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On the Complex WKB Analysis for a Schrödinger Equation with a General Three-Segment Characteristic Polygon

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Abstract. Asymptotic property of solutions of a Schrödinger equation with a turning point is known to be characterized literally by their characteristic polygon (Iwano-Sibuya [4]).

The simplest case is the Airy equation that has a one-segment characteristic polygon with a turning point and without any secondary turning point. The second simplest is a differential equation with a two-segment characteristic polygon studied in Nakano [5], Nakano et al. [8].

The third simplest is treated in Roos [12] which studies one with a three-segment characteristic polygon connecting four points. Here, we study one with a general three-segment characteristic polygon connecting much more than four points.

1. Introduction

1.1. We study the following Schrödinger equation containing a small parameter ε:

$$\varepsilon^{2h}\,\frac{d^2y}{dx^2} = a(x,\varepsilon)\,y \quad (x, y \in \mathbb{C};\ D : 0 \le |x| \le x_0;\ \ 0 < \varepsilon \le \varepsilon_0), \tag{1.1}$$

$$a(x,\varepsilon) := \sum_{j=0}^{k} a_j \varepsilon^j x^{h+m+k-2j} + \sum_{j=k+1}^{h} a_j \varepsilon^j\, x^{h+m-j}, \quad a_j \in \mathbb{C}\ \ (\forall a_j \neq 0), \tag{1.2}$$

where $x_0,\ \varepsilon_0$ are small constants, and $h,\ k$ and m are integers such that

$$h = 3, 4, 5, \cdots;\ \ 1 \le k \le h-1;\ \ 0 \le m \le h-3. \tag{1.3}$$

The zero $x = 0$ of $a(x,0)$ $(= a_0\, x^{h+m+k})$ is called a *turning point* of (1.1). Our aim is to get asymptotic property of solutions of (1.1) in D by using the concept of the characteristic polygon for (1.1) and by applying what we call the stretching-matching method (Nakano [5 - 7], Nishimoto [9], Wasow [14]).

We define the following points on the (X,Y)-plane according to the indices of ε and x of $a(x,\varepsilon)$

$$P_j := \begin{cases} \left(\dfrac{j}{2},\ \dfrac{h+m+k-2j}{2}\right) & (j = 0,1,2,\cdots,k), \\ \left(\dfrac{j}{2},\ \dfrac{h+m-j}{2}\right) & (j = k+1,k+2,\cdots,h), \end{cases} \qquad R := (h,-1). \tag{1.4}$$

The *characteristic polygon* for (1.1) is, by definition (Iwano-Sibuya [4]), given by a polygon with segments connecting $P_0, P_1, P_2, \cdots, P_h$ and R in order. P_0 is on the Y-axis and it corresponds to the first term in $a(x,\varepsilon)$. P_k and P_h correspond respectively to the $(k+1)$-st term, i.e., the middle term of $a(x,\varepsilon)$ and the $(h+1)$-st term, i.e., the last term of $a(x,\varepsilon)$. This characteristic polygon is convex downward and snaps at P_k and P_h. On *the first segment* connecting P_0 and P_k are $k+1$ points $P_0, P_1, \cdots, P_k$, and on *the second segment* connecting P_k and P_h are $h-k+1$ points $P_k, P_{k+1}, \cdots, P_h$. P_h and R are on *the third segment.*

Fig. 1.2

Fig. 1.1

The Roos' equation (Roos [12]) is

$$\varepsilon^4\, \frac{d^2 y}{dx^2} = (x^5 + \varepsilon x^2 + \varepsilon^2)\, y, \tag{1.5}$$

whose characteristic polygon consists of three segments connecting four points $P_0 := (0, 5/2),\ P_1 := (1/2, 1),\ P_2 := (1, 0)$ and $R := (2, -1)$.

1.2. The contents of this paper are as follows. In Sec. 2, we reduce (1.1) asymptotically to the simpler differential equations in appropriate subdomains of D (Theorem 2.1). In Sec. 3, the WKB approximations of the reduced differential equations are got (Theorem 3.2). In Sec. 4 we construct the Stokes curve configurations for the reduced differential equations, and in Sec. 5 we define the canonical domains for the reduced differential equations (Fig. 4.1 - 4.3). In Sec. 6, by using WKB approximations all the solutions of the reduced differential equations are matched, i.e., they are connected linearly each other, and also we calculate the matching matrices (Theorem 6.1 - 6.2).

2. The Asymptotic Reductions of (1.1)

2.1. Each term of $a(x, \varepsilon)$ can be regarded as "the asymptotically dominant term" in some circular subdomain of D. First, we can consider the first term $a_0\, x^{h+m+k}$ to be the dominant term by writing $a(x, \varepsilon)$ as follows

$$a(x, \varepsilon) = x^{h+m+k}\Big(a_0 + \sum_{j=1}^{k} a_j(\varepsilon x^{-2})^j + \sum_{j=k+1}^{h} a_j(\varepsilon x^{-2})^j \cdot x^{j-k}\Big).$$

In fact, when $\varepsilon\, x^{-2} \to 0$, two Σ's tend to zero. Then (1.1) can be asymptotically reduced to

$$\varepsilon^{2h}\frac{d^2y}{dx^2} = a_0 x^{h+m+k} y \quad \text{for} \quad (x, \varepsilon) : K_1\varepsilon^{1/2} \leq |x| \leq x_0, \tag{2.1}$$

where K_1 is a sufficiently large constant.

2.2. We rewrite $a(x, \varepsilon)$ to consider the $(k+1)$-st term to be the dominant term

$$a(x, \varepsilon) = \varepsilon^k x^{h+m-k}\Big(\sum_{j=0}^{k-1} a_j(\varepsilon^{-1}x^2)^{k-j} + a_k\Big) + x^{h+m} \sum_{j=k+1}^{h} a_j\, (\varepsilon\, x^{-1})^j,$$

then the first Σ tends to zero when $\varepsilon^{-1}\, x^2 \to 0$ and the second Σ also tends to zero when $\varepsilon\, x^{-1} \to 0$. Thus (1.1) is asymptotically reduced to

$$\varepsilon^{2h-k}\frac{d^2y}{dx^2} = a_k x^{h+m-k} y \quad \text{for} \quad (x, \varepsilon) : K_2\, \varepsilon \leq |x| \leq k_1\, \varepsilon^{1/2}, \tag{2.2}$$

where K_2 is a sufficiently large constant and k_1 is a sufficiently small constant.

2.3. In order to consider the last term of $a(x, \varepsilon)$ to be the dominant term, we represent $a(x, \varepsilon)$ in

$$a(x, \varepsilon) = \varepsilon^h\, x^m\Big(\sum_{j=0}^{k-1} a_j\, (\varepsilon^{-1}x)^{h-j} \cdot x^{k-j} + \sum_{j=k+1}^{h-1} a_j\, (\varepsilon^{-1}\, x)^{h-j} + a_h\Big).$$

Then two Σ's tend to zero when $\varepsilon^{-1}\,x \to 0$. Thus (1.1) can be asymptotically reduced to

$$\varepsilon^h \frac{d^2\,y}{d\,x^2} = a_h\,x^m\,y \quad \text{for} \quad (x,\varepsilon) : 0 \le |x| \le k_2\,\varepsilon, \tag{2.3}$$

where k_2 is a sufficiently small constant. (2.3) is sometimes called a generalized Airy equation.

We can know all information near $x = 0$, the turning point of (1.1), from the solution of (2.3) because it can be represented by the Bessel functions (Nakano [5]). However, we cannot match or connect linearly two sets of solutions of (2.2) and (2.3) since their subdomains do not possess common interior points. Then, in order to know information of solutions for x belonging to an intermediate domain between (2.2) and (2.3), we need an intermediate equation ((2.5) below) in a region between these two subdomains. By the same reason, we need another intermediate equation ((2.4) below) between subdomains for (2.1) and (2.2).

2.4. In order to get the first intermediate equation in the circular subdomain $k_1\varepsilon^{1/2} \le |x| \le K_1\varepsilon^{1/2}$ between domains of (2.1) and (2.2), we apply the so-called *stretching transformation*

$$x := t\varepsilon^{1/2} \quad (k_1 \le |t| \le K_1)$$

to (1.1). Then $a(x,\varepsilon)$ can be represented in

$$a(x,\varepsilon) \equiv a(t\varepsilon^{1/2},\varepsilon) = \varepsilon^{(h+m+k)/2}\Big(\sum_{j=0}^{k} a_j\,t^{h+m+k-2j} + \sum_{j=k+1}^{h} a_j\,\varepsilon^{(j-k)/2}\,t^{h+m-j}\Big).$$

All the terms in the second Σ become small for t $(k_1 \le |t| \le K_1)$ when $\varepsilon \to 0$. Thus, (1.1) can be asymptotically reduced to

$$\varepsilon^{(3h-m-k-2)/2}\,\frac{d^2y}{dt^2} = \Big(\sum_{j=0}^{k} a_j t^{h+m+k-2j}\Big)y \;\text{ for }\; t : k_1 \le |t| \le K_1 \quad (t := x\,\varepsilon^{-1/2}). \tag{2.4}$$

The zeros of the coefficient of (2.4) are turning points of (2.4) itself, and they are called *secondary turning points* of (1.1). Nakano et al. [8] studied first the differential equation with secondary turning points.

The stretching transformation shows that the origin $x = 0$, which is a turning point of (1.1), corresponds to $t = 0$. Then, if we can obtain a solution of (2.4) for $|t| \ge 0$ we can know every information near $x = 0$, but appropriate special functions of (2.4) are not known (cf. Sibuya [13]). Then we need the stretching-matching method.

2.5. By applying another stretching transformation

$$x := t\,\varepsilon \quad (k_2 \le |t| \le K_2)$$

to (1.1), we represent $a(x,\varepsilon)$ as follows:

$$a(x,\varepsilon) \equiv a(t\,\varepsilon,\varepsilon) = \varepsilon^{h+m} \Big(\sum_{j=0}^{k-1} a_j\, \varepsilon^{k-j}\, t^{h+m+k-2j} + \sum_{j=k}^{h} a_j\, t^{h+m-j} \Big).$$

Then the first Σ is small for $t\,(k_2 \le |t| \le K_2)$ when $\varepsilon \to 0$. Thus (1.1) can be asymptotically reduced to

$$\varepsilon^{h-m-2} \frac{d^2 y}{dt^2} = \Big(\sum_{j=k}^{h} a_j\, t^{h+m-j} \Big) y \quad \text{for} \quad t : k_2 \le |t| \le K_2 \quad (t := x\,\varepsilon^{-1}). \tag{2.5}$$

2.6. Summing up the above results, we get the

Theorem 2.1. *The differential equation* (1.1) *can be asymptotically reduced to* (2.j)*'s* ($j = 1, 2, \cdots, 5$)*, which correspond to the points on the characteristic polygon as follows:*
(2.1) *corresponds to* P_0*,* (2.2) *corresponds to* P_k*,* (2.3) *corresponds to* P_h*,* (2.4) *corresponds to* $P_0, P_1, \cdots, P_k$ *on the first segment and* (2.5) *corresponds to* $P_k, P_{k+1}, \cdots, P_h$ *on the second segment. There is no reduced differential equation corresponding to the third segment.*

3. The WKB Approximations

3.1. All the reduced differential equations (2.j)'s possess the common form of singular perturbation

$$\varepsilon^2 \frac{d^2 y}{dx^2} = Q(x) y \quad (0 \le |x| < \infty, \;\; 0 < \varepsilon \le \varepsilon_1), \tag{3.1}$$

where $Q(x)$ is a polynomial. The point at infinity is an irregular singular point for (3.1). Zeros of $Q(x)$ are called *turning points* of (3.1). The *WKB approximations* $\tilde{y}^{\pm}(x,\varepsilon)$ of (3.1) are defined by

$$\tilde{y}^{\pm}(x,\varepsilon) := \frac{C^{\pm}}{\sqrt{Q(x)}} \exp\Big[\pm \frac{1}{\varepsilon}\, \xi(a,x) \Big] \quad (C_{\pm} = \text{const.}), \tag{3.2}$$

where

$$\xi(a,x) := \int_a^x \sqrt{Q(x)}\, dx. \tag{3.3}$$

The curve in the complex x-plane defined by the equation

$$\Re\, \xi(a,x) = 0 \quad (Q(a) = 0) \tag{3.4}$$

is called a *Stokes curve* for (3.1), and the curve defined by the equation

$$\Im\, \xi(a,x) = 0 \quad (Q(a) = 0) \tag{3.5}$$

is called an *anti-Stokes curve* for (3.1).

The function $\xi := \xi(a, x)$ is a conformal mapping from the x-plane to the ξ-plane. It is not conformal at the turning points because $d\xi/dx = \sqrt{Q(x)} = 0$ only at zeros of $Q(x)$.

Lemma 3.1. (Fedoryuk) *There exist an x-domain $\mathcal{D}^{can}$ and true solutions $y^{\pm}(x, \varepsilon)$ of* (3.1) *such that*

$$y^{\pm}(x,\varepsilon) \ \sim \ \tilde{y}^{\pm}(x,\varepsilon) \quad as \ \begin{cases} x \to \infty \ \ in \ \ \mathcal{D}^{can}, & 0 < \varepsilon \le \varepsilon_1, \\ \varepsilon \to 0, \quad x \in \mathcal{D}^{can}. \end{cases} \tag{3.6}$$

If $\mathcal{D}^{can}$ is bounded from $x = \infty$, the first relation is naturally vacant.

The domain $\mathcal{D}^{can}$ in Lemma 3.1 is bounded by several Stokes curves and it is called a *canonical domain* for (3.1) if $\mathcal{D}^{can}$ is maximal (Sec. 5). Thus the canonical domain is the maximal existence domain of the true solutions $y^{\pm}(x, \varepsilon)$ having the WKB approximations $\tilde{y}^{\pm}(x, \varepsilon)$ as (the leading term of) their asymptotic expansion. The relation (3.6) is called the *double asymptotic property* of the WKB approximations (Fedoryuk [2]). Examples of canonical domains for various differential equations are given in Fedoryuk [2], Nakano [5 - 7] and Wasow [15].

3.2. Before getting WKB approximations for $(2.j)$'s, we should specify the coefficients a_j's in $a(x, \varepsilon)$. For arbitrary values of a_j's, clearly we cannot determine any turning points and any Stokes curves, then any canonical domains for especially (2.4) and (2.5) cannot be constructed.

Thus, we specify a_j $(j = 0, 1, 2, \cdots, k)$ such that

$$\sum_{j=0}^{k} a_j t^{h+m+k-2j} \equiv t^{h+m-k}(t^2+1)^k, \tag{3.7}$$

from which we can know $a_j = {}_kC_j$ $\ (j = 0, 1, 2, \cdots, k)$, especially $a_0 = a_k = 1$, and other a_j's such that

$$\sum_{j=k}^{h} a_j \, t^{h+m-j} \equiv t^m \, (t+1)^{h-k}, \tag{3.8}$$

from which we can know $a_j = {}_{h-k}C_{j-k}$ $\ (j = k+1, k+2, \cdots, h)$, especially $a_h = 1$. Thus we can get

Theorem 3.1. *Let a_j's satisfy* (3.7) *and* (3.8). *Then* (1.1) *can be reduced asymptotically to*

$$\varepsilon^{2h} \, \frac{d^2y}{dx^2} = x^{h+m+k} y \quad in \quad D_1 : K_1 \, \varepsilon^{1/2} \le |x| \le x_0 \cdots [1] \tag{3.9}$$

$$\varepsilon^{(3h-m-k-2)/2} \, \frac{d^2y}{dt^2} = t^{h+m-k} \, (t^2+1)^k \, y \quad in \ \ D_2 : \ \ k_1 \le |t| \le K_1 \quad (t := x\,\varepsilon^{-1/2}) \cdots [2] \tag{3.10}$$

$$\varepsilon^{2h-k}\,\frac{d^2y}{dx^2} = x^{h+m-k}\,y \quad in\ \ D_3:\ \ K_2\,\varepsilon \le |x| \le k_1\,\varepsilon^{1/2}\cdots[3] \tag{3.11}$$

$$\varepsilon^{h-m-2}\,\frac{d^2y}{dt^2} = t^m\,(t+1)^{h-k}\,y \quad in\ \ D_4:\ \ k_2 \le |t| \le K_2\ \ (t := x\,\varepsilon^{-1})\cdots[4] \tag{3.12}$$

and

$$\varepsilon^{h}\,\frac{d^2y}{dx^2} = x^m\,y \quad in \quad D_5: 0 \le |x| \le k_2\,\varepsilon\cdots[5] \tag{3.13}$$

We denote $y_j^{\pm}(\ \cdot\ ,\varepsilon)$ *for the true solutions of* $[j]$ *and* $\tilde{y}_j^{\pm}(\ \cdot\ ,\varepsilon)$ *for the WKB approximations of* $[j]$.

The points $t = 0,\ \pm i,\ -1$ are secondary turning points of (1.1). Here, we must notice that two t-domains D_2 and D_4 in Theorem 3.1 are bounded. But they should be unbounded such as

$$D_j^{\infty}:\ 0 < |t| < \infty \quad (j = 2, 4) \tag{3.14}$$

if we want to match, i.e., connect linearly, two sets of solutions of neighboring differential equations. When we match two sets of solutions $y_1^{\pm}(x,\varepsilon)$ and $y_2^{\pm}(t,\varepsilon)$, the domain D_2 should be expanded to an unbounded domain because D_1 and D_2 must have common interior points for a small ε. Furthermore D_2 and D_3 must have common interior points in order to match two sets of solutions $y_2^{\pm}(t,\varepsilon)$ and $y_3^{\pm}(x,\varepsilon)$. Similarly, D_4 must be extended to an unbounded domain in order to match two sets of solutions $y_3^{\pm}(x,\varepsilon)$ and $y_4^{\pm}(t,\varepsilon)$, and to match two sets of solutions $y_4^{\pm}(t,\varepsilon)$ and $y_5^{\pm}(x,\varepsilon)$.

3.3. Due to (3.1) and (3.2), we can get all the WKB approximations $\tilde{y}_j^{\pm}$'s for $[j]$.

Theorem 3.2. *The differential equation* $[j]$ $(j = 1, 2, \cdots, 5)$ *possesses the WKB approximations:*

$$\tilde{y}_1^{\pm}(x,\varepsilon) := x^{-(h+m+k)/4}\ \exp\Big[\pm\,\frac{2}{h+m+k+2}\,\frac{x^{(h+m+k+2)/2}}{\varepsilon^h}\Big], \tag*{(3.9)$^{\sim}$}$$

$$\tilde{y}_2^{\pm}(t,\varepsilon) :=$$
$$\{t^{h+m-k}(t^2+1)^k\}^{-1/4}\ \exp\Big[\pm\,\frac{1}{\varepsilon^{(3h-m-k-2)/4}}\int^t \{\ t^{h+m-k}(t^2+1)^k\ \}^{1/2}\ dt\Big], \tag*{(3.10)$^{\sim}$}$$

$$\tilde{y}_3^{\pm}(x,\varepsilon) :=\ x^{-(h+m-k)/4}\ \exp\Big[\pm\ \frac{2}{h+m-k+2}\,\frac{x^{(h+m-k+2)/2}}{\varepsilon^{h-k/2}}\Big], \tag*{(3.11)$^{\sim}$}$$

$$\tilde{y}_4^{\pm}(t,\varepsilon) := \{t^m\ (t+1)^{h-k}\}^{-1/4}\exp\Big[\pm\,\frac{1}{\varepsilon^{(h-2-m)/2}}\int^t\Big\{t^m\ (t+1)^{h-k}\Big\}^{1/2}dt\Big], \tag*{(3.12)$^{\sim}$}$$

and

$$\tilde{y}_5^{\pm}(x,\varepsilon) := x^{-m/4}\ \exp\Big[\pm\ \frac{2}{m+2}\ \frac{x^{(m+2)/2}}{\varepsilon^{h/2}}\Big]. \qquad (3.13)^{\sim}$$

Let $\mathcal{D}_j^{can}$*'s* $(j = 1, 2, \cdots, 5)$ *be canonical domains for* $[j]$*. Then* $\tilde{y}_j^{\pm}(x,\varepsilon)$ *are* (*the leading term of*) *the asymptotic expansion of* $y_j^{\pm}(x,\varepsilon)$ *such that*

$$y_j^{\pm}(x,\varepsilon) \sim \tilde{y}_j^{\pm}(x,\varepsilon) \quad as\ \ \varepsilon \to 0,\ \ x \in \mathcal{D}_j^{can} \quad (j = 1, 3, 5). \qquad (3.15)$$

And $\tilde{y}_j^{\pm}(t,\varepsilon)$ $(j = 2, 4)$ *are* (*the leading term of*) *the asymptotic expansion of* $y_j^{\pm}(t,\varepsilon)$ *such that*

$$y_j^{\pm}(t,\varepsilon)\ \sim\ \tilde{y}_j^{\pm}(t,\varepsilon)\quad as\ \begin{cases} t \to \infty\ \ in\ \ \mathcal{D}_j^{can}, & 0 < \varepsilon \le \varepsilon_0 \\ \varepsilon \to 0, \quad t \in \mathcal{D}_j^{can} & \end{cases} \quad (j = 2, 4). \qquad (3.16)$$

We notice that $\mathcal{D}_j^{can}$ $(j = 1, 3, 5)$ is bounded because D_j $(j = 1, 3, 5)$ is bounded, and $\mathcal{D}_j^{can}$ $(j = 2, 4)$ is the unbounded subset of D_j^{∞} $(j = 2, 4)$. All the canonical domains will be constructed or defined in Sec. 5.

4. Stokes Curve Configurations

4.1. In order to get $\mathcal{D}_j^{can}$'s we need to know topology of Stokes curves. The general properties of Stokes curve configuration for (3.1) are well known as shown below (Evgrafov-Fedoryuk [1], Fedoryuk [2] and Nakano [7]). See Hukuhara [3] or Paris-Wood [11] for local Stokes curves.

Lemma 4.1. *The Stokes and anti-Stokes curves for* (3.1) *possess the following properties*:

- (i) *If* $x = a$ *is a turning point for* (3.1) *such as* $Q(x) = O\big((x-a)^l\big)$ $(x \to a)$, *then* $l+2$ *Stokes curves and* $l+2$ *anti-Stokes curves emerge from* $x = a$.
- (ii) *If* $x = \infty$ *is an irregular singular point for* (3.1) *such as* $Q(x) = O(x^l)$ $(x \to \infty)$, *then* $l+2$ *Stokes and* $l+2$ *anti-Stokes curves emerge from* (*or tend to*) $x = \infty$.
- (iii) *Any Stokes curve cannot cross other Stokes curves except for at turning points or at* $x = \infty$.
- (iv) *Any Stokes or any anti-Stokes curve emerging from turning points tends to another turning points or to* $x = \infty$.
- (v) *Any Stokes or any anti-Stokes curve does not cross itself.*
- (vi) *There are no* (*sums of several*) *Stokes or anti-Stokes curves homotopic to a circle.*

Proof.
(i) Let

$$Q(x) = \alpha\,(x-a)^l + (\text{higher order terms})\ \ (x \sim a)$$

with a complex constant $\alpha\,(\neq 0)$. Then, we can get different $l+2$ arguments near $x = a$ after the following computation:

$$\Re\,\xi(a,x) = \Re\Big(\frac{2\sqrt{\alpha}}{l+2}\,(x-a)^{(l+2)/2} + \cdots\Big) = 0.$$

$$\therefore\ \cos\Big[\arg\Big(\frac{2\sqrt{\alpha}}{l+2}\,(x-a)^{(l+2)/2}\Big)\Big] \longrightarrow 0 \quad (x \to a).$$

$$\therefore \arg\Big(\sqrt{\alpha}\,(x-a)^{(l+2)/2}\Big) = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\cdots.$$

Similarly, we get different $l+2$ arguments near $x = a$ for anti-Stokes curves.
(ii) Let

$$Q(x) = \beta\, x^l + (\text{lower order terms}) \quad (x \sim \infty)$$

with a complex constant $\beta\,(\neq 0)$. In the same way as (i), we can get different $l+2$ arguments near $x = \infty$ from

$$\Re\int^x \sqrt{Q(x)}\,dx \;=\; \Re\,\Big(\frac{2\sqrt{\beta}}{l+2}\,x^{(l+2)/2} \;+\; \cdots\Big).$$

For other properties, see the references cited above. ■

4.2. We will first analyze a Stokes curve configuration for (3.12). Put, instead of $Q(x)$ in (3.1),

$$Q_4(t) := t^p\,(t+1)^q \quad (p := m = 0,1,2,\cdots;\ \ q := h-k \in \mathbb{N}), \tag{4.1}$$

$$\xi_4(a,t) := \int_a^t \sqrt{Q_4(t)}\,dt \quad (a = 0,\ -1). \tag{4.2}$$

From Lemma 4.1 (i), we see that $q+2$ Stokes curves emerge from the turning point $t=-1$ with arguments $(p\pm1)\pi/(q+2),\ (p\pm3)\pi/(q+2),\ (p\pm5)\pi/(q+2),\ \cdots$, and $p+2$ Stokes curves emerge from the turning point $t=0$ with arguments $\pm\pi/(p+2),\ \pm3\pi/(p+2),\ \pm5\pi/(p+2),\ \cdots$.

Since $Q_4(t) \sim t^{p+q}\ (x\to\infty)$, $p+q+2$ Stokes curves tend to (or emerge from) $x=\infty$ with arguments $\pm\pi/(p+q+2),\ \pm3\pi/(p+q+2),\ \pm5\pi/(p+q+2),\ \cdots$. Anti-Stokes curves emerge from the turning points and $x=\infty$ with middle arguments between neighboring arguments for Stokes curves.

In order to know the global properties of a Stokes curve configuration, we check whether an interval on the real axis of the t-plane can become a Stokes curve or an anti-Stokes curve. We study the following five cases.

(i) $p =$ even, $q =$ even.
Since we see that $Q_4(t) \geq 0$ for $t\in\mathbb{R}$, the integral (4.2) takes only real values for $t\in\mathbb{R}$, i.e., $\Im\,\xi_4(a,t) = 0$ for $t\in\mathbb{R}$, and so the whole real axis of the t-plane is an anti-Stokes curve.

(ii) $p =$ even, $q =$ odd.
Since we see that $Q_4(t) < 0$ for $t < -1$, then the integral (4.2) takes only pure imaginary values, i.e., $\Re\,\xi_4(-1,t) = 0$ for $t < -1$, and the half real axis $(t \leq -1)$ of the t-plane is a Stokes curve.

(iii) $p =$ odd, $q =$ even.

The half real axis ($t \leq 0$) is a Stokes curve because $Q_4(t) < 0$ for $t < 0$ and so $\Re\, \xi_4(0,t) = 0$ for $t < 0$.

(iv) $p =$ odd, $q =$ odd.
The interval ($-1 \leq t \leq 0$) is a Stokes curve because $Q_4(t) < 0$ for $t:\ -1 < t < 0$.

(v) any p, any q.
Since $Q_4(t) > 0$ ($t > 0$), the positive real axis ($t \geq 0$) is an anti-Stokes curve.

Putting $t := i\,\tau$ ($\tau \in \mathbb{R}$), we see that any interval on the imaginary axis of the t-plane cannot be a Stokes or an anti-Stokes curve because $Q_4 = i^p\,\tau^p\,(i\,\tau + 1)^q$ takes both real and imaginary values for any p and any q.

Summing up the above result, we get

Theorem 4.1. *The Stokes curve configuration for* (3.12) (*or* [4]) *has the following properties.*

(1) *The local properties are as follows:*

(a) $m+2$ *Stokes curves tend to* (*or emerge from*) $t=0$ *with arguments*

$$\pm\theta_5,\ \ \pm 3\theta_5,\ \ \pm 5\theta_5,\ \ \cdots \quad (\theta_5 := \pi/(m+2)).$$

(a)' $m+2$ *anti-Stokes curves tend to* (*or emerge from*) $t=0$ *with arguments*

$$\pm\, 2\theta_5,\ \ \pm 4\theta_5,\ \ \pm 6\theta_5,\ \ \cdots .$$

(b) $h-k+2$ *Stokes curves tend to* (*or emerge from*) $t=-1$ *with arguments*

$$(m\pm 1)\pi/(h-k+2),\ \ (m\pm 3)\pi/(h-k+2),\ \ (m\pm 5)\pi/(h-k+2),\cdots .$$

(b)' $h-k+2$ *anti-Stokes curves tend to* (*or emerge from*) $t=-1$ *with arguments*

$$m\pi/(h-k+2),\ \ (m\pm 2)\pi/(h-k+2),\ \ (m\pm 4)\pi/(h-k+2),\ \ \cdots .$$

(c) $h+m-k+2$ *Stokes curves tend to* (*or emerge from*) $t=\infty$ *with arguments*

$$\pm\theta_3,\ \ \pm 3\theta_3,\ \ \pm 5\theta_3,\ \ \cdots \ \ (\theta_3 := \pi/(h+m-k+2)).$$

(c)' $h+m-k+2$ *anti-Stokes curves tend to* (*or emerge from*) $t=\infty$ *with arguments*

$$\pm 2\theta_3,\ \ \pm 4\theta_3,\ \ \pm 6\theta_3,\ \ \cdots .$$

(2) An interval on the real or imaginary axis becomes a Stokes curve or an anti-Stokes curves as follows ($q := h-k$):

(a) *The positive real axis is an anti-Stokes curve for any* m *and any* q.

(b) *There exist neither Stokes nor anti-Stoke curves on the imaginary axis for any* m *and any* q.

(c) *An interval on the real axis is a Stokes or an anti-Stokes curve such that*

(i) m = *even*, q = *even* $\Longrightarrow$ *the whole real axis* $-\infty < t < \infty$ *is an anti-Stokes curve.*
(ii) m = *even*, q = *odd* $\Longrightarrow$ *the half real axis* $t \leq -1$ *is a Stokes curve.*
(iii) m = *odd*, q = *even* $\Longrightarrow$ *the half real axis* $t \leq 0$ *is a Stokes curve.*
(iv) m = *odd*, q = *odd* $\Longrightarrow$ *the interval* $-1 \leq t \leq 0$ *is a Stokes curve.*

All the Stokes curves and all the anti-Stokes curves emerging from a turning point must tend to ∞ if they do not tend to other turning points along the real axis. Then we can construct a whole Stokes curve configuration by applying the above theorem and the general properties in Lemma 4.1. Some examples are shown in Fig. 4.1 (a) for $h = 5,\ m = 2,\ k = 3$ and in Fig. 4.1 (b) for $h = 5,\ m = 2,\ k = 4$. The solid lines represent the Stokes curves and the broken lines represent the anti-Stokes curves.

4.3. In the similar way, by putting

$$\xi_2(a,t) := \int_a^t \sqrt{Q_2(t)}dt \quad (a = 0, \pm i), \quad Q_2(t) := t^p(t^2+1)^q \quad (p := h+m-k, q := k) \tag{4.3}$$

Fig. 4.1 (a)
$Q_4(t) = t^2(t+1)^2 (h=5, m=2, k=4)$

Fig. 4.1 (b)
$Q_4(t) = t^2(t+1)^2 (h=5, m=2, k=3)$

Fig. 4.1 $(a)', (b)'$

we can get a Stokes curve configuration for (3.10) (or [2]) as follows:

Theorem 4.2. *The Stokes curve configuration for* (3.10) (*or* [2]) *has the following properties.*

(1) *The local properties are as follows:*

(a) $h+m-k+2$ *Stokes curves tend to* (*or emerge from*) $t=0$ *with arguments* $\pm\theta_3,\ \pm 3\theta_3,\ \pm 5\theta_3,\ \cdots\ (\theta_3 := \pi/(h+m-k+2)$.

(a)' $h+m-k+2$ *anti-Stokes curves tend to* (*or emerge from*) $t=0$ *with arguments* $\pm 2\theta_3,\ \pm 4\theta_3,\ \pm 6\theta_3,\ \cdots$.

(b) $k+2$ *Stokes curves tend to* (*or emerge from*) $t=i$ *with arguments* $\{\pm 2\cdot 1-(h+m)\}\pi/2(k+2),\ \{\pm 2\cdot 3-(h+m)\}\pi/2(k+2),\ \{\pm 2\cdot 5-(h+m)\}\pi/2(k+2),\ \cdots$.

(b)' $k+2$ *anti-Stokes curves tend to* (*or emerge from*) $t=i$ *with arguments* $\{-(h+m)\}\pi/2(k+2),\ \{\pm 2\cdot 2-(h+m)\}\pi/2(k+2),\ \{\pm 2\cdot 4-(h+m)\}\pi/2(k+2),\ \cdots$.

(c) $k+2$ *Stokes curves tend to* (*or emerge from*) $t=-i$ *with arguments* $\{\pm 2\cdot 1+(h+m)\}\pi/2(k+2),\ \{\pm 2\cdot 3+(h+m)\}\pi/2(k+2),\ \{\pm 2\cdot 5+(h+m)\}\pi/2(k+2),\ \cdots$.

(c)' $k+2$ *anti-Stokes curves tend to* (*or emerge from*) $t=-i$ *with arguments* $(h+m)\pi/2(k+2),\ \{\pm 2\cdot 2+(h+m)\}\pi/2(k+2),\ \{\pm 2\cdot 4+(h+m)\}\pi/2(k+2),\ \cdots$.

(d) $h+m+k+2$ *Stokes curves tend to* (*or emerge from*) $t=\infty$ *with arguments* $\pm\theta_1,\ \pm 3\theta_1,\ \pm 5\theta_1,\ \cdots\ (\theta_1 := \pi/(h+m+k+2))$.

(d)' $h+m+k+2$ *anti-Stokes curves tend to* (*or emerge from*) $t=\infty$ *with arguments* $\pm 2\theta_1,\ \pm 4\theta_1,\ \pm 6\theta_1,\ \cdots$.

(2) An interval on the real axis or on the imaginary axis of the t-plane becomes a Stokes or an anti-Stokes curve as follows ($p := h+m-k$):

(a) *An interval on the real axis of the t-plane is a Stokes or an anti-Stokes curve such that*

(i) $p=$ *even,* $k\in\mathbb{N} \implies$ *the real axis* $-\infty<t<\infty$ *is an anti-Stokes curve.*

(ii) $p=$ *odd,* $k\in\mathbb{N} \implies$ *the positive real axis* $t\geq 0$ *is an anti-Stokes curve, and the negative real axis* $t\leq 0$ *is a Stokes curve.*

(iii) $p\in\mathbb{N},\ k\in\mathbb{N} \implies$ *the positive real axis* $t\geq 0$ *is an anti-Stokes curve.*

(b) *An interval on the imaginary axis of the t-plane is a Stokes or an anti-Stokes curve such that*

(i) $p=4n\ (n\in\mathbb{N}),\ k=$ *even* $\implies$ *the imaginary axis* $-\infty<\Im t<\infty$ *is a Stokes curve.*

(ii) $p=4n\ (n\in\mathbb{N}),\ k=$ *odd* $\implies$ *the interval* $-1\leq\Im t\leq 1$ *is a Stokes curve.*

(iii) $p=4n-2\ (n\in\mathbb{N}),\ k=$ *even* $\implies$ *the imaginary axis* $-\infty<\Im t<\infty$ *is an anti-Stokes curve.*

(iv) $p=4n-2\ (n\in\mathbb{N}),\ k=$ *odd* $\implies$ *the half imaginary axes* $\Im t\geq 1$ *and* $\Im t\leq -1$ *are Stokes curves.*

All the Stokes and anti-Stokes curves emerging from the turning points must tend to ∞ if they do not tend to other turning points along the imaginary axis. Then we can construct a whole Stokes curve configuration for (3.10) (or [2]) by applying the above theorem and the general properties in Lemma 4.1. Some examples are shown in Fig. 4.2 (a) for $h = 5$, $m = 2$, $k = 3$ and in Fig. 4.2 (b) for $h = 5$, $m = 2$, $k = 4$. The solid lines represent the Stokes curves and the broken lines represent the anti-Stokes curves.

The Stokes curve configurations for (3.9) (or [1]), (3.11) (or [3]) and (3.13) (or [5]) are very simple because they have a monomial coefficient with a turning point at $x = 0$.

5. Canonical Domains

5.1. The function $\xi := \xi(a, x)$ defined by (3.3) is a conformal mapping from the x-plane to the ξ-plane (Sec. 3.1). We prepare several definitions. A *Stokes domain* for (3.1) is, by definition, a simply connected set on the x-plane bounded by several Stokes curves without any Stokes curve in it. There are two types of Stokes domains. One is called a *half-plane type* and it is mapped onto the half ξ-plane $\big(\Re\xi \le C$ or $\Re\xi \ge C$ ($C = \text{const.}$)$\big)$ by (3.3), and the other is called a *strip type* and it is mapped onto the strip domain such as $C_1 \le \Re\xi \le C_2$ ($C_1, C_2 = \text{const.}$). A *canonical domain* for (3.1) is, by definition, a simply connected set on the x-plane mapped onto the whole ξ-plane with cuts mapped by (3.3).

Fig. 4.2 (a)
$Q_2(t) = t^4(t^2+1)^3 (h=5, m=2, k=3)$

Fig. 4.2 (b)
$Q_2(t) = t^3(t^2+1)^4 (h=5, m=2, k=4)$

Fig. 4.2 (a)'

Fig. 4.2 (b)'

Thus, a canonical domain consists of two half-plane type Stokes domains with or without strip type ones. Put

$$\xi_j(0,x) := \int_0^x \sqrt{Q_j(x)}dx \quad \big(Q_1(x) := x^{h+m+k},\ Q_3(x) := x^{h+m-k},\ Q_5(x) := x^m\big). \tag{5.1$_x$}$$

Clearly $Q_j(x) > 0$ for $x > 0$, and we choose a branch of the square roots of $Q_j(x)$ such that $\sqrt{Q_j(x)} > 0$ for $x > 0$. Then, we can see that $\Re\,\xi_j(0,x) \geq 0$ for $x \geq 0$ and the positive real axis $x \geq 0$ is an anti-Stokes curve, i.e., $\Im\,\xi_j(0,x) = 0\ \ (x \geq 0)$, for $[1,3,5]$.

As mentioned just below the theorems 4.1 and 4.2, the positive real axis of each of two t-planes is an anti-Stokes curve for any indices p and q. Therefore, we see $\Im\,\xi_j(0,t) = 0$ for $t > 0$, where

$$\xi_j(0,t) := \int_0^t \sqrt{Q_j(t)}\,dt \quad \big(Q_2(t) := t^{h+m-k}(t^2+1)^k,\ \ Q_4(t) := t^m(t+1)^{h-k}\big). \tag{5.1$_t$}$$

We should choose a branch of $\sqrt{Q_j(t)}$ such that $\sqrt{Q_j(t)} > 0$ for $t > 0$ in order to match the signs with $\Re_j(0,x)$, then $\Re\,\xi_j(0,t) > 0$ for $t > 0$.

5.2. Let us consider the differential equation

$$\varepsilon^{2h}\,\frac{d^2y}{dx^2} = x^{h+m+k}y \quad (0 \leq |x| < \infty). \tag{3.9$'$}$$

Stokes curves for $(3.9)'$ are direct lines emerging from the turning point $x = 0$ and tending to the irregular singular point $x = \infty$ with the arguments $\pm\,\theta_1,\ \pm\,3\,\theta_1,\ \pm\,5\,\theta_1, \cdots$. The anti-Stokes curves are also direct lines connecing $x = 0$ and $x = \infty$ with the arguments $0,\ \pm\,2\,\theta_1,\ \pm\,4\,\theta_1,\ \pm\,6\,\theta_1, \cdots$. A Stokes domain for (3.9)' is a sector with an angle $2\,\theta_1$ around $x = 0$ between any two neighboring Stokes curves. Then, from the choice of the branch $\sqrt{Q_1(x)}$, we can see that

$$\Re\,\xi_1(0,x)\begin{cases} < 0 & \text{for} \quad x : \theta_1 < \arg x < 3\theta_1, \\ > 0 & \text{for} \quad x : -\theta_1 < \arg x < \theta_1. \end{cases} \tag{5.2$_1$}$$

From Lemma 4.1, Theorem 4.2 and from the choice of the branch $\sqrt{Q_2(t)}$ we can see that

$$\Re\,\xi_2(0,t)\begin{cases} < 0 & \text{for} \quad t : -\theta_1 < \arg t < 3\theta_1 \\ > 0 & \text{for} \quad t : -\theta_1 < \arg t < \theta_1 \end{cases} \quad (t \to \infty) \tag{5.2$_2^\infty$}$$

and

$$\Re\,\xi_2(0,t)\begin{cases} < 0 & \text{for} \quad t : \theta_3 < \arg t < 3\theta_3 \\ > 0 & \text{for} \quad t : -\theta_3 < \arg t < \theta_3 \end{cases} \quad (t \to 0). \tag{5.2$_2^0$}$$

Next, we consider a Stokes curve configuration of the differential equation

$$\varepsilon^{2h-k}\,\frac{d^2y}{dx^2} = \ x^{h+m-k}\ y \quad (0 \leq |x| < \infty). \tag{3.11$'$}$$

It is very similar to one for $(3.9)'$. Therefore, in the same way as $(5.2)_1$, we can see that

$$\Re\, \xi_3(0,x) \begin{cases} < 0 & \text{for} \quad x : \theta_3 < \arg x < 3\theta_3, \\ > 0 & \text{for} \quad x : -\theta_3 < \arg x < \theta_3. \end{cases} \tag{5.2$_3$}$$

In the similar way to $(5.2)_2$ we can see that

$$\Re\, \xi_4(0,t) \begin{cases} < 0 & \text{for} \quad t : \theta_3 < \arg t < 3\theta_3 \\ > 0 & \text{for} \quad t : -\theta_3 < \arg t < \theta_3 \end{cases} \quad (t \to \infty) \tag{5.2$_4^\infty$}$$

and

$$\Re\, \xi_4(0,t) \begin{cases} < 0 & \text{for} \quad t : \theta_5 < \arg t < 3\theta_5 \\ > 0 & \text{for} \quad t : -\theta_5 < \arg t < \theta_5 \end{cases} \quad (t \to 0). \tag{5.2$_4^0$}$$

Also we can see that

$$\Re\, \xi_5(0,x) \begin{cases} < 0 & \text{for} \quad x : \theta_5 < \arg x < 3\theta_5, \\ > 0 & \text{for} \quad x : -\theta_5 < \arg x < \theta_5. \end{cases} \tag{5.2$_5$}$$

We should notice the correspondence relations of the angles between neighboring arguments up and down.

5.3. Now, under the preparation in the previous sections for the Stokes curve configurations, we are constructing canonical domains $\mathcal{D}_1^{can}$ to $\mathcal{D}_5^{can}$ for (3.9) (or [1]) to (3.13) (or [5]). The shadow zones in Figs. 4.1 (a) - 4.3 show the canonical domains and their maps under $\xi := \xi_j$ are given in Figs. 4.1 (a)' - 4.3', respectively. Notice that the same letters are used for the images.

First, we define $\mathcal{D}_1^{can}$ for (3.9). In order to define it we need to know the Stokes curve configuration for (3.9). Stokes curves and anti-Stokes curves for (3.9) are the limited lines of ones for $(3.9)'$ in D_1 (see (3.9)) and a Stokes domain for (3.9) is a part of one for $(3.9)'$ limited in D_1.

A canonical domain for (3.9) consists of any two neighboring Stokes domains. Then, by refering $(5.2)_1$ we define $\mathcal{D}_1^{can}$ as follows:

$$\mathcal{D}_1^{can} := \left\{ x : -\theta_1 < \arg x < 3\,\theta_1,\ K_1 \varepsilon^{1/2} \le |x| \le x_0 \right\} \tag{5.3}$$

which is a subset of the domain D_1. Let $l_1^{(0)}$ and $l_1^{(2)}$ be Stokes curves, and $L_1^{(1)}$ and $L_1^{(2)}$ be anti-Stokes curves such that

$$\begin{cases} l_1^{(0)} : x = r\, e^{-i\,\theta_1}, & l_1^{(2)} : x = r\, e^{i\,3\,\theta_1}, \\ L_1^{(1)} : x = r, & L_1^{(2)} : x = r\, e^{i2\,\theta_1} \quad (K_1 \varepsilon^{1/2} \le r \le x_0). \end{cases} \tag{5.4}$$

Thus, $\mathcal{D}_1^{can}$ is bounded by $l_1^{(0)}$, $l_1^{(2)}$ and $|x| = x_0$, $|x| = K_1 \varepsilon^{1/2}$ (see Fig. 4.3 $(j = 1)$).

5.4. As the shadow zones shown in Fig. 4.2, we can define a canonical domain $\mathcal{D}_2^{can}$ for (3.10) (or [2]) to be the set

$$\mathcal{D}_2^{can} := \mathcal{D}_2^{(1)} \cup l_2^{(1)} \cup \mathcal{D}_2^{(2)} \quad \text{or} \quad \mathcal{D}_2^{(1)} \cup l_2^{(1)} \cup \mathcal{D}_2^{(3)} \cup l_2^{(3)} \cup \mathcal{D}_2^{(2)}, \tag{5.5}$$

which is an unbounded subset of D_2^{∞} $(:= \{0 < |t| < \infty\})$. It intersects $\mathcal{D}_1^{can}$. $\mathcal{D}_2^{(1)}$ and $\mathcal{D}_2^{(2)}$ are half-plane type Stokes domains and $D_2^{(3)}$ is a strip type Stokes domain. Strip type Stokes domains appear possibly for $p \neq 4n$ $(n \in \mathbb{N})$ (Theorem 4.2. (2), (b)).

The first canonical domain in (5.5) is mapped onto the whole ξ-plane with a downward cut by the mapping defined by $\xi := \xi_2(0, t)$ in the manner such that

(i) *The anti-Stokes curve $L_2^{(1)}$ is mapped onto the positive real axis of the ξ-plane.*

(ii) *The anti-Stokes curve $L_2^{(2)}$ is mapped onto the negative real axis.*

(iii) *The Stokes curve $l_2^{(1)}$ is mapped onto the positive imaginary axis.*

(iv) *Two Stokes curves $l_2^{(0)}$, $l_2^{(2)}$ (or $l_2^{(2)} \cup l_2^{(3)}$) are mapped onto the negative imaginary axis. We consider that $l_2^{(0)}$ is mapped onto the right side of the negative imaginary axis, and $l_2^{(2)}$ (or $l_2^{(2)} \cup l_2^{(3)}$) onto the left side. Thus, the cut on the ξ-plane is made from the 'gap' of images of $l_2^{(0)}, l_2^{(2)}$ (or $l_2^{(2)} \cup l_2^{(3)}$).*

As a consequence, we see

(v) *Two Stokes domains $\mathcal{D}_2^{(1)}$ and $\mathcal{D}_2^{(2)}$ are mapped onto the right and the left half plane, respectively.*

We notice that the interval $0 \leq \Im\, \xi_2(0, t) \leq 1$ is a Stokes curve in the case where $p = 4n$ $(n \in \mathbb{N})$, then $l_2^{(2)} \cup l_2^{(3)}$ is mapped onto the left side of the imaginary axis of the ξ-plane (Fig. 4.2 $(a)'$).

The second canonical domain in (5.5) is mapped onto the ξ-plane with two downward cuts in the manner such that

(i) *$L_2^{(1)}$ is mapped onto the positive real axis.*

(ii) *$L_2^{(2)}$ is mapped onto the negative real axis.*

(iii) *$l_2^{(1)}$ is mapped onto the positive imaginary axis.*

(iv) *$l_2^{(0)}, l_2^{(4)}$ are mapped onto the negative imaginary axis.*

(v) *$l_2^{(2)}, l_2^{(5)}$ are mapped onto the downward line and $l_2^{(3)}$ is mapped onto the upward line, parallel to the negative imaginary axis, emerging from the image of i in the fourth quadrant. Two cuts on the ξ-plane are made from the 'gaps' of images of $l_2^{(0)}, l_2^{(4)}$ and $l_2^{(2)}, l_2^{(5)}$.* As a consequence, we see

(vi) *$\mathcal{D}_2^{(1)}$ is mapped onto the right half plane $(\Re\, \xi > 0)$, $\mathcal{D}_2^{(2)}$ is mapped onto the left half plane $\big(\Re\, \xi < \Re\, \xi_2(0, i)\big)$, $\mathcal{D}_2^{(3)}$ is mapped onto the strip domain between the images of other two Stokes domains $\big(\Re\, \xi_2(0, i) < \Re\, \xi < 0\big)$* (Fig. 4.2 $(b)'$).

5.5. A canonical domain $\mathcal{D}_3^{can}$ for (3.11) (or [3]) is a subset of one for $(3.11)'$ (Sec. 5.2) just like $\mathcal{D}_1^{can}$. Thus, by refering $(5.2)_3$, let $l_3^{(0)}$ and $l_3^{(2)}$ be Stokes curves, and $L_3^{(1)}$ and $L_3^{(2)}$ be anti-Stokes curves such that

$$\begin{cases} l_3^{(0)} : x = r\,e^{-i\,\theta_3}, & l_3^{(2)} : x = r\,e^{i\,3\,\theta_3}, \\ L_3^{(1)} : x = r, & L_3^{(2)} : x = r\,e^{i\,2\,\theta_3} \ \ (K_2\varepsilon \le r \le k_1\varepsilon^{1/2}). \end{cases} \tag{5.6}$$

We define a canonical domain $\mathcal{D}_3^{can}$ for (3.11) (or [3]) such that

$$\mathcal{D}_3^{can} := \left\{x : -\theta_3 < \arg x < 3\,\theta_3,\ K_2\varepsilon \le |x| \le k_1\varepsilon^{1/2}\right\}, \tag{5.7}$$

which is a subset of the domain $D_3 : K_2\varepsilon \le |x| \le k_1\varepsilon^{1/2}$ bounded by two Stokes curves $l_3^{(0)}$ and $l_3^{(2)}$ (Fig. 4.3 ($j = 3$)). Notice that the canonical domain $\mathcal{D}_3^{can}$ intersects the canonical domain $\mathcal{D}_2^{can}$ (Sec. 3.2 and 3.3) and that the arguments of boundaries of $\mathcal{D}_3^{can}$ and $\mathcal{D}_2^{can}$ coincide.

5.6. A canonical domain $\mathcal{D}_4^{can}$ for (3.12) (or [4]) is defined by

$$\mathcal{D}_4^{can} := \mathcal{D}_4^{(1)} \cup l_4^{(1)} \cup \mathcal{D}_4^{(2)}, \tag{5.8}$$

which is a subset of $D_4^{\infty} : 0 < |t| < \infty$ and intersects $\mathcal{D}_3^{can}$ (Sec. 3.2, 3.3). $\mathcal{D}_4^{(1)}$ and $\mathcal{D}_4^{(2)}$ are half-plane type Stokes domains. In this case, any strip type Stokes domain does not appear because we need not consider a canonical domain near $t = -1$. The shadow zones show $\mathcal{D}_4^{can}$ in Fig. 4.1 $(a), (b)$.

The canonical domain $\mathcal{D}_4^{can}$ is mapped onto the whole ξ-plane with a downward cut by the mapping defined by $\xi := \xi_4(0, t)$ in the manner such that:

(i) *The anti-Stokes curve $L_4^{(1)}$ is mapped onto the positive real axis of the ξ-plane.*

(ii) *The anti-Stokes curve $L_4^{(2)}$ is mapped onto the negative real axis.*

(iii) *The Stokes curve $l_4^{(1)}$ is mapped onto the positive imaginary axis.*

(iv) *Two Stokes curves $l_4^{(0)}$, $l_4^{(2)}$ are mapped onto the negative imaginary axis. We consider that $l_4^{(0)}$ is mapped onto the right side of the negative imaginary axis, and $l_4^{(2)}$ onto the left side. Thus, the cut on the ξ-plane is made from the 'gap' of images of $l_4^{(0)}, l_4^{(2)}$.* As a consequence, we see

(v) *Two Stokes domains $\mathcal{D}_4^{(1)}$ and $\mathcal{D}_4^{(2)}$ are mapped onto the right and the left half plane, respectively* (Figs. 4.1 $(a)', (b)'$).

5.7. By refering $(5.2)_5$, we define the last canonical domain, i.e., $\mathcal{D}_5^{can}$ for (3.9) (or [5]), in the similar way to $\mathcal{D}_1^{can}$ and $\mathcal{D}_3^{can}$, to be

$$\mathcal{D}_5^{can} := \left\{x : -\theta_5 < \arg x < 3\theta_5,\ \ 0 \le |x| \le k_2\varepsilon\right\} \tag{5.9}$$

which is a subset of $D_5 : 0 \le |x| \le k_2\varepsilon$ and intersects $\mathcal{D}_4^{can}$ (Secs. 3.2, 3.3). Notice that the arguments of $x \in \mathcal{D}_5^{can}$ coincide with ones of t of $\mathcal{D}_4^{can}$ small t.

Fig. 4.3. *Fig. 4.3'*

Let $l_5^{(0)}$ and $l_5^{(2)}$ be Stokes curves, and $L_5^{(1)}$ and $L_5^{(2)}$ be anti-Stokes curves such that

$$\begin{cases} l_5^{(0)} : x = re^{-i\theta_5}, & l_5^{(2)} : x = re^{i\,3\,\theta_5}, \\ L_5^{(1)} : x = r, & L_5^{(2)} : x = r\,e^{i2\,\theta_5} \quad (0 \le r \le k_2\varepsilon). \end{cases} \tag{5.10}$$

Then $\mathcal{D}_5^{can}$ is bounded by $l_5^{(0)}$, $l_5^{(2)}$ and $0 \le |x| \le k_2\varepsilon$ and it must include the origin though Fig. 4.3 excludes its neighbor for $j = 5$.

Thus we could get one set of five canonical domains as follows:

Theorem 5.1. *By refering how to connect boundaries of neighboring two canonical domains, there exists a set of five canonical domains* $\mathcal{D}_1^{can}, \cdots, \mathcal{D}_5^{can}$ *for the reduced differential equations* (3.9) *or* [1], $\cdots$, (3.13) *or* [5], *respectively.*

6. Matching Matrices

6.1. First, we will connect two sets of solutions $y_1^{\pm}(x,\varepsilon)$ and $y_2^{\pm}(t,\varepsilon)$. The matching matrix $M_{1,2}$ matching these two sets of solutions is, by definition, given by

$$M_{1,2} \cdot \begin{pmatrix} y_1^+(x,\varepsilon) \\ y_1^-(x,\varepsilon) \end{pmatrix} = \begin{pmatrix} y_2^+(t,\varepsilon) \\ y_2^-(t,\varepsilon) \end{pmatrix}, \quad M_{1,2} := \begin{pmatrix} a(\varepsilon) & b(\varepsilon) \\ c(\varepsilon) & d(\varepsilon) \end{pmatrix} \quad (x = t\,\varepsilon^{1/2}). \tag{6.1}$$

Theorem 6.1. *The matching matrix* $M_{1,2}$ *is given by*

$$M_{1,2} \;\sim\; \varepsilon^{(h+m+k)/8} \cdot E \qquad (\varepsilon \to 0), \tag{6.2}$$

where E *is the* 2×2 *identitiy matrix.*

Proof. Substituting the corresponding WKB approximations (Theorem 3.2) for the true solutions in (6.1) we get an asymptotic relation

$$M_{1,2} \cdot \begin{pmatrix} y_1^+(x, \widetilde{y}_1^+(x,\varepsilon) \\ \widetilde{y}_1^-(x,\varepsilon) \end{pmatrix} \sim \begin{pmatrix} \widetilde{y}_2^+(t,\varepsilon) \\ \widetilde{y}_2^-(t,\varepsilon) \end{pmatrix} \quad (\varepsilon \to 0), \tag{6.3}$$

from which the following relations hold

$$a\,\frac{\tilde{y}_1^+}{\tilde{y}_2^+} \;+\; b\,\frac{\tilde{y}_1^-}{\tilde{y}_2^+} \;\sim\; 1, \quad c\,\frac{\tilde{y}_1^+}{\tilde{y}_2^-} \;+\; d\,\frac{\tilde{y}_1^-}{\tilde{y}_2^-} \;\sim\; 1 \quad (\varepsilon \to 0). \tag{6.4}$$

Since x and t are combined by the equality $x = t\,\varepsilon^{1/2}$, we split $\varepsilon^{1/2}$ such that $x = t\,\varepsilon^{1/4}\cdot\varepsilon^{1/4}$ and put

$$t := \eta\,\varepsilon^{-1/4}, \quad x := \eta\,\varepsilon^{1/4} \quad (|\eta| = 1), \tag{6.5}$$

where a new parameter η is determined later.

Since we see that $t^{h+m-k}(t^2+1)^k = O(t^{h+m+k})$ $(t \to \infty)$ in the WKB approximations $\tilde{y}_2^{\pm}$, we let $t \to \infty$ and substitute (6.5) in $\tilde{y}_1^+ \,/\, \tilde{y}_2^+$ of (6.4) to get

$$\begin{aligned}\frac{\tilde{y}_1^+}{\tilde{y}_2^+} &\sim \frac{t^{(h+m+k)/4}}{x^{(h+m+k)/4}}\exp\left[\frac{2}{h+m+k+2}\left(\frac{x^{(h+m+k+2)/2}}{\varepsilon^h}\right.\right.\\ &\qquad \left.\left. - \frac{t^{(h+m+k+2)/2}}{\varepsilon^{(3h-m-k-2)/4}}\right)\right] \;(\varepsilon \to 0)\\ &= \varepsilon^{-(h+m+k)/8}\cdot\exp(0) = \varepsilon^{-(h+m+k)/8},\end{aligned} \tag{6.6}$$

because two indices of ε of the exponential parts of $\tilde{y}_1^+$ and $\tilde{y}_2^+$ are equal to $(-7h+m+k+2)/8$.

Next, letting $t \to \infty$ and substituting (6.5) in $\tilde{y}_1^- \,/\, \tilde{y}_2^+$ of (6.4), we similarly get

$$\frac{\tilde{y}_1^-}{\tilde{y}_2^+} \sim \varepsilon^{-(h+m+k)/8}\exp\left[-\frac{4}{h+m+k+2}\varepsilon^{(-7h+m+k+2)/8}\eta^{(h+m+k+2)/2}\right] \;(\varepsilon \to 0), \tag{6.7}$$

whose magnitude tends to $+\infty$ if we take η such that $\Re\,\eta^{(h+m+k+2)/2} < 0$. Notice that the index of ε of the exponential part is negative from the inequality $-7h+m+k+2 < -5h-2 < 0$. The new parameter η with the property $\Re\,\eta^{(h+m+k+2)/2} < 0$ can be chosen in $\mathcal{D}_2^{can}$ such as $\eta := \eta_{-\infty} := e^{i\,2\,\theta_1}$, i.e., $\arg\eta_{-\infty} = 2\,\theta_1 =$ "the angle of the anti-Stokes curve $L_2^{(2)}$ at $t = \infty$" (Fig. 4.2.). Thus, from (6.6), (6.7) and from the first relation of (6.4), we get the relation

$$a\cdot\varepsilon^{-(h+m+k)/8} \;+\; b\cdot\infty \;\sim\; 1 \quad (\varepsilon \to 0). \tag{6.8}$$

Then, we can get $a(\varepsilon) \sim \varepsilon^{(h+m+k)/8}$, $b(\varepsilon) = 0$ $(\varepsilon \to 0)$.

Similarly, we get the following relations:

$$\begin{aligned}\frac{\tilde{y}_1^+}{\tilde{y}_2^-} &\sim \varepsilon^{-(h+m+k)/8}\exp\left[\frac{4}{h+m+k+2}\varepsilon^{(-7h+m+k+2)/8}\,\eta^{(h+m+k+2)/2}\right] \;(\varepsilon \to 0)\\ &\to \infty \quad (\Re\,\eta^{(h+m+k+2)/2} > 0,\;\; \varepsilon \to 0).\end{aligned} \tag{6.9}$$

The new parameter η with the property $\Re\,\eta^{(h+m+k+2)/2} > 0$ can be chosen in $\mathcal{D}_2^{can}$ such as $\eta := \eta_{+\infty} := 1$, i.e., $\arg \eta_{+\infty} = 0 =$ "the angle of the anti-Stokes curve $L_2^{(1)}$ at $t = \infty$" (Fig. 4.2). Also, we can obtain

$$\frac{\tilde{y}_1^-}{\tilde{y}_2^-} \sim \varepsilon^{-(h+m+k)/8} \quad (\varepsilon \to 0). \tag{6.10}$$

Thus, the second relation of (6.4) becomes

$$c \cdot \infty \ + \ d \cdot \varepsilon^{-(h+m+k)/8} \ \sim \ 1 \quad (\varepsilon \to 0), \tag{6.11}$$

which induces $c(\varepsilon) = 0,\ \ d(\varepsilon) \sim \varepsilon^{(h+m+k)/8}\ \ (\varepsilon \to 0)$. Thus, we get (6.2). ■

6.2. In the similar way to $M_{1,2}$, we can compute other matching matrices. We will compute another matching matrix $M_{2,3}$ connecting two sets of solutions $y_2^{\pm}(t,\varepsilon)$ and $y_3^{\pm}(x,\varepsilon)$. $M_{2,3}$ must satisfy the following asymptotic relation by using WKB approximations :

$$M_{2,3} \cdot \begin{pmatrix} \widetilde{y}_2^+(t,\varepsilon) \\ \widetilde{y}_2^-(t,\varepsilon) \end{pmatrix} \sim \begin{pmatrix} \widetilde{y}_3^+(x,\varepsilon) \\ \widetilde{y}_3^-(x,\varepsilon) \end{pmatrix} \quad (x = t\,\varepsilon^{1/2}, \quad \varepsilon \to 0). \tag{6.12}$$

We split $\varepsilon^{1/2}$ in $x = t\,\varepsilon^{1/2}$ such that $x = t\,\varepsilon^{-1/4} \cdot \varepsilon^{3/4}$ and put

$$t := \eta\,\varepsilon^{1/4}, \quad x := \eta\,\varepsilon^{3/4} \quad (|\eta| = 1), \tag{6.13}$$

where η is a new parameter considered as being on the anti-Stokes curves. In the just same way as getting $M_{1,2}$, we can get

$$M_{2,3} \sim \varepsilon^{-(h+m-k)/8}\, E \quad (\varepsilon \to 0). \tag{6.14}$$

6.3. Similarly, we can get other matching matrices.

Theorem 6.2. *Let $y_j^{\pm}$ be the true solutions of the differential equation $[j]$ $(j = 2, \cdots, 5)$ in Theorem 3.1 and let $M_{j,j+1}$ be the matching matrix connecting two sets of solutions $y_j^{\pm}$ and $y_{j+1}^{\pm}$ such that*

$$M_{j,j+1} \cdot \begin{pmatrix} y_j^+ \\ y_j^- \end{pmatrix} = \begin{pmatrix} y_{j+1}^+ \\ y_{j+1}^- \end{pmatrix}. \tag{6.15}$$

Then, the matching matrices are given by

$$M_{2,3} \sim \varepsilon^{-(h+m-k)/8}\, E,\ \ M_{3,4} \sim \varepsilon^{(h+m-k)/4}\, E,\ \ M_{4,5} \sim \varepsilon^{-m/4}\, E\ \ (\varepsilon \to 0). \tag{6.16}$$

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