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Asymptotically Periodic Solutions of Volterra Difference Equations

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Abstract. In this paper, we discuss the existence of asymptotically periodic solutions of asymptotically periodic Volterra difference equations by using Schauder's first fixed point theorem, and show two examples of a linear equation and a nonlinear equation. Next, similarly we discuss the existence of periodic solutions of periodic Volterra difference equations obtained as limiting equations of the asymptotically periodic Volterra difference equations. Finally, we discuss relations between solutions of the two equations.

1. Introduction

Many results have been obtained for periodic and asymptotically periodic solutions of integral and integro-differential equations. In [1], we studied the behavior of solutions of

$$x(t) = a(t) - \int_0^t D(t, s, x(s))ds$$

and

$$x(t) = p(t) - \int_{-\infty}^t P(t, s, x(s))ds,$$

and their relations to each other. In this paper, corresponding to these Volterra equations, we study the behavior of solutions of the Volterra difference equation

$$x_{n+1} = a(n) - \sum_{k=0}^{n} D(n, k, x_k), \ \ n \in \mathbb{Z}^+ \tag{1}$$

and the Volterra difference equation

$$x_{n+1} = p(n) - \sum_{k=-\infty}^{n} P(n, k, x_k),\ n \in \mathbb{Z}, \tag{2}$$

and their relations to each other, where

$$\begin{gathered} a : \mathbb{Z}^+ \to \mathbb{R}^d, \\ p : \mathbb{Z} \to \mathbb{R}^d, \\ D : \Delta^+ \times \mathbb{R}^d \to \mathbb{R}^d, \\ P : \Delta \times \mathbb{R}^d \to \mathbb{R}^d, \end{gathered}$$

and where $\mathbb{Z}^+$ and $\mathbb{Z}$ denote the sets of nonnegative integers and integers respectively, and

$$\Delta^+ := \{(n, k) \mid 0 \le k \le n,\ k \in \mathbb{Z}^+,\ n \in \mathbb{Z}^+\}$$

$$\Delta := \{(n, k) \mid k \le n,\ k \in \mathbb{Z},\ n \in \mathbb{Z}\},$$

and for any fixed (n, k), $D(n, k, x)$ and $P(n, k, x)$ are continuous in x. Other conditions on a, p, D and P are given later. Throughout this paper, k and n denote integers.

Eq. (1) and Eq. (2) have two very different types of solutions. Eq. (1) may have a solution $\{x_n\}$ $(n \in \mathbb{Z}^+)$ satisfying Eq. (1) on $\mathbb{Z}^+$; similarly, Eq. (2) may have a solution $\{x_n\}$ $(n \in Z)$ satisfying Eq. (2) on $\mathbb{Z}$, such as a periodic solution of Eq. (2). On the other hand, given an initial sequence

$$\phi = \{\phi_n : 0 \le n \le n_0\},\ n_0 \in \mathbb{Z}^+ \quad \text{and} \quad n_0 \ge 1,$$

we write Eq. (1) as

$$x_{n+1} = a(n) - \sum_{k=0}^{n_0-1} D(n, k, \phi_k) - \sum_{k=n_0}^{n} D(n, k, x_k) =: \Phi(n) - \sum_{k=n_0}^{n} D(n, k, x_k).$$

There is a solution $\{x_n(n_0, \phi)\}$ with

$$x_n(n_0, \phi) = \phi_n \quad for \quad 0 \le n \le n_0,$$

and $x_n(n_0, \phi)$ satisfies Eq. (1) for $n > n_0$. Similarly, for a given initial sequence

$$\phi = \{\phi_n : n \le n_0\},\ n_0 \in \mathbb{Z},$$

there is a solution $\{x_n(n_0, \phi)\}$ of Eq. (2) as just described.

In this paper, first we show that if Eq. (1) has an asymptotically N-periodic solution, then Eq. (2) has an N-periodic solution. Next, we use Schauder's first fixed point theorem to show that Eq. (1) has an asymptotically N-periodic solution, thus yielding an N-periodic solution of Eq. (2). We also infer directly that Eq. (2) has N-periodic solutions using Schauder's theorem and a growth condition on P. Finally, we give a detailed list of relations between solutions of Eq. (1) and Eq. (2).

2. Preliminaries

We consider Eq. (1) and Eq. (2), and suppose that

$$p(n+N) = p(n), \quad \text{and} \quad q(n) := a(n) - p(n) \to 0 \quad as \quad n \to \infty, \tag{3}$$

where N is a constant positive integer,

$$P(n+N, k+N, x) = P(n, k, x), \quad and \quad Q(n, k, x) := D(n, k, x) - P(n, k, x), \tag{4}$$

and for any $J > 0$ there are functions

$$P_J : \Delta \to \mathbb{R}^+ := [0, \infty)$$

and

$$Q_J : \Delta^+ \to \mathbb{R}^+$$

such that

$$P_J(n+N, k+N) = P_J(n, k) \quad \text{if} \quad (n, k) \in \Delta,$$

$$|P(n, k, x)| \le P_J(n, k) \quad \text{if} \quad (n, k) \in \Delta \quad \text{and} \quad |x| \le J,$$

where $|\cdot|$ denotes a norm on $\mathbb{R}^d$, and

$$|Q(n, k, x)| \le Q_J(n, k) \quad \text{if} \quad (n, k) \in \Delta^+ \quad and \quad |x| \le J,$$

$$\sum_{k=-\infty}^{n} P_J(n+\nu, k) \to 0 \quad \text{uniformly for} \quad n \in \mathbb{Z} \quad as \quad \nu \to \infty \tag{5}$$

and

$$\sum_{k=0}^{n} Q_J(n, k) \to 0 \quad \text{as} \quad n \to \infty. \tag{6}$$

In this paper, we discuss the existence of asymptotically periodic and periodic solutions of Eq. (1) and Eq. (2) using the following theorem, which we state without proof (cf. [3]).

Theorem 2.1. (Schauder's first theorem). *Let $(C, \|\cdot\|)$ be a normed space, and let S be a compact convex nonempty subset of C. Then every continuous mapping of S into S has a fixed point.*

3. Asymptotically Periodic Solutions

Let C be a set of bounded sequences $\xi = \{\xi_n\} : \mathbb{Z}^+ \to \mathbb{R}^d$. For any $\xi \in C$, define $\|\xi\|_+$ by

$$\|\xi\|_+ := \sup\{|\xi_n| : n \in \mathbb{Z}^+\}.$$

Then clearly $\|\cdot\|_+$ is a norm on C, and $(C, \|\cdot\|_+)$ is a Banach space. Let $n_0 \in \mathbb{Z}^+$ be a fixed integer. For any $\xi \in C$ define a mapping $H = H_{n_0}$ on C by

$$(H\xi)_n := \begin{cases} \xi_n, & 0 \le n \le n_0 \\ a(n-1) - \sum_{k=0}^{n-1} D(n-1, k, \xi_k), & n > n_0. \end{cases}$$

Moreover, for any $J > 0$ let $C_J := \{\xi \in C : \|\xi\|_+ \le J\}$. First we have the following lemma.

Lemma 3.1. *If* (3) - (6) *hold, then for any asymptotically N-periodic sequence $\xi = \{\xi_n\}$ on $\mathbb{Z}^+$ such that $\xi_n = \pi_n + \rho_n$, ξ, $\pi \in C$, $\pi_{n+N} = \pi_n$ on $\mathbb{Z}^+$ and $\rho_n \to 0$ as $n \to \infty$, the sequence $\delta = \{\delta_n\}$ with*

$$\delta_{n+1} := \sum_{k=0}^{n} D(n, k, \xi_k), \quad n \in \mathbb{Z}^+$$

is asymptotically N-periodic, and the N-periodic part of δ is given by

$$\Big\{ \sum_{k=-\infty}^{n} P(n, k, \pi_k) \Big\}.$$

Proof. By (4) - (6), one can easily check that the sequences δ and $\phi = \{\phi_n\}$ with

$$\phi_{n+1} := \sum_{k=-\infty}^{n} P(n, k, \pi_k), \quad n \in \mathbb{Z}^+$$

are bounded and that $\phi_{n+N} = \phi_n$ for $n \ge 1$. Therefore, in order to establish the lemma, it is sufficient to show that $\delta_n - \phi_n \to 0$ as $n \to \infty$. Let $J > 0$ be a number with $\|\xi\|_+ \le J$. Then clearly we have $\|\pi\|_+ \le J$. From (5), for any $\epsilon > 0$ there is a positive $\nu_1 \in \mathbb{Z}^+$ with

$$\sum_{k=-\infty}^{n} P_J(n + \nu_1, k) < \epsilon \quad \text{if} \quad n \in \mathbb{Z}^+.$$

Let $n \ge \nu_1$. Then we obtain

$$\begin{aligned} |\delta_{n+1} - \phi_{n+1}| &= \Big| \sum_{k=0}^{n} P(n, k, \xi_k) - \sum_{k=-\infty}^{n} P(n, k, \pi_k) + \sum_{k=0}^{n} Q(n, k, \xi_k) \Big| \\ &\le \sum_{k=0}^{n-\nu_1} P_J(n, k) + \sum_{k=-\infty}^{n-\nu_1} P_J(n, k) \\ &\quad + \sum_{k=n-\nu_1+1}^{n} |P(n, k, \xi_k) - P(n, k, \pi_k)| + \sum_{k=0}^{n} Q_J(n, k). \end{aligned}$$

Since $P(n, k, x)$ is continuous in x and N-periodic, for the ϵ there is a $\gamma > 0$ with

$$|P(n, k, x) - P(n, k, y)| < \frac{\epsilon}{\nu_1}$$

if $n \in \mathbb{Z},\ n - \nu_1 + 1 \le k \le n,\ |x| \le J,\ |y| \le J$ and $|x - y| < \gamma$. Moreover, since $\rho_n \to 0$ as $n \to \infty$, and since we have (6), for the γ and the ϵ there is a $\nu_2 \in \mathbb{Z}^+$ with

$$|\rho_n| = |\xi_n - \pi_n| < \gamma \quad if \quad n \ge \nu_2$$

and

$$\sum_{k=0}^{n} Q_J(n,k) < \epsilon \quad if \quad n \ge \nu_2.$$

Then, we obtain

$$|\delta_n - \phi_n| < 4\epsilon \quad \text{if} \quad n > \nu_1 + \nu_2,$$

which proves that $\delta_n - \phi_n \to 0$ as $n \to \infty$. ∎

From this lemma, we have the following theorem.

Theorem 3.2. *If* (3) - (6) *hold, and if Eq.* (1) *has an asymptotically N-periodic solution with an initial time n_0 in $\mathbb{Z}^+$, then the N-periodic extension to $\mathbb{Z}$ of its N-periodic part is an N-periodic solution of Eq.* (2). *In particular, if the asymptotically N-periodic solution of Eq.* (1) *is asymptotically constant, then Eq.* (2) *has a constant solution.*

Proof. Let $x = \{x_n\}$ be an asymptotically N-periodic solution of Eq. (1) with an initial time $n_0 \in \mathbb{Z}^+$ such that $x_n = y_n + z_n,\ x,\ y \in C,\ y_{n+N} = y_n$ on $\mathbb{Z}^+$ and $z_n \to 0$ as $n \to \infty$. Then we have

$$y_{n+1} + z_{n+1} = p(n) + q(n) - \sum_{k=0}^{n} D(n,k,x_k), \quad n \ge n_0. \tag{7}$$

From Lemma 3, taking the N-periodic part of the both sides of Eq. (7) we obtain

$$y_{n+1} = p(n) - \sum_{k=-\infty}^{n} P(n,k,y_k), \quad n \ge n_0.$$

From this, it is easy to see that the N-periodic extension to $\mathbb{Z}$ of $\{y_n\}$ is an N-periodic solution of Eq. (2).

The latter part follows easily from the above conclusion. ∎

In order to prove the existence of an asymptotically N-periodic solution of Eq. (1) using Schauder's first theorem, we need additional assumptions. In addition to (3) - (6), suppose that for some $n_0 \in \mathbb{Z}^+$ and $J > 0$ the inequality

$$\|a\|_{n_0} + \sum_{k=0}^{n} P_J(n,k) + \sum_{k=0}^{n} Q_J(n,k) \le J \quad \text{if} \quad n \ge n_0 \tag{8}$$

holds, where

$$\|a\|_{n_0} := \sup\{|a(n)| : n \ge n_0\},$$

and that there are functions

$$L_J : \Delta \to \mathbb{R}^+$$

and

$$q_J : \{n \in \mathbb{Z} : n \geq n_0 - 1\} \to \mathbb{R}^+$$

such that $L_J(n+N, k+N) = L_J(n,k)$ and

$$|P(n,k,x) - P(n,k,y)| \leq L_J(n,k)|x-y| \quad \text{if} \quad (n,k) \in \Delta, \ |x| \leq J, |y| \leq J, \quad (9)$$

$$q_J(n) \to 0 \quad \text{as} \quad n \to \infty, \quad (10)$$

and

$$\begin{aligned} |q(n)| &+ \sum_{k=-\infty}^{n_0-1} P_J(n,k) + \sum_{k=0}^{n_0-1} P_J(n,k) \\ &+ \sum_{k=0}^{n} Q_J(n,k) + \sum_{k=n_0}^{n} L_J(n,k) q_J(k-1) \leq q_J(n) \quad \text{if} \quad n \geq n_0. \end{aligned} \quad (11)$$

Then we have the following theorem.

Theorem 3.3. *If* (3) - (6) *and* (8) - (11) *with some* $n_0 \in \mathbb{Z}^+$ *and* $J > 0$ *hold, then for any initial sequence* $\phi^0 = \{\phi_n^0\}$ *with* $\sup\{|\phi_n^0| : 0 \leq n \leq n_0\} \leq J$, $x = \{x_n\} = \{x_n(n_0, \phi^0)\} = y + z$ *is an asymptotically* N*-periodic solution of Eq.* (1), *and the* N*-periodic extension to* $\mathbb{Z}$ *of* y *is an* N*-periodic solution of Eq.* (2).

Proof. Let S be a set of sequences $\xi \in C_J$ such that $\xi = \pi + \rho, \xi, \pi \in C_J, \xi_n = \phi_n^0$ for $0 \leq n \leq n_0, \pi_{n+N} = \pi_n$ on $\mathbb{Z}^+$ and

$$|\rho_{n+1}| \leq q_J(n) \quad \text{if} \quad n \geq n_0. \quad (12)$$

First we prove that S is a compact convex nonempty subset of C. Since any $\xi \in C_J$ such that $\xi_n = \phi_n^0$ for $0 \leq n \leq n_0$ and $\xi_n \equiv \xi_{n_0}$ for $n > n_0$ is contained in S, S is nonempty. Clearly S is a convex subset of C. In order to prove the compactness of S, let $\{\xi^k\}$ be an infinite sequence in S such that $\xi^k = \pi^k + \rho^k$, ξ^k, $\pi^k \in C_J$, $\pi_{n+N}^k = \pi_n^k$ on $\mathbb{Z}^+$ and $|\rho_{n+1}^k| \leq q_J(n)$ for $n \geq n_0$. Taking a subsequence if necessary, we may assume that the sequence $\{\pi^k\}$ converges to a $\pi \in C_J$ uniformly on $\mathbb{Z}^+$, and the sequence $\{\rho^k\}$ converges to a $\rho \in C$ uniformly on any finite subset of $\mathbb{Z}^+$. Clearly π_n is N-periodic on $\mathbb{Z}^+$, and ρ_n satisfies (12), and hence the sequence $\{\xi^k\}$ converges to the asymptotically N-periodic sequence $\xi := \pi + \rho$ uniformly on any finite subset of $\mathbb{Z}^+$ as $k \to \infty$. It is clear that $\xi \in S$. Now we show that $\|\rho^k - \rho\|_+ \to 0$ as $k \to \infty$. From (10), for any $\epsilon > 0$ there is a $\nu_1 \in \mathbb{Z}^+$ with $\nu_1 \geq n_0$ and

$$q_J(n) < \frac{\epsilon}{2} \quad \text{if} \quad n \geq \nu_1,$$

which yields

$$|\rho_{n+1}^k - \rho_{n+1}| \leq 2q_J(n) < \epsilon \quad \text{if} \quad n \geq \nu_1. \quad (13)$$

On the other hand, since ρ_n^k converges to ρ_n uniformly for $0 \leq n \leq \nu_1$ as $k \to \infty$, for the ϵ there is a $\kappa \in \mathbb{Z}^+$ with

$$|\rho_n^k - \rho_n| < \epsilon \quad \text{if} \quad k \geq \kappa \quad \text{and} \quad 0 \leq n \leq \nu_1,$$

which together with (13), implies $\|\rho^k - \rho\|_+ < \epsilon$ if $k \geq \kappa$. This yields $\|\rho^k - \rho\|_+ \to 0$ as $k \to \infty$, and hence

$$\|\xi^k - \xi\|_+ \to 0 \quad \text{as} \quad k \to \infty.$$

Thus S is compact.

Next we prove that H maps S into S continuously. For any $\xi \in S$ such that $\xi = \pi + \rho,\ \ \xi,\ \ \pi \in C_J,\ \ \pi_{n+N} = \pi_n$ on $\mathbb{Z}^+$ and $|\rho_{n+1}| \leq q_J(n)$ for $n \geq n_0$, let $\phi := H\xi$. Then from (8), for $n \geq n_0$ we have

$$\begin{aligned}|\phi_{n+1}| &\leq |a(n)| + \sum_{k=0}^{n} |P(n,k,\xi_k)| + \sum_{k=0}^{n} |Q(n,k,\xi_k)| \\ &\leq \|a\|_{n_0} + \sum_{k=0}^{n} P_J(n,k) + \sum_{k=0}^{n} Q_J(n,k) \leq J,\end{aligned}$$

which together with $\xi \in C_J$, implies that $\phi \in C_J$. Now from Lemma 3.1, ϕ has the unique decomposition $\phi = \psi + \mu,\ \ \phi,\ \ \psi \in C_J,\ \ \psi_{n+N} = \psi_n$ on $\mathbb{Z}^+$, and $\mu_n \to 0$ as $n \to \infty$, where the restriction of $\mu = \{\mu_n\}$ for $n > n_0$ is given by

$$\begin{aligned}\mu_{n+1} := q(n) &+ \sum_{k=-\infty}^{n_0-1} P(n,k,\pi_k) - \sum_{k=0}^{n_0-1} P(n,k,\xi_k) \\ &- \sum_{k=0}^{n} Q(n,k,\xi_k) - \sum_{k=n_0}^{n} (P(n,k,\xi_k) - P(n,k,\pi_k)), \quad n \geq n_0.\end{aligned}$$

Thus from (11), for $n \geq n_0$ we obtain

$$\begin{aligned}|\mu_{n+1}| &\leq |q(n)| + \sum_{k=-\infty}^{n_0-1} |P(n,k,\pi_k)| + \sum_{k=0}^{n_0-1} |P(n,k,\xi_k)| \\ &\quad + \sum_{k=0}^{n} |Q(n,k,\xi_k)| + \sum_{k=n_0}^{n} |P(n,k,\xi_k) - P(n,k,\pi_k)| \\ &\leq |q(n)| + \sum_{k=-\infty}^{n_0-1} P_J(n,k) + \sum_{k=0}^{n_0-1} P_J(n,k) \\ &\quad + \sum_{k=0}^{n} Q_J(n,k) + \sum_{k=n_0}^{n} L_J(n,k) q_J(k-1) \leq q_J(n).\end{aligned}$$

Thus H maps S into S. We must prove that H is continuous. For any $\xi^i \in S\ \ (i = 1, 2)$ and $n > n_0$ we have

$$\begin{aligned}|(H\xi^1)_{n+1} - (H\xi^2)_{n+1}| &\leq \sum_{k=0}^{n} |D(n,k,\xi_k^1) - D(n,k,\xi_k^2)| \\ &\leq \sum_{k=0}^{n} |P(n,k,\xi_k^1) - P(n,k,\xi_k^2)| + \sum_{k=0}^{n} |Q(n,k,\xi_k^1) - Q(n,k,\xi_k^2)|.\end{aligned} \tag{14}$$

From (5), for any $\epsilon > 0$ there is a positive $\nu_2 \in \mathbb{Z}^+$ with

$$\sum_{k=-\infty}^{n-\nu_2} P_J(n,k) < \frac{\epsilon}{6} \quad \text{if} \quad n \in \mathbb{Z}. \tag{15}$$

Since $P(n,k,x)$ is continuous in x and N-periodic, for the ϵ there is a $\delta_1 > 0$ with

$$|P(n,k,x) - P(n,k,y)| < \frac{\epsilon}{6\nu_2} \tag{16}$$

if $n \in \mathbb{Z},\ n - \nu_2 + 1 \le k \le n,\ |x| \le J,\ |y| \le J$ and $|x - y| < \delta_1$. From this, for the ϵ we obtain

$$\sum_{k=0}^{n} |P(n,k,\xi_k^1) - P(n,k,\xi_k^2)| < \frac{\epsilon}{6} \tag{17}$$

if $n_0 \le n \le \nu_2$ and $\|\xi^1 - \xi^2\|_+ < \delta_1$. If $n > \nu_2$ and $\|\xi^1 - \xi^2\|_+ < \delta_1$, then from (15) and (16) we have

$$\begin{aligned}
&\sum_{k=0}^{n} |P(n,k,\xi_k^1) - P(n,k,\xi_k^2)| \\
&\le 2\sum_{k=-\infty}^{n-\nu_2} P_J(n,k) + \sum_{k=n-\nu_2+1}^{n} |P(n,k,\xi_k^1) - P(n,k,\xi_k^2)| < \frac{\epsilon}{2},
\end{aligned}$$

which together with (17), yields

$$\sum_{k=0}^{n} |P(n,k,\xi_k^1) - P(n,k,\xi_k^2)| < \frac{\epsilon}{2} \quad \text{if} \quad \|\xi^1 - \xi^2\|_+ < \delta_1. \tag{18}$$

Next from (6), for the ϵ there is a positive $\nu_3 \in \mathbb{Z}^+$ with

$$\sum_{k=0}^{n} Q_J(n,k) < \frac{\epsilon}{4} \quad \text{if} \quad n \ge \nu_3,$$

which implies

$$\sum_{k=0}^{n} |Q(n,k,\xi_k^1) - Q(n,k,\xi_k^2)| \le 2\sum_{k=0}^{n} Q_J(n,k) < \frac{\epsilon}{2} \quad \text{if} \quad n \ge \nu_3. \tag{19}$$

Since $Q(n,k,x)$ is continuous in x, for the ϵ there is a $\delta_2 > 0$ with

$$|Q(n,k,x) - Q(n,k,y)| < \frac{\epsilon}{2\nu_3}$$

if $0 \le k \le n \le \nu_3,\ \ |x| \le J,\ \ |y| \le J$ and $|x - y| < \delta_2$, which yields

$$\sum_{k=0}^{n} |Q(n,k,\xi_k^1) - Q(n,k,\xi_k^2)| < \frac{\epsilon}{2} \quad \text{if} \quad 0 \le n \le \nu_3 \quad \text{and} \quad \|\xi_1 - \xi_2\|_+ < \delta_2.$$

This together with (19), implies

$$\sum_{k=0}^{n} |Q(n,k,\xi_k^1) - Q(n,k,\xi_k^2)| < \frac{\epsilon}{2} \quad \text{if} \quad \|\xi^1 - \xi^2\|_+ < \delta_2. \tag{20}$$

Thus, from (14), (18) and (20), for the $\delta := \min(\delta_1, \delta_2)$ we obtain

$$\|H\xi^1 - H\xi^2\| < \epsilon \ \text{if} \ \xi^1, \quad \xi^2 \in S \quad \text{and} \quad \|\xi^1 - \xi^2\|_+ < \delta,$$

and hence H is continuous.

Finally, applying Theorem 2.1, H has a fixed point in S, which is a desired asymptotically N-periodic solution of Eq. (1). The latter part is a direct consequence of Theorem 3.2. ∎

From this theorem and the argument in the proof of Theorem 2.1 in [2], we have the following corollary.

Corollary 3.4. *If Eq.* (2) *has a unique* $\mathbb{Z}$*-bounded solution* $\eta = \{\eta_n\}$ *such that* $|\eta_n| \leq J$ *on* $\mathbb{Z}$ *and* η_n *satisfies Eq.* (2) *on* $\mathbb{Z}$*, then* η_n *is* N*-periodic on* $\mathbb{Z}$*, and for any solution* $x = \{x_n\} = \{x_n(n_0, \phi)\}$ *of Eq.* (1)*,* $x_n - \eta_n \to 0$ *as* $n \to \infty$*, provided that* $\|x\|_+ \leq J$*.*

Now we show two examples of a linear equation and a nonlinear equation.

Linear Example. Consider the scalar linear equation

$$x_{n+1} = p(n) + \rho e^{-n} - \frac{1}{3}\sum_{k=0}^{n} ((-1)^k e^{k-n} + \beta e^{-k-n}) x_k, \quad n \in \mathbb{Z}^+, \tag{21}$$

where $p : \mathbb{Z} \to \mathbb{R}$ is a 2-periodic function, and ρ and β are constants. Eq. (21) is obtained from Eq. (1) taking $d = 1$, $N = 2$, $a(n) = p(n) + \rho e^{-n}$, $q(n) = \rho e^{-n}$, $D(n,k,x) = ((-1)^k e^{k-n} + \beta e^{-k-n})x/3$, $P(n,k,x) = (-1)^k e^{k-n} x/3$ and $Q(n,k,x) = \beta e^{-k-n} x/3$. Define a number α by

$$\alpha := \frac{1}{3}\sum_{k=-\infty}^{n} e^{k-n} = \frac{e}{3(e-1)}. \tag{22}$$

Then clearly α satisfies $0 < \alpha < 2/3$. For this α, we take $\beta = (2 - 3\alpha)/(3\alpha)$. For $J := 3(\|p\| + |\rho|)$ with $\|p\| := sup\{|p(n)| : n \in \mathbb{Z}\}$, we can take the following functions as P_J, Q_J and L_J.

$$P_J(n,k) := \frac{J}{3} e^{k-n} \quad \text{if} \quad (n,k) \in \Delta,$$
$$Q_J(n,k) := \frac{\beta J}{3} e^{-k-n} \quad \text{if} \quad (n,k) \in \Delta^+,$$

and

$$L_J(n,k) := \frac{1}{3} e^{k-n} \quad \text{if} \quad (n,k) \in \Delta.$$

It is easy to see that the above functions satisfy (3) - (6) and (9). Theorem 3.3 is true under (8) with $n_0 = 0$ and

$$\begin{aligned} |q(n)| &+ \sum_{k=-\infty}^{-1} P_J(n,k) + \sum_{k=0}^{n} Q_J(n,k) \\ &+ \sum_{k=0}^{n} L_J(n,k) q_J(k-1) \le q_J(n), \quad n \in \mathbb{Z}^+ \end{aligned} \tag{23}$$

instead of (11). It is easy to see that (8) with $n_0 = 0$ holds for the J, since we have $\|a\|_+ \le \|p\| + |\rho|$, $\sum_{k=0}^{n} P_J(n,k) \le \alpha J$ and $\sum_{k=0}^{n} Q_J(n,k) \le \alpha\beta J$ on $\mathbb{Z}^+$. Now define a function $q_J : \{-1\} \cup \mathbb{Z}^+ \to \mathbb{R}^+$ by

$$q_J(n) := \frac{3\gamma}{3-e}\left(\frac{2}{e}\right)^n, \quad n \in \{-1\} \cup \mathbb{Z}^+,$$

where $\gamma := |\rho| + 2J/3$. Clearly (10) holds. Moreover it is easily seen that for any $n \in \mathbb{Z}^+$ we have

$$\begin{aligned} |q(n)| + &\sum_{k=-\infty}^{-1} P_J(n,k) + \sum_{k=0}^{n} Q_J(n,k) + \sum_{k=0}^{n} L_J(n,k) q_J(k-1) \\ &\le \gamma e^{-n} + \frac{1}{3}\sum_{k=0}^{n} e^{k-n} q_J(k-1) \le q_J(n), \end{aligned}$$

that is, (23) holds. Thus by Theorem 3.3, for any x_0 with $|x_0| \le J$, $x = \{x_n\} = \{x_n(0,x_0)\} = y + z$ is an asymptotically 2-periodic solution of Eq. (21) such that $x,\ y \in C_J,\ y_{n+2} = y_n$ and $|z_{n+1}| \le q_J(n)$ on $\mathbb{Z}^+$, and the 2-periodic extension of y to $\mathbb{Z}$ is a 2-periodic solution of the equation

$$x_{n+1} = p(n) - \frac{1}{3}\sum_{k=-\infty}^{n} (-1)^k e^{k-n} x_k, \quad n \in \mathbb{Z}.$$

Nonlinear Example. Corresponding to Eq. (21), consider the scalar nonlinear equation

$$x_{n+1} = p(n) + \rho e^{-n} - \sum_{k=0}^{n} (\sigma(-1)^k e^{k-n} + \beta e^{-k-n}) x_k^2, \quad n \in \mathbb{Z}^+, \tag{24}$$

where $p : \mathbb{Z} \to \mathbb{R}$ is a 2-periodic function, and ρ, σ and β are constants such that $0 < 12\alpha(\|p\| + |\rho|)(|\sigma| + |\beta|) < 1$ and $0 < (e - 3\alpha)|\sigma| < 3\alpha|\beta|$, where α is the number defined in (22). Eq. (24) is obtained from Eq. (1) taking $d = 1,\quad N = 2, a(n) = p(n) + \rho e^{-n}, (n) = \rho e^{-n}, D(n,k,x) = (\sigma(-1)^k e^{k-n} + \beta e^{-k-n})x^2, P(n,k,x) = \sigma(-1)^k e^{k-n} x^2$ and $Q(n,k,x) = \beta e^{-k-n} x^2$. Let J be a number defined by

$$J := \frac{1 - \sqrt{1 - 12\alpha(\|p\| + |\rho|)(|\sigma| + |\beta|)}}{6\alpha(|\sigma| + |\beta|)}. \tag{25}$$

Then it is easy to see that $\|p\|+|\rho|+3\alpha(|\sigma|+|\beta|)J^2 = J$ and $0 < 2e|\sigma|J < 1$. For this J, we can take the following functions as P_J, Q_J and L_J.

$$P_J(n,k) := J^2|\sigma|e^{k-n} \quad \text{if} \quad (n,k) \in \Delta,$$
$$Q_J(n,k) := J^2|\beta|e^{-k-n} \quad \text{if} \quad (n,k) \in \Delta^+,$$

and

$$L_J(n,k) := 2J|\sigma|e^{k-n} \quad \text{if} \quad (n,k) \in \Delta.$$

It is easy to see that these functions satisfy (3) - (6) and (9). Moreover (25) implies (8) with $n_0 = 0$, since we have $\|a\|_+ \le \|p\| + |\rho|$, $\sum_{k=0}^{n} P_J(n,k) \le 3\alpha|\sigma|J^2$ and $\sum_{k=0}^{n} Q_J(n,k) \le 3\alpha|\beta|J^2$ on $\mathbb{Z}^+$. Now define a function $q_J : \{-1\} \cup \mathbb{Z}^+ \to \mathbb{R}^+$ by

$$q_J(n) := \frac{\gamma}{1-2e|\sigma|J}\left(\frac{2}{e}\right)^n, \quad n \in \{-1\} \cup \mathbb{Z}^+,$$

where $\gamma := |\rho| + 3\alpha J^2(|\sigma|/e + |\beta|)$. As in Linear Example, it is easily seen that (10) and (23) hold. Thus by Theorem 3.3, for any x_0 with $|x_0| \le J$, $x = \{x_n\} = \{x_n(0,x_0)\} = y + z$ is an asymptotically 2-periodic solution of Eq. (24) such that $x,\ y \in C_J$, $y_{n+2} = y_n$ and $|z_{n+1}| \le q_J(n)$ on $\mathbb{Z}^+$, and the 2-periodic extension of y to $\mathbb{Z}$ is a 2-periodic solution of the equation

$$x_{n+1} = p(n) - \sum_{k=-\infty}^{n} \sigma(-1)^k e^{k-n} x_k^2, \quad n \in \mathbb{Z}.$$

4. Periodic Solutions

Although Theorem 3.3 assures the existence of an N-periodic solution of Eq. (2), we can prove directly the existence of an N-periodic solution of Eq. (2) under weaker assumptions than those in Theorem 3.3 using Schauder's first theorem.

Let $(\mathcal{P}_N, \|\cdot\|)$ be the Banach space of N-periodic functions $\xi : \mathbb{Z} \to \mathbb{R}^d$ with the supremum norm. For any $\xi \in \mathcal{P}_N$, define a mapping H on $\mathcal{P}_N$ by

$$(H\xi)_{n+1} := p(n) - \sum_{k=-\infty}^{n} P(n,k,\xi_k), \quad n \in \mathbb{Z}.$$

Then we have the following theorem.

Theorem 4.1. *In addition to* (3) - (5) *with* $Q(n,k,x) \equiv 0$, *suppose that for some* $J > 0$ *the condition*

$$\|p\| + \sum_{k=-\infty}^{n} P_J(n,k) \le J \quad \text{if} \quad n \in \mathbb{Z} \tag{26}$$

holds. Then Eq. (2) *has an* N*-periodic solution* $x = \{x_n\}$ *with* $\|x\| \le J$.

Proof. Let S be a set of functions $\xi \in \mathcal{P}_N$ such that $\|\xi\| \le J$. First it can be easily seen that S is a compact convex nonempty subset of $\mathcal{P}_N$. Next we prove that H maps S into S. For any $\xi \in S$, let $\phi := H\xi$. Then, clearly ϕ_n is N-periodic. In addition, from (26) we have

$$|\phi_{n+1}| \leq \|p\| + \sum_{k=-\infty}^{n} P_J(n,k) \leq J \quad \text{if} \quad n \in \mathbb{Z},$$

and hence $\|\phi\| \leq J$. Thus H maps S into S. The continuity of H can be proved similarly as in the proof of Theorem 3.3. Finally, applying Theorem 2.1 we can conclude that H has a fixed point x in S, which is an N-periodic solution of Eq. (2) with $\|x\| \leq J$. ∎

Remark. In addition to the continuity of the mapping H, we can easily prove that H maps each bounded set of $\mathcal{P}_N$ into a compact set of $\mathcal{P}_N$. Thus Theorem 4.1 can be proved using Schauder's second theorem.

5. Relations Between Two Equations

In Theorem 3.2, we showed a relation between an asymptotically N-periodic solution of Eq. (1) and an N-periodic solution of Eq. (2). Moreover, concerning relations between solutions of Eq. (1) and Eq. (2) we have the following theorem.

Theorem 5.1. *Under the assumptions* (3) - (6), *the following five conditions are equivalent.*

(i) *Eq.* (2) *has an N-periodic solution.*
(ii) *For some $q(n)$ and $Q(n,k,x) \equiv 0$, Eq.* (1) *has an N-periodic solution which satisfies Eq.* (1) *on $\mathbb{Z}^+$.*
(iii) *For some $q(n)$ and $Q(n,k,x) \equiv 0$, Eq.* (1) *has an asymptotically N-periodic solution with an initial time in $\mathbb{Z}^+$.*
(iv) *For some $q(n)$ and $Q(n,k,x)$, Eq.* (1) *has an N-periodic solution which satisfies Eq.* (1) *on $\mathbb{Z}^+$.*
(v) *For some $q(n)$ and $Q(n,k,x)$, Eq.* (1) *has an asymptotically N-periodic solution with an initial time in $\mathbb{Z}^+$.*

Proof. First we prove that (i) implies (ii). Let π_n be an N-periodic solution of Eq. (2), and let

$$q(n) := -\sum_{k=-\infty}^{-1} P(n,k,\pi_k), \quad n \in \mathbb{Z}^+.$$

Then, clearly $q(n) \to 0$ as $n \to \infty$. Thus it is easy to see that for the $q(n)$ and $Q(n,k,x) \equiv 0$, Eq. (1) has an N-periodic solution π_n, which satisfies Eq. (1) on $\mathbb{Z}^+$. Next, it is clear that (ii) and (iii) imply (iii) and (v) respectively. Moreover, from Theorem 3.2, (v) yields (i).

Finally, since it is trivial that (ii) implies (iv), we prove that (iv) yields (ii). Let ψ_n be an N-periodic solution of Eq. (1) with some $q(n)$ and $Q(n,k,x)$ which satisfies Eq. (1) on $\mathbb{Z}^+$, and let

$$r(n) := -\sum_{k=0}^{n} Q(n,k,\psi_k), \quad n \in \mathbb{Z}^+.$$

Then, clearly $r(n) \to 0$ as $n \to \infty$. Thus it is easy to see that for the $a(n) := p(n) + q(n) + r(n)$ and $Q(n,k,x) \equiv 0$, Eq. (1) has an N-periodic solution ψ_n which satisfies Eq. (1) on $\mathbb{Z}^+$. ∎

Remark. The equivalence among (i)-(iii) can be obtained without the assumption (6).

In [2], we discussed a relation between the equation

$$x_{n+1} = a(n) - \sum_{k=0}^{n} E(n,k)x_k - \sum_{k=0}^{n} Q(n,k,x_k), \quad n \in \mathbb{Z}^+, \tag{27}$$

and the linear equation

$$x_{n+1} = p(n) - \sum_{k=-\infty}^{n} E(n,k)x_k, \quad n \in \mathbb{Z},$$

where a, p and Q satisfy (3) and (6), and $E : \Delta \to \mathbb{R}^{d\times d}$ satisfies

$$E(n+N, k+N) = E(n,k)$$

and

$$\sum_{k=-\infty}^{n} |E(n+\nu,k)| \to 0 \quad \text{uniformly for} \quad n \in \mathbb{Z} \quad \text{as} \quad \nu \to \infty,$$

where $|E| := \sup\{|Ex| : |x| = 1\}$. Concerning Eq. (27) and Eq. (28), we state a theorem. For the proof, see Lemma 2.1 and Theorem 5.1 in [2].

Theorem 5.2. [2] *Under the above assumptions for Eq.* (27) *and Eq.* (28), *the following hold.*

(i) *If Eq.* (27) *has a $\mathbb{Z}^+$-bounded solution with an initial time in $\mathbb{Z}^+$, then Eq.* (28) *has a $\mathbb{Z}$-bounded solution which satisfies Eq.* (28) *on $\mathbb{Z}$.*

(ii) *If Eq.* (28) *has a $\mathbb{Z}$-bounded solution which satisfies Eq.* (28) *on $\mathbb{Z}$, then Eq.* (28) *has an N-periodic solution.*

Now we have the following theorem concerning relations between Eq. (27) and Eq. (28).

Theorem 5.3. *Under the above assumptions for Eq.* (27) *and Eq.* (28), *the following eight conditions are equivalent.*

(i) *Eq.* (28) *has an N-periodic solution.*

(ii) *For some $q(n)$ and $Q(n,k,x) \equiv 0$, Eq.* (27) *has an N-periodic solution which satisfies Eq.* (27) *on $\mathbb{Z}^+$.*

(iii) *For some $q(n)$ and $Q(n,k,x) \equiv 0$, Eq.* (27) *has an asymptotically N-periodic solution with an initial time in $\mathbb{Z}^+$.*

(iv) *For some* $q(n)$ *and* $Q(n,k,x) \equiv 0$*, Eq.* (27) *has a* $\mathbb{Z}^+$*-bounded solution with an initial time in* $\mathbb{Z}^+$*.*

(v) *For some* $q(n)$ *and* $Q(n,k,x)$*, Eq.* (27) *has an* N*-periodic solution which satisfies Eq.* (27) *on* $\mathbb{Z}^+$*.*

(vi) *For some* $q(n)$ *and* $Q(n,k,x)$*, Eq.* (27) *has an asymptotically* N*-periodic solution with an initial time in* $\mathbb{Z}^+$*.*

(vii) *For some* $q(n)$ *and* $Q(n,k,x)$*, Eq.* (27) *has a* $\mathbb{Z}^+$*-bounded solution with an initial time in* $\mathbb{Z}^+$*.*

(viii) *Eq.* (28) *has a* $\mathbb{Z}$*-bounded solution which satisfies Eq.* (28) *on* $\mathbb{Z}$*.*

Proof. The equivalence among (i)-(iii), (v) and (vi) is a direct consequence of Theorem 5.1. It is clear that (iii) and (iv) imply (iv) and (vii) respectively. Next, from Theorem 5.2(i), (vii) yields (viii). Moreover, from Theorem 5.2(ii), (viii) implies (i), which completes the proof. ■

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