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Spectrum and (Almost) Periodic Solutions of Functional Differential Equations

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Abstract. We give a short summary of recent development of spectral analysis of autonomous or periodic linear functional differential equations with infinite delay on the abstract phase space with uniformly fading memory property. The considered functional differential equation is transformed into an operator equation on function spaces. The admissibility theory of function spaces is reduced to the problem of the solvability of this operator equation. The spectra of bounded solutions are studied from the decomposition technique of solutions. The method of commuting operators are effectively used by combining the general result about the distribution of the spectrum of the generator of solution semigroup of autonomous equation.

1. Phase Space, Process and Fading Memory

We will give an introduction to several results about the admissibility and the Massera type theorem for functional differential equations obtained in the joint works with our colleagues; Hino, Furumochi, Murakami, Miyazaki, and Hamaya, etc.

We are interested in the following linear equation:

$$u'(t) = Au(t) + F(t, u_t) + f(t), \tag{1}$$

where A is the infinitesimal generator of a C_0 semigroup $T(t), t \geq 0$, on a Banach space E, u_t is a function segment of u at time t defined as $u_t(\theta) = u(t+\theta), -r < \theta \leq 0$, where r is independent of solutions u and $0 < r \leq \infty$. The function $f(t)$ is continuous and periodic or almost periodic.

The space to which u_t belongs is called the phase space of the equation. If r

is finite, we say that the delay is finite, and as the phase space we usually take $C([-r,0],E)$, the space of continuous functions mapping $[-r,0]$ to E with the supremum norm.

If the $r=\infty$, we say the delay is infinite. The typical example of the equation is the differential and integral equation such that

$$u'(t)=Au(t)+\int_{-\infty}^{0}Be^{-\gamma\theta}u_t(\theta)ds+f(t),$$

where B is a bounded linear operator on E and γ is a real constant. The integral in the right side exists if the segment $u_t(\theta)$ belongs to one of the following spaces; $\mathcal{C}_{\gamma+\epsilon}$, the space of continuous functions $\phi\in C((-\infty,0],E)$ such that

$$\|\phi\|_{C_{\gamma+\epsilon}}:=\sup\{e^{-(\gamma+\epsilon)\theta}|\phi(\theta)|_E:\theta\in(-\infty,0]\}$$

is finite for some $\epsilon>0$, or $\mathcal{L}_\gamma$, the space of measurable functions $\psi(\theta)$ such that

$$\|\psi\|_{L_\gamma}:=|\psi(0)|_E+\int_{-\infty}^{0}e^{-\gamma\theta}|\psi(\theta)|_E d\theta$$

is finite.

As is shown in this example, we have several possibilities in the choice of the phase space for the equation with infinite delay. However we do not use so many properties of these spaces to investigate the equation. Hale and Kato [5] presented some axioms for the phase space $\mathcal{B}$: they are satisfied by several examples. We partially refer the statement of the axioms from the book by Hale and Lunel [6].

At first, we want a solution to be continuous to the right of the initial time and we desire certain continuity properties of the solutions. $\mathcal{B}$ is a linear space with a (semi)norm $\|\cdot\|_B$ consisting of some functions $\phi:(-\infty,0]\to E$, and satisfying the following axioms:

(H) *There is a positive constant H and functions $K,M:\mathbb{R}^+\to\mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma\in R, a>0$, if $x:(-\infty,\sigma+a)\to E, x_\sigma\in\mathcal{B}$, and x is continuous on $[\sigma,\sigma+a)$, then for every $t\in[\sigma,\sigma+a)$, the following condition holds:*
(i) $x_t\in\mathcal{B}$,
(ii) $|x(t)|_E\le H\|x_t\|_B$,
(iii) $\|x_t\|_B\le K(t-\sigma)\sup\{|x(s)|_E:\sigma\le s\le t\}+M(t-\sigma)\|x_\sigma\|_B$

(H-1) *For the function x in (H), x_t is a $\mathcal{B}$-valued continuous function for $t\in[\sigma,\sigma+a)$.*

We will employ this abstract phase space $\mathcal{B}$ for the equations with infinite delay.

Assume that $F(t,\phi)$ is continuous for $(t,\phi)\in\mathbb{R}\times\mathcal{B}$ and that $F(t,\phi)$ is linear in ϕ. Under this axiom we can show the convergence of the successive approximation of the following integral equation which is an integral version of the original equation (1):

$$u(t) = \begin{cases} T(t-\sigma)\phi(0) + \int_{\sigma}^{t} T(t-s)(F(s,u_s)+f(s))ds & t \geq \sigma \\ \phi(t-\sigma) & t \leq \sigma. \end{cases} \tag{2}$$

Furthermore, the solution of this equation is determined for $t \geq \sigma$ uniquely with respect to the initial function $\phi \in \mathcal{B}$. The unique solution is denoted by $u(t,\sigma,\phi,f)$ and called the *mild solution* of Eq. (1). We deal with this mild solution hereafter. Let $u_t(\sigma,\phi,0)$ be the t-segment of $u(t,\sigma,\phi,0)$. We define solution operators $V(t,\sigma)$ for $\sigma \leq t$ by

$$V(t,\sigma)\phi = u_t(\sigma,\phi,0), \quad \phi \in \mathcal{B}.$$

They are bounded linear operators on $\mathcal{B}$ satisfying the properties of evolution process on $\mathcal{B}$, that is,

(i) For a fixed $\phi \in \mathcal{B}$, $V(t,\sigma)\phi$ is continuous for (t,σ).
(ii) $V(\sigma,\sigma) = I$(the identity on $\mathcal{B}$).
(iii) $V(t,s)V(s,\sigma) = V(t,\sigma)$ for $\sigma \leq s \leq t$.

We call the family of $V(t,\sigma)$ the *solution process* of the equation. To deal with such a family by applying the known results in functional analysis, we add the following axiom.

(H-2) *$\mathcal{B}$(or the quotient space $\mathcal{B}/\|\cdot\|_B$) is complete.*

In addition, $V(t,\sigma)$ has the property

(iv) If F is τ-periodic in t, $V(t+\tau,\sigma+\tau) = V(t,\sigma)$.

In this case the process is called τ-periodic. If F is independent of t, we can take $\tau = -\sigma$ for every $\sigma > 0$; as a result,

(v) If F is independent of t, $V(t,\sigma) = V(t-\sigma,0)$ for $t \geq \sigma$.

Hence, if we define $V(t) = V(t,0)$, then the family $\{V(t)\}, t \geq 0$, is a C_0-semigroup on $\mathcal{B}$. We call it the *solution semigroup* of the autonomous, linear equation.

The trivial equation of the autonomous, linear equation is $u'(t) = 0$; its solution semigroup is denoted by $S(t)$. Let $S_0(t)$ be the restriction of $S(t)$ to the closed subspace $\mathcal{B}_0 := \{\phi \in \mathcal{B} : \phi(0) = 0\}$. Then $\{S_0(t)\}$ is a C_0-semigroup on $\mathcal{B}_0$, and $\|S_0(t)\| \leq M(t)$.

Let C_{00} be the family of continuous functions on $(-\infty,0]$ to E with compact support. Axiom (H) implies that $C_{00} \subset \mathcal{B}$. If we want to have more functions in the space $\mathcal{B}$, we add the following axiom which guarantees the consistency of the convergence in $\mathcal{B}$ with the convergence in the compact open topology.

(H-3) *If $\{\phi^n\}$ is a Cauchy sequence in $\mathcal{B}$, and if $\{\phi^n(\theta)\}$ converges to a function $\phi(\theta)$ uniformly on any compact set of $(-\infty,0]$, then $\phi \in \mathcal{B}$ and $\lim_{n\to\infty}\|\phi^n - \phi\| = 0$.*

If the space $\mathcal{B}$ satisfies (H-1,2,3) and if $K(t) \equiv K$ (a constant),$\lim_{t\to\infty} M(t) = 0$, we call it a *uniform fading memory space*. Such a space $\mathcal{B}$ contains the space BC, the space of bounded continuous functions on $(-\infty,0]$ to E with the supremum norm, as a closed subspace, and the embedding is continuous, that is, $\|\phi\|_B \leq K\|\phi\|_{BC}$ for $\phi \in BC$. Also, it is clear that $\lim_{t\to\infty} \|S_0(t)\| = 0$. In the following sections, we assume that $\mathcal{B}$ is a uniform fading memory space.

Let us give examples of the space satisfying Axioms. For any positive continuous function g on $(-\infty, 0]$, let

$$C_g := \{\phi \in C((-\infty, 0], E) : \|\phi\|_g < \infty\},$$

where $\|\phi\|_g := \sup\{|\phi(\theta)|_E/g(\theta) : -\infty < \theta \leq 0\}$. Let

$$UC_g := \{\phi \in C_g : \phi/g \quad \text{is uniformly continuous on ;} \quad (-\infty, 0]\}$$
$$LC_g := \{\phi \in C_g : \lim_{\theta\to-\infty} \phi(\theta)/g(\theta) \quad \text{exists in} \quad E\}$$
$$LC_g^0 := \{\phi \in C_g : \lim_{\theta\to-\infty} \phi(\theta)/g(\theta) = 0\}.$$

In the case $g(\theta) = e^{\gamma\theta}$, we write $C_\gamma := C_g$. etc. Suppose that

$$G(t) := \sup\{g(\theta + t)/g(\theta) : -\infty < \theta \leq -t\}$$

is locally bounded for $t \in [0, \infty)$. This condition is satisfied if $g(\theta)$ is nonincreasing. Then UC_g, LC_g, LC_g^0 satisfies axioms (H, H-1,2,3) with

$$K(t) = \sup\{1/g(\theta) : -t \leq \theta \leq 0\}, \quad M(t) = G(t).$$

If $\gamma < 0$, $UC_\gamma, LC_\gamma, LC_\gamma^0$ are uniform fading memory spaces.

For other examples of the phase space, we refer the reader to [5, 8].

2. Autonomous Equations

For the stability theory of the autonomous, linear equation

$$u'(t) = Au(t) + F(u_t), \tag{3}$$

we consider $\sigma(G)$, the spectrum of the generator G of the solution semigroup $V(t)$ of Eq. (3). If $d := \dim E$ is finite, and if the delay r is finite, the following results are well known. $V(t)$ is compact for $t \geq r$, $\sigma(G) = P_\sigma(G)$, the point spectrum of G. A complex number λ is in $P_\sigma(G)$ if and only if the $d \times d$ matrix $\Delta(\lambda) := \lambda I - A - F(e^{\lambda\cdot} I)$ is not invertible, that is, $\det \Delta(\lambda) = 0$. $P_\sigma(G)$ is contained in a left half plane. If we set

$$\omega := \sup\{\Re\lambda : \lambda \in P_\sigma(G)\}$$

then, for any $\epsilon > 0$ there is an M_ϵ such that, for $t \geq 0$, $\|V(t)\| \leq M_\epsilon e^{(\omega+\epsilon)t}$.

If $\dim E = \infty$, $\Delta(\lambda)$ is a closed linear operator in E with some restrictions on the domain. For $\lambda \in \mathbb{C}, x \in E$, denote by $\omega(\lambda)x$ the exponential function such that

$$(\omega(\lambda)x)(\theta) = e^{\lambda\theta}x, \quad \theta \leq 0.$$

In the case of finite delay, $\omega(\lambda)x$ is in the phase space for every λ and x. However, in the case of infinite delay there is a restriction for λ such that this function is in the phase space. For example, $\omega(\lambda)x \in C_\gamma$ if and only if $\Re\lambda \geq \gamma$. If $\dim E$ is finite, the number for $\mathcal{B}$ corresponding to γ of C_γ is already given as follows in [14]. To explain it, we introduce the essential spectral radius of linear operators. For instance, $\lambda \in E_\sigma(A)$ if and only if one of the following conditions holds:

(i) The generalized eigenspace $N_\lambda := \cup_{n\geq 1} N((\lambda I - A)^n)$ is of infinite dimension.

(ii) The range $R(\lambda I - A)$ is not closed.

(iii) λ is an accumulating point of $\sigma(A)$.

The point $\lambda \notin E_\sigma(A)$ is called a normal point of A; it has nice properties in the spectral analysis of linear operators. The spectral radius of C_0 semigroup $T(t)$ is given by $r_\sigma(T(t)) = e^{wt}$ for $t \geq 0$, where $w = w(T(\cdot))$ is the growth bound of $T(t)$, that is,

$$w = \lim_{t\to\infty} (\log \|T(t)\|)/t = \inf_{t>0} (\log \|T(t)\|)/t.$$

In the similar manner the essential spectral radius is given by $r_e(T(t)) = \exp(w_e t)$, where $w_e = w_e(T(\cdot))$ is the essential growth bound of $T(t)$ given by

$$w_e := \lim_{t\to\infty} (\log \alpha(T(t)))/t = \inf_{t>0} (\log \alpha(T(t)))/t.$$

where $\alpha(T(t))$ is the seminorm induced from the Kuratowski measure of noncompactness, cf. [24, pp. 170-175].

Let β be the essential growth bound for the semigroup $S(t)$ on $\mathcal{B}$:

$$\beta := \lim_{t\to\infty} (\log \alpha(S(t)))/t = \inf_{t>0} (\log \alpha(S(t)))/t.$$

If $\dim E < \infty$, it is proved in [14] that $r_e(V(t)) = r_e(S(t)) = e^{\beta t}$ *and* $\omega(\lambda)x \in \mathcal{B}$ *for any* $x \in E$ *and* λ *such that* $\Re\lambda > \beta$. This follows from the fact that $V(t)$ is always a compact perturbation of $S(t)$ provided $\dim E < \infty$.

In the case that $\dim E = \infty$ and $r = \infty$, we have not so many information in advance whether $\omega(\lambda)x \in \mathcal{B}$ or not. Hence in the beginning we consider that

$$\Delta(\lambda)x := \lambda x - Ax - F(\omega(\lambda)x)$$

is defined only for $x \in E$ such that $x \in D(A)$ and $\omega(\lambda)x \in \mathcal{B}$. Nevertheless, we have the following result as a starting point.

Theorem 2.1. $\lambda \in P_\sigma(G)$ *if and only if* $N(\Delta(\lambda)) \neq \{0\}$, *and*

$$N(\lambda I - G) = \{\omega(\lambda)x \in \mathcal{B} : x \in N(\Delta(\lambda))\}.$$

If $\omega(\lambda)x \notin \mathcal{B}$ for $x \neq 0$, this theorem implies that $\lambda \notin P_\sigma(G)$. Thus G will have another type of spectrum. However, to proceed further, we need more information for the function $\omega(\lambda)x$. So we assume the following additional axiom.

(H-4) *There exists a constant* γ *such that, if* $\Re\lambda > \gamma$, $\omega(\lambda)x \in \mathcal{B}$ *for any* $x \in E$, *and* $\sup\{\|\omega(\lambda)x\|_B : x \in E, |x|_E \leq 1\}$ *is bounded for* $\Re\lambda > \gamma_1$ *and for some* $\gamma_1 > \gamma$.

Then we have the following result about $\rho(G)$, the resolvent set of G, in [15].

Theorem 2.2. *If $\Re\lambda \in \rho(G), \Delta(\lambda)$ is a closed linear operator having a bounded inverse operator $\Delta(\lambda)^{-1}$ defined on E. For each $x \in E$, $\Delta(\lambda)^{-1}x$ is holomorphic for $\lambda \in \rho(G)$. The set $E, D(A)$ as well as $D(\Delta(\lambda))$ are isomorphic to each other; in particular, there is an $x \neq 0$ such that $\omega(\lambda)x \in \mathcal{B}$.*

Suppose that λ is a point such that $\omega(\lambda)x \notin \mathcal{B}$ for any $x \neq 0$. Then Theorem 2.1 implies $\lambda \notin P_\sigma(G)$; Theorem 2.2 implies that $\lambda \notin \rho(G)$. Hence λ is in the continuous spectrum or in the residual spectrum of G. We have not general results for this point now. But for the spaces UC_γ and LC_γ we have the following result about $R_\sigma(G)$, the residual spectrum of G [18].

Theorem 2.3. *Let $\mathcal{B} = UC_\gamma$ or LC_γ in Eq. (3). Then the open left half plane $\{\lambda : \Re\lambda < \gamma\}$ is contained in $R_\sigma(G)$. As a result, the closed left half plane $\{\lambda : \Re\lambda \leq \gamma\}$ is contained in $E_\sigma(G)$.*

The spectral mapping theorem implies $\exp(t\sigma(G)) \subset \sigma(V(t)), \exp(tE_\sigma(G)) \subset E_\sigma(T(t))$. Hence we have the lower bound for $\alpha(V(t))$ and $\|V(t)\|$.

Corollary 2.4. *Let $\mathcal{B} = UC_\gamma$ or LC_γ. Then $e^{t\gamma} \leq \alpha(V(t)) \leq \|V(t)\|$.*

This implies the following result. If $\|V(t)\| \to 0$, then $\gamma < 0$; that is, if we want the uniform asymptotic stability of the zero solution of Eq. (3) with respect to the norm of UC_γ or LC_γ, we have to take negative γ.

3. Method of Commuting Operators

3.1. Admissibility

Let $BUC(\mathbb{R}, E)$ be the space of bounded uniformly continuous functions on $\mathbb{R}$ to E and $AP(E)$ its subspace of almost periodic functions. We consider the equation

$$u'(t) = Au(t) + F(u_t) + f(t), \tag{4}$$

where A is as in the previous section, and $F \in L(\mathcal{B}, E)$, the class of bounded linear operators on $\mathcal{B}$ into E. Furthermore, we assume that $\mathcal{B}$ is a uniform fading memory space.

Suppose that $\mathcal{M} \subset AP(E) \subset BUC(R, E)$. If for every $f \in \mathcal{M}$ Eq. (4) has a unique solution $u \in \mathcal{M}$, $\mathcal{M}$ is said to be addmissible with respect to Eq. (4).

To find the admissibility condition for $\mathcal{M}$, we rewrite Eq. (4) as an operator equation on the space $\mathcal{M}$. To do so, let us define the operators $\mathcal{D}, \mathcal{A}, \mathcal{F}$ acting on $BUC(\mathbb{R}, E)$ by

$$(\mathcal{D}u)(t) = u'(t), \quad (\mathcal{A}u)(t) = Au(t), \quad (\mathcal{F}u)(t) = F(u_t) \quad t \in \mathbb{R},$$

where $u \in D(\mathcal{D})$ if and only if $u, u' \in BUC(\mathbb{R}, E)$; $u \in D(\mathcal{A})$ if and only if $u(t) \in D(A)$ and $(Au)(\cdot) \in BUC(\mathbb{R}, E)$.

Since $BUC((-\infty,0],E)$ is embedded continuously into the uniform fading memory space $\mathcal{B}$, $\mathcal{F}u$ belongs to $BUC(\mathbb{R},E)$ for every $u \in BUC(\mathbb{R},E)$ and $\mathcal{F} \in L(BUC(\mathbb{R},E))$. Furthermore, $\mathcal{S}^h\mathcal{F}u = \mathcal{F}\mathcal{S}^h u, \quad h \in \mathbb{R}$, where $\mathcal{S}^h, h \in \mathbb{R}$, is the translation group on $BUC(\mathbb{R},E)$ defined by

$$(\mathcal{S}^h u)(t) = u(h+t) \quad h,t \in \mathbb{R}.$$

If $\mathcal{F}$ satisfies this commutativeness condition, we say that $\mathcal{F}$ is an *autonomous operator* on $BUC(\mathbb{R},E)$. In this sense, $\mathcal{A}$ is also an autonomous operator.

Let $\mathcal{M}$ be a translation invariant closed subspace of $BUC(\mathbb{R},E)$, and $\mathcal{D}_M, \mathcal{A}_M$, $\mathcal{F}_M$ be the restrictions of these operators to $\mathcal{M}$. Then Eq. (4) with $f \in \mathcal{M}$ becomes the equation

$$(\mathcal{D}_M - \mathcal{A}_M - \mathcal{F}_M)u = f.$$

Hence, if $0 \in \rho(\mathcal{D}_M - \mathcal{A}_M - \mathcal{F}_M)$, u is solved uniquely as

$$u = (\mathcal{D}_M - \mathcal{A}_M - \mathcal{F}_M)^{-1} f$$

for every $f \in \mathcal{M} \subset BUC(\mathbb{R},E)$. The problem is reduced to the study of the spectrum of the operator $\mathcal{D}_M - \mathcal{A}_M - \mathcal{F}_M$ which is a sum of $\mathcal{D}_M, \mathcal{A}_M$ and $\mathcal{F}_M$. In general, we say that two operators A and B on a Banach space with non-empty resolvent sets commute if their resolvents commute. This is equivalent to that: if $x \in D(A)$, then $R(\mu,B)x \in D(A)$ and $AR(\mu,B)x = R(\mu,B)Ax$ for some (all) $\mu \in \rho(B)$. Since $\mathcal{A}_M$ is an autonomous operator, $\mathcal{A}_M$ and $\mathcal{D}_M$ commute. From the same reason, $\mathcal{F}_M$ and $\mathcal{D}_M$ commute.

For $\alpha \in (0,\pi), R > 0$, set $\Sigma(\alpha,R) := \{z \in \mathbb{C} : |z| > R, |\arg z| \le \alpha\}$.

Theorem 3.1. [1] *Assume that A and B commute. Then*

(i) *If one of A and B is bounded, $\sigma(A+B) \subset \sigma(A) + \sigma(B)$.*

(ii) *Assume that there exist $R > 0$ and $\theta, \theta', 0 < \theta' < \theta < \pi/2$ such that*

$$\Sigma(\pi/2+\theta,R) \subset \rho(A) \quad \text{and} \quad \sup\{\|\lambda R(\lambda,A)\| : \lambda \in \Sigma(\pi/2+\theta,R)\} < \infty,$$

$$\Sigma(\pi/2-\theta',R) \subset \rho(B) \quad \text{and} \quad \sup\{\|\lambda R(\lambda,B)\| : \lambda \in \Sigma(\pi/2-\theta',R)\} < \infty.$$

Then $A+B$ is A-closable, and $\sigma(\overline{A+B}^A) \subset \sigma(A)+\sigma(B)$. In particular, if $D(A)$ is dense, then $\overline{A+B}^A = \overline{A+B}$, where $\overline{A+B}^A$ is the closed extension of $A+B$ with respect to the norm $\|x\|_A := \|R(\lambda,A)x\|$.

In our case, we assume that the original semigroup $T(t)$ is analytic. Then we can apply this theorem for $\mathcal{A}_M + \mathcal{F}_M$ and $\mathcal{D}_M$, that is

$$\sigma(\overline{\mathcal{D}_M - \mathcal{A}_M - \mathcal{F}_M}) \subset \sigma(\mathcal{D}_M) - \sigma((\mathcal{A}+\mathcal{F})_M).$$

The operator $\overline{\mathcal{D}_M - \mathcal{A}_M - \mathcal{F}_M}$ is realized as follows. Since $\mathcal{F}_M$ is bounded,

$$\overline{\mathcal{D}_M - \mathcal{A}_M - \mathcal{F}_M} = \overline{\mathcal{D}_M - \mathcal{A}_M} - \mathcal{F}_M.$$

The operator $-\overline{\mathcal{D}_M - \mathcal{A}_M}$ is a generator of the following semigroup. We define a family of operators $T^h, h \ge 0$, on $\mathcal{M}$ as

$$(T^h u)(t) = T(h)u(t-h) \quad t \in \mathbb{R}, u \in \mathcal{M}.$$

Then T^h becomes a C_0-semigroup on $\mathcal{M}$. It is called an evolutionary semigroup associated with $T(t)$. T^h is a composition of the multiplication operator $\mathcal{T}(h)$ and the translation operator $\mathcal{S}^{-h}, h \in \mathbb{R}$, that is, $T^h = \mathcal{T}(h)\mathcal{S}^{-h}$. $\mathcal{T}(h)$ is a C_0-semigroup on $\mathcal{M}$ with the generator $\mathcal{A}_M$; $\mathcal{S}^h$ is a C_0-group on $\mathcal{M}$ with the generator $\mathcal{D}_M$. Since $\mathcal{T}(h)S^{-h} = S^{-h}\mathcal{T}(h)$ for $h \geq 0$, from the fundamental result in [13, pp. 23-24], $\overline{-\mathcal{D}_M + \mathcal{A}_M}$ is the generator of T^h.

On the other hand, the generator of T^h is given as follows.

Lemma 3.2. *Let $f \in AP(E)$ and H be the infinitesimal generator of T^h. Then the following statements are equivalent:*

(i) $u \in D(H), Hu = -f$.

(ii) *u is an almost periodic solution of the equation*

$$x(t) = T(t-s)x(s) + \int_s^t T(t-r)f(r)dr \quad \forall t \geq s, \tag{5}$$

where $\forall t \geq s$ is the abbreviation of " for every $t, s \in \mathbb{R}$ such that $t \geq s$."

This is proved only by using the definition of the generator of T^h.

From this fundamental lemma, we define an operator $\mathcal{L}_M$ on $\mathcal{M}$ as follows: $u \in D(\mathcal{L}_M)$ if and only if there is a function $f \in \mathcal{M}$ such that $u \in \mathcal{M}$ is a solution of Eq. (5), and $\mathcal{L}_M u := f$ for $u \in D(\mathcal{L}_M)$.

Corollary 3.3. $\overline{\mathcal{D}_M - \mathcal{A}_M} = \mathcal{L}_M$.

Therefore we naturally arrives at the notion of mild solution. A function u on $\mathbb{R}$ to E is called a mild solution of Eq. (4) on $\mathbb{R}$ if

$$u(t) = T(t-s)u(s) + \int_s^t T(t-r)[F(u_r) + f(r)]dr, \quad \forall t \geq s.$$

Then $u \in \mathcal{M}$ is a mild solution if and only if $(\mathcal{L}_M - \mathcal{F}_M)u = f$, or $(\overline{\mathcal{D}_M - \mathcal{A}_M} - \mathcal{F}_M)u = f$.

Summarizing these results, we obtain the following theorem.

Theorem 3.4. [12] *Let A be the infinitesimal generator of an analytic semigroup, and $\mathcal{M}$ be a closed translation invariant subspace of $AP(E)$. Moreover, assume that $\sigma(\mathcal{D}_M) \cap \sigma(\mathcal{A} + \mathcal{F}) = \emptyset$. Then $\mathcal{M}$ is admissible with respect to mild solutions, that is, for every $f \in \mathcal{M}$ there is a unique mild solution $u_f \in \mathcal{M}$ of Eq.* (4).

To apply this theorem for the existence of almost periodic solutions, the space $\mathcal{M}$ is taken as follows. For $u \in BUC(\mathbb{R}, E)$, the Fourier-Carleman transform of u is defined as

$$\hat{u}(\lambda) = \begin{cases} \int\limits_0^\infty e^{-\lambda t} u(t)dt & \Re\lambda > 0 \\ -\int\limits_0^\infty e^{\lambda t} u(-t)dt & \Re\lambda < 0. \end{cases}$$

If $\hat{u}(\lambda)$ has no holomorphic extension to a neighborhood of purely imaginary point $i\xi, \xi \in \mathbb{R}$, ξ is said to be a point of Carleman spectrum $sp(u)$. For example, (i) if $u(t) = e^{i\xi t}$, $sp(u) = \{\xi\}$; (ii) if $u(t)$ is τ-periodic, $sp(u) \subset \{\xi : \tilde{u}(\xi) \neq 0\}$, where $\tilde{u}$ is the Fourier coefficient of u; (iii) if $u \in AP(E)$, $sp(f) = \overline{\sigma_b(u)}$, where $\sigma_b(u)$ is the set of ξ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-i\xi t} u(t)dt \neq 0.$$

In turn, the Fourier-Carleman spectrum of u coincides with its Arveson spectrum

$$isp(u) = \sigma(\mathcal{D}_{M(u)}),$$

where $M(u)$ is the closed subspace of $BUC(\mathbb{R}, E)$ generated by the family $\{\mathcal{S}^h u : h \in \mathbb{R}\}$. For a closed subset $\Lambda \subset \mathbb{R}$, set $\Lambda(E) := \{u \in BUC(\mathbb{R}, E) : sp(f) \subset \Lambda\}$. It is a closed subspace which is translation invariant, and

$$\sigma(\mathcal{D}_{\Lambda(E)}) = i\Lambda.$$

Hence we have the following result.

Theorem 3.5. *Let A be the infinitesimal generator of an analytic C_0-semigroup, $\Lambda \subset \mathbb{R}$ a closed subset and $\mathcal{M}$ a closed translation invariant subspace of $AP(E)$. Moreover, assume that $i\Lambda \cap \sigma(\mathcal{A}_M + \mathcal{F}_M) = \emptyset$. Then $\Lambda(E) \cap \mathcal{M}$ is admissible with respect to mild solutions, that is, for every $f \in \Lambda(E) \cap \mathcal{M}$ there is a unique mild solution $u_f \in \Lambda(E) \cap \mathcal{M}$ of Eq.* (4).

3.2. Decomposition Theorem

Suppose that $u \in BUC(\mathbb{R}, E)$ is a mild solution of Eq. (4). We will study $sp(u)$ in general, and deduce Massera type theorems. Since $\mathcal{L}u - \mathcal{F}u = f$, we have

$$sp(f) = sp(\mathcal{L}u - \mathcal{F}u) = sp\left(\lim_{h\to 0+} \frac{T^h u - u}{h} - \mathcal{F}u\right) \subset sp(u).$$

On the other hand, if u is a mild solution, then, for all $t \in \mathbb{R}$,

$$u(t) - u(0) = A\int_0^t u(s)ds + \int_0^t [Fu_s + f(s)]ds.$$

Taking the Fourier-Carleman transform we have

$$\Delta(\lambda)\hat{u}(\lambda) = u(0) + F(M_\lambda u_0) + \hat{f}(\lambda), \tag{6}$$

where $M_\lambda \in L(\mathcal{B})$ is defined as

$$(M_\lambda \phi)(\theta) = \int_\theta^0 e^{\lambda(\theta - s)} \phi(s) ds, \quad \phi \in \mathcal{B}.$$

We assume that $\mathcal{B}$ is a uniform fading memory space. Then $M_\lambda \in L(\mathcal{B})$ and it is holomorphic for λ in a right half plane containing the imaginary axis [15]. Recall that $\Delta(\lambda)^{-1}$ exists in $L(E)$ and holomorphic for $\lambda \in \rho(G)$, where G is the generator of the solution semigroup $V(t)$. Thus if $i\xi \in i\rho(f) \cap \rho(G), \xi \in \mathbb{R}, \hat{u}(\lambda)$ has a holomorphic extension around $i\xi$, where $\rho(f) = \mathbb{R} \setminus sp(f)$. Hence we have the following result. Let $\sigma(\Delta) = \{\lambda : \Delta(\lambda) \text{ has not an inverse operator in } L(E)\}$. Then $\sigma(\Delta) \subset \sigma(G)$. Set

$$\sigma_i(\Delta) = \{\xi \in \mathbb{R} : i\xi \in \sigma(\Delta)\}, \quad \sigma_i(G) = \{\xi \in \mathbb{R} : i\xi \in \sigma(G)\}.$$

Lemma 3.6. *If $u \in BUC(\mathbb{R}, E)$ is a mild solution of Eq.* (4),

$$sp(f) \subset sp(u) \subset sp(f) \cup \sigma_i(\Delta) \subset sp(f) \cup \sigma_i(G).$$

From the mild solution $u \in BUC(\mathbb{R}, E)$, we can take out a mild solution v such that $\overline{e^{isp(v)}} \subset \overline{e^{isp(f)}}$ by the following decomposition technique. Let us consider the subspace

$$\mathcal{M} = \{v \in BUC(\mathbb{R}, E) : \sigma(v) := \overline{e^{isp(v)}} \subset S_1 \cup S_2\},$$

where S_1 and S_2 are disjoint closed subsets of the unit circle in the complex plane. Let $\mathcal{M}_i := \{v \in \mathcal{M} : \sigma(v) \subset S_i\}, i = 1, 2$.

Lemma 3.7. [19] *$\mathcal{M}$ is decomposed as $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Moreover, any autonomous operator leaves invariant $\mathcal{M}$ as well as $\mathcal{M}_i, i = 1, 2$.*

Let γ be a contour enclosing S_1 and disjoint from S_2. Define a projection P_v^1 on $\mathcal{M}(v)$ by

$$P_v^1 := \frac{1}{2\pi i} \int_\gamma R(\lambda, S(1)|_{M(v)}) d\lambda.$$

Then $v_1 := P_v^1 v$ is the component for the decomposition $v = v_1 + v_2, v_i \in \mathcal{M}_i, i = 1, 2$.

Theorem 3.8. [4] *Let the following condition be satisfied: $\overline{e^{i\sigma_i(\Delta)}} \setminus \overline{e^{isp(f)}}$ is closed and Eq.* (4) *has a bounded uniformly continuous mild solution on the whole line $\mathbb{R}$. Then there exists a bounded uniformly continuous mild solution w of Eq.* (4) *such that $\overline{e^{isp(w)}} \subset \overline{e^{isp(f)}}$. Moreover, if $\overline{e^{i\sigma_i(\Delta)}} \cap \overline{e^{isp(f)}} = \emptyset$, such a solution w is unique in the sense that if there exists a mild solution v to Eq.* (4) *such that $\overline{e^{isp(v)}} \subset \overline{e^{isp(f)}}$, then $v = w$.*

3.3. Floquet Theory and Decomposition

We consider the periodic equation

$$u'(t) = Au(t) + F(t)u_t + f(t), \tag{7}$$

where $F(t)$ and $f(t)$ are τ-periodic. We assume that $\mathcal{B}$ is a uniform fading memory space. Let $V(t,s)$ be the solution operator for the homogeneous equation. Recall that $u(t,\sigma,\phi,f)$ denotes the mild solution of Eq. (7) such that $u_\sigma = \phi \in \mathcal{B}$.

Suppose that v is a bounded mild solution on $\mathbb{R}$. From the superposition principle, it follows that

$$v(t) = u(t,s,v_s,f) = u(t,s,v_s,0) + u(t,s,0,f).$$

Let $w(t,s) = u(t,s,0,f)$. Then $v_t = V(t,s)v_s + w_t(\cdot,s)$ and $w(t) := w(t,s)$ satisfies the equation

$$w(t) = \begin{cases} \int_t^s T(t-r)F(r)[w_r + f(r)]dr & t \geq s \\ 0 & t < s \end{cases}$$

This equation for w has a unique solution which is independent of u. It is easy to see $w(t+\tau, s+\tau) = w(t,s)$. Hence the function $g(t) := w(t, t-\tau)$ is a τ-periodic function, and

$$v_t = V(t, t-\tau)v_{t-\tau} + g_t \tag{8}$$

We call the operator $P(t) := V(t, t-\tau)$ the *monodromy operator* of the equation, which is τ-periodic. As a standard hypothesis we assume that the monodromy operator is compact. For example, this hypothesis holds provided that $T(t)$ is a C_0-compact semigroup, and that $\mathcal{B}$ is a uniform fading memory space. For other sufficient conditions for this property, we refer the reader to [23, Theorem 3.5]. It is known that $\sigma(P(t))\backslash\{0\}$ is independent of t; see ([7, Lemma 7.3.2, p. 197]).

From the Floquet theory, Eq. (8) is transformed into the system of equations on the phase space such that

$$\begin{aligned} v_1(t) &= Bv_1(t-\tau) + F_1(t) \\ v_2(t) &= B(t)v_2(t-\tau) + F_2(t), \end{aligned}$$

where $v_1(t)$ lies in a finite dimensional space,

$$\sigma(B) = \{\lambda \in \sigma(P) : |\lambda| \geq 1\}$$

and $\|B(t)\| < Me^{-\tau\omega}, t \in \mathbb{R}, \omega > 0$, $F_1(t), F_2(t)$ are τ-periodic.

From the contraction principle, if $Me^{-\tau\omega} < 1$, the equation for $v_2(t)$ has a unique bounded continuous solution, and it is τ-periodic. For the equation to $v_1(t)$, we have the following general result.

Lemma 3.9. *Let f be τ-periodic and continuous, and let u be a bounded uniformly continuous solution to the functional equation $u(t) = Bu(t-\tau) + f(t)$. Then $e^{i\tau sp(u)} \subset \{1\} \cup \{\sigma(B) \cap S_1\}$.*

Combining these results and the decomposition technique, we have the following result. cf. [17].

Theorem 3.10. *Assume that $T(t)$ is a compact C_0-semigroup on E. Let u be a bounded uniformly continuous mild solution to Eq.* (7). *Then u can be represented in the form*

$$u(t) = u^0(t) + \sum_{k=1}^{M} e^{i\lambda_k t} u^k(t),$$

where u^0 is τ-periodic mild solution to Eq. (7) *and $u(t) - u^0(t)$ is a quasi periodic solution to its homogeneous equation. The Fourier coefficients $a_k, k \in \mathbb{Z}$, of u_0 are determined by the formula*

$$a_k = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-(2k\pi i/\tau)t} u(t)dt, \qquad k \in \mathbb{Z}.$$

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