(x),\ z_2(x);\ x \in \Omega)$, $z(t) = (z(t,x),\ x \in \Omega)$, $z(t,x) = (z_1(t,x), z_2(t,x))$; $z_1(t,x) = y(t,x)$, $z_2(t,x) = y_t(t,x)$; $G = \{z :\ |z_1(x)| \leq L_0,\ |z_2(x)| \leq L_1;\ x \in \Omega\}$ be a vicinity of $z = 0$,

$$||z(t)|| = \max\left\{\max_{x\in\Omega}|z_1(t,x)|, \max_{x\in\Omega}|z_2(t,x)|\right\}.$$

Definition 6. *A functional*

$$u(z),\ z \in G, \tag{34}$$

is called a damping feedback control if
(1) *equation* (33) *at* $u(t) = u(z(t)),\ t \geq 0$, *has a solution* $y(t,x),\ x \in \Omega,\ t \geq 0$;
(2) $||z(t)|| \to 0$ *under* $t \to \infty$.

From the practical point of view it is important that the feedback is bounded. Let

$$|u(z)| \leq L,\ z \in G,\ L = \text{const.} \tag{35}$$

Bounded damping feedback (34), (35) providing at a given $L \geq 0$ sufficiently large domain of damping G has very complex structure. Its realization is the difficult problem due to the extreme demand to computer storage. In this connection to construct the feedback in question one can use measurements $y(t,x),\ x = kh,\ k = 1,...,m-1;\ y_t(t,x),\ x = kh,\ k = 0,...,m-1,\ t \geq 0;\ h = l/m,\ m > 1$.

Denote $z_{(m)} = (z_{1(m)},\ z_{2(m)})$, $z_{1(m)} = (z_1(x_k),\ k = 1,...,m-1)$, $z_{2(m)} = (z_2(x_k),\ k = 0,...,m-1)$, $z_{1(m)}(t) = (y_k(t),\ k = 1,...,m-1)$, $z_{2(m)}(t) = (y_{tk}(t),\ k = 0,...,m-1)$, $z_{(m)}(t) = (z_{1(m)}(t),\ z_{2(m)}(t))$, $y_k(t) = y(t,x_k)$, $y_{tk} = y_t(t,x_k)$, $G_m = \{z_{(m)} :\ ||z_{1(m)}|| \leq L_0,\ ||z_{2(m)}|| \leq L_1\}$.

Definition 7. *A function*

$$u(z_{(m)}),\ z_{(m)} \in G_m,$$

is called to be a finite bounded damping feedback if it possesses the following properties
(1) $|u(z_{(m)})| \leq L,\ z_{(m)} \in G_m$;
(2) *problem* (33) *at* $u(t) = u(z_{(m)}(t)),\ t \geq 0$, *has the solution* $y(t,x),\ x \in \Omega,\ t \geq 0$;
(3) $||z(t)|| \to 0$ *if* $t \to \infty$.

As in the previous consideration we shall use discrete control functions. Let control signals are fed to (33) only at discrete moments: $t_i = i\nu,\ i = 0, 1, \ldots$ ($\nu > 0$ is a quantization period).

Definition 8. *A function*

$$u_\nu(z_{(m)}),\ z_{(m)} \in G_m \tag{36}$$

is called a bounded discrete damping feedback if

(1) $|u_\nu(z_{(m)})| \le L,\ z_{(m)} \in G_m$;

(2) *problem* (33) *at* $u(t) = u_\nu(z_{(m)}(t_i)),\ t \in [i\nu, (i+1)\nu[,\ i = 0, 1, \ldots$ *has the solution* $y(t,x),\ x \in \Omega,\ t \ge 0$;

(3) $||z(t)|| \to 0$ *if* $t \to \infty$.

Suppose that a finite bounded discrete damping feedback (36) was constructed. Close system (33), put $z_0^* = (\varphi^*(x),\ \psi^*(x),\ x \in \Omega)$, and obtain the transient $z^*(t),\ t \ge 0$. In the process under question the left end of the string $y^*(t,0),\ t \ge 0$, is moving according to the law

$$u^*(t) = u_\nu(z^*_{(m)}(t_i)),\ t \in [i\nu, (i+1)\nu[,\ i = 0, 1, \ldots .$$

Definition 9. *The function* $u^*(t),\ t \ge 0$, *is called a realization of* (36) *in the concrete process of damping oscillations of the string.*

A device which is able to compute in real time the realization $u^*(t),\ t = i\nu,\ i = 0, 1, 2, \ldots$ in the course of any concrete process of damping is called *Damper*.

So the problem is reduced to constructing algorithms for *Damper*.

Algorithms suggested are based on positional solutions of auxiliary optimal control problems to a finite dimensional dynamic system. Choose an integer $m > 1$ and approximate equation (33) by ordinary differential equations

$$\frac{d^2 y_k(t)}{dt^2} = a^2 \frac{y_{k-1}(t) - 2y_k(t) + y_{k+1}(t)}{h^2}, \quad k = 1, \ldots, m-1; \qquad (37)$$
$$y_0(t) = u(t),\ y_m(t) = 0,\ t \ge 0.$$

Denote

$$y_k(0) = z_1(x_k),\ \frac{dy_k(0)}{dt} = z_2(x_k),\ ;\ æ_{2k-1} = y_{2k-1},\ æ_{2k} = \frac{dy_{2k-1}}{dt},$$
$$g = g(z_{(m)}) = (g_{2k-1}, g_{2k}),\ ;\ g_{2k-1} = z_1(x_k),$$
$$g_{2k} = z_2(x_k),\ k = 1, \ldots, m-1,\ \ n = 2m,$$

and write system (37) as a linear system of ordinary n-order differential equations

$$\dot{æ} = Aæ + bu,\ æ(0) = g. \qquad (38)$$

Here $æ = æ(t)$ is a n-vector of the state of (38) at the moment t. Choose an integer $N < \infty$, a real $\nu > 0$, put $\Theta = N\nu$. A piecewise-constant function $u(t),\ t \in T = [0, \Theta] :\ u(t) = u_j,\ t \in [(j-1)\nu, j\nu[,\ j = 1, \ldots N$, is called accessible control. An accessible control $u(t), t \in T$, is said to be admissible for a state $g \in \mathbb{R}^n$ if the corresponding trajectory $æ(t),\ t \in T$, of (38) satisfies the condition $æ(\Theta) = 0$. Introduce the cost function

$$\rho(u) = \max_{t\in[0,\Theta]} |u(t)|.$$

An admissible control $u^0(t) = u^0(t|g),\ t \in T$, is called the optimal open-loop control for $g \in \mathbb{R}^n$ if

$$\rho(u^0) = \min_u \max_{t\in[0,\Theta]} |u(t)|.$$

Thus, the optimal open-loop control is a solution of the following problem

$$\begin{gathered}\rho(g) = \min \rho, \quad \dot{æ} = Aæ + bu, \\ æ(0) = g,\ æ(\Theta) = 0, \quad |u(t)| \le \rho,\ t \in T.\end{gathered} \tag{39}$$

Problem (39) is called an accompanying optimal control problem. Introduce the set $G_{mu} = \{g:\ \rho(g) \le L\}$. Let $[G_{mu}]_\varepsilon \subset S_L$, be an ε-vicinity of G_{mu}, S_L be a set of controllability of (39) with $|u(t)| \le L,\ t \in T$.

Definition 10. *The function*

$$u^0_\nu(g) = u^0(0|g),\ g \in G_{mu} \tag{40}$$

is called the start bounded (finite, discrete) feedback.

Fig. 3

It can be proved [14] that the start feedback possesses the damping property. Below we give a short description of realization of the damping feedback.

1. $\tau = 0$. The task of Damper is to construct the optimal open-loop control to (38) for $g^* = g(z^*_{(m)})$ at $m,\ N,\ \nu$ given beforehand, corresponding to a realized initial state $(\varphi^*,\ \phi^*)$. It can be done by LP methods [7, 9].
2. The control $u(t) = u^*(t) = u^0(0|g^*),\ t \in [0, \nu[$, is fed to the input of system (33) and generates $z^*(\nu) = (y^*(\nu, x),\ y^*_t(\nu, x),\ x \in \Omega)$.
3. Suppose that Damper worked at $0, \nu, \dots, (s-1)\nu$ and system (33), closed by (36) is turned out to be at $z^*(s\nu)$. The vectors $z^*_{(m)}(s\nu)$ and $g^*(s\nu)$ (the state of (38)) correspond to $z^*(s\nu)$).

4. At $[s\nu, (s+1)\nu[$ Damper uses the value of the optimal open-loop control to problem (39) $u^*(t) = u^*(s\nu) = u^0(0|g^*(z^*_{(m)}(s\nu)))$, $t \in [0, \nu[$, to feed it to system (33) on the segment $[s\nu, (s+1)\nu[$.

More details on the problem are given in [14].

Computer experiments. Parameters: $m = 2$, $\Theta = 6$, $\nu = 0.3$, $L = 1$. Let $l = 1$, $a = 1$, $\varphi^*(x) = \sin(\pi x)$, $\psi^*(x) = -0.3$. The transient is given on Fig. 3. As is seen from Fig. 3 Damper successfully damps the oscillations.

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