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Bounds for Cohomological Deficiency Functions of Projective Schemes Over Artinian Rings*

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Abstract. Let X be a projective scheme over an artinian commutative ring R_0 and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. We give bounds on the so called cohomological deficiency functions $\Delta^i_{X,\mathcal{F}}$ and the cohomological postulation numbers $\nu^i_{X,\mathcal{F}}$ of the pair $(X, \mathcal{F})$. As bounding invariants we use the "cohomology diagonal" $\left(h^j_{X,\mathcal{F}}(-j)\right)_{j\leq i}$ at and below level i and the i-th "cohomological Hilbert polynomial" $p^i_{X,\mathcal{F}}$ of the pair $(X, \mathcal{F})$. Our bounds present themselves as a quantitative and extended version of the vanishing theorem of Severi – Enriques – Zariski – Serre.

0. Introduction

In this paper, we continue the investigation begun in [7]. For the reader's convenience, we first recall a few basic notions which were used in the first part of our work. Let $\mathcal{C}$ be the class of all pairs $(X, \mathcal{F})$ in which

$X = \text{Proj}\,(R)$ is the projective scheme induced by a positively graded homogeneous noetherian ring $R = \oplus_{n\geq 0} R_n$ with artinian base ring R_0. (0.1)

$\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules. (0.2)

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For any pair $(X, \mathcal{G}) \in \mathcal{C}$ and any $i \in \mathbb{N}_0$, the i-th cohomology module $H^i(X, \mathcal{G})$ of X with coefficients in $\mathcal{G}$ is a finitely generated R_0-module (cf. [16, III, Theorem 5.2], [22, § 66, Théorème 1]) and hence of finite length. So, for each $i \in \mathbb{N}_0$ and each pair $(X, \mathcal{F}) \in \mathcal{C}$ we may introduce the i-th *cohomological Hilbert function of* (X *with respect to*) $\mathcal{F}$

$$h^i_{X,\mathcal{F}} := h^i_{\mathcal{F}} : \mathbb{Z} \longrightarrow \mathbb{N}_0, \ \ (i \in \mathbb{N}_0) \tag{0.3}$$

defined by

$$h^i_{X,\mathcal{F}}(n) := h^i_{\mathcal{F}}(n) := \ell_{R_0}\big(H^i(X, \mathcal{F}(n))\big), \ \ (n \in \mathbb{Z}) \tag{0.4}$$

where $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ is the n-th *twist of* $\mathcal{F}$ and where $\ell_{R_0}(N)$ denotes the length of the R_0-module N.

Let us also recall the notion of *subdepth of* (X *with respect to*) $\mathcal{F}$:

$$\delta(\mathcal{F}) := \min\{\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) | x \in X, \quad x \text{ closed}\}, \tag{0.5}$$

the concept of *regularity of* $\mathcal{F}$ *above level* k:

$$\mathrm{reg}_k(\mathcal{F}) := \inf\{r \in \mathbb{Z} | h^i_{\mathcal{F}}(n - i) = 0, \quad \forall n \geq r, \ \ \forall i > k\}, \tag{0.6}$$

and the concept of *coregularity of* $\mathcal{F}$ *at and below level* k*:*

$$\mathrm{coreg}^k(\mathcal{F}) := \sup\{c \in \mathbb{Z} | h^i_{\mathcal{F}}(n - i) = 0, \quad \forall n \leq c, \ \ \forall i \leq k\}, \tag{0.7}$$

defined whenever $(X, \mathcal{F}) \in \mathcal{C}$.

In [7] we studied – for arbitrary pairs $(X, \mathcal{F}) \in \mathcal{C}$:
Bounds of *Castelnuovo type* which, for each $i > 0$ bound the values $h^i_{\mathcal{F}}(n)$ in the range $n \geq -i$ in terms of the *diagonal values* $\big(h^j_{\mathcal{F}}(-j)\big)_{j \geq i}$ *at and above level* i. (0.8)

Bounds of *Severi type* which, for each $i < \delta(\mathcal{F})$ bound the values $h^i_{\mathcal{F}}(n)$ in the range $n \leq -i$ in terms of the *diagonal values* $\Big(h^j_{\mathcal{F}}(-j)\Big)_{j \leq i}$ *at and below level* i. (0.9)

The bounds mentioned under (0.8) in particular give rise to bounds on the regularities $\mathrm{reg}_k(\mathcal{F})$, whereas the bounds of (0.9) induce bounds on the coregularities $\mathrm{coreg}^i(\mathcal{F})$ for all $i < \delta(\mathcal{F})$.

One appearent disadvantage of our bounds of Severi type is the fact that they apply only if $i < \delta(\mathcal{F})$. One way to get bounding results (in the range $n \leq -i$) for $i \geq \delta(\mathcal{F})$ is to estimate the so called cohomological deficiency functions. For a pair $(X, \mathcal{F}) \in \mathcal{C}$ and for $i \in \mathbb{N}_0$, the i-th *cohomological deficiency function of (X with respect to)* $\mathcal{F}$ is defined by

$$\Delta^i_{X,\mathcal{F}} := \Delta^i_{\mathcal{F}} := h^i_{\mathcal{F}} - p^i_{\mathcal{F}}, \tag{0.10}$$

where $p^i_{\mathcal{F}}(n)$ is the i-th *cohomological Hilbert polynomial* of $\mathcal{F}$, i.e. the unique polynomial in $\mathbb{Q}[\mathbf{x}]$ such that

$$p^i_{\mathcal{F}}(n) = h^i_{\mathcal{F}}(n), \quad \forall n \ll 0. \tag{0.11}$$

These deficiency functions were investigated by the second author in [20] for projective schemes over an algebraically closed field. The aim of the present paper is to give some *a priori bounds* on the i-th deficiency function $\Delta^i_{X,\mathcal{F}} = \Delta^i_{\mathcal{F}}$ and the so called i-th *cohomological postulation number of* (X *with respect to*) $\mathcal{F}$

$$\nu^i_{X,\mathcal{F}} := \nu^i_{\mathcal{F}} := \inf\left\{m \in \mathbb{Z} | \Delta^i_{X,\mathcal{F}}(m+1) \neq 0\right\} \tag{0.12}$$

for abitrary pairs $(X,\mathcal{F}) \in \mathcal{C}$.

More precisely, we shall define bounding functions

$$E^{(i)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}^{i+1} \times \mathbb{Z}_{\leq -i} \longrightarrow \mathbb{Z},$$
$$G^{(i)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}^{i+1} \times \mathbb{Z}_{\leq -i} \longrightarrow \mathbb{Z},$$
$$F^{(i)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}^{i+1} \longrightarrow \mathbb{Z}_{\leq -2i-1},$$

such that (cf. Corollary 3.1 a), c)):

For each pair $(X,\mathcal{F})$ (0.13)

a) $G^{(i)}(\underline{h}^{(i)}_{\mathcal{F}}, \underline{p}^{(i)}_{\mathcal{F}}; n) \leq \Delta^i_{\mathcal{F}}(n) \leq E^{(i)}(\underline{h}^{(i)}_{\mathcal{F}}, \underline{p}^{(i)}_{\mathcal{F}}; n)$ for all $n \leq -i$,

b) $\nu^i_{\mathcal{F}} \geq F^{(i)}(\underline{h}^{(i)}_{\mathcal{F}}, \underline{p}^{(i)}_{\mathcal{F}}) + 1$,

where

a) $\underline{h}^{(i)}_{\mathcal{F}} := (h^0_{\mathcal{F}}(0), h^1_{\mathcal{F}}(-1), \ldots, h^i_{\mathcal{F}}(-i))$, (0.14)

b) $\underline{p}^{(i)}_{\mathcal{F}} := (p^i_{\mathcal{F}}(0), p^i_{\mathcal{F}}(-1), \ldots, p^i_{\mathcal{F}}(-i))$.

Our bounding functions $E^{(i)}, G^{(i)}$ and $F^{(i)}$ are defined recursively (cf. (3.1)–(3.9)). Sometimes one might prefer to dispose on explicit bounds - even if they are weaker. As an example of this latter type of bounds we shall establish the following estimates (cf. Proposition 3.3 a), c))

a) $|\Delta^i_{\mathcal{F}}(n)| < \frac{1}{2}\big(2(1+S^{(i)}_{\mathcal{F}})\big)^{2^i}$ for all $n \leq -i$, (0.15)

b) $\nu^i_{\mathcal{F}} \geq -\big(2(1+S^{(i)}_{\mathcal{F}})\big)^{2^i} + 2$,

where

$$S^{(i)}_{\mathcal{F}} := \sum_{j=0}^{i} \binom{i}{j} \big(h^j_{\mathcal{F}}(-j) + |p^i_{\mathcal{F}}(-j)|\big). \tag{0.16}$$

The estimates given in (0.13) b) and in (0.15) b) show in particular, that the $2i+2$ invariants

a) $h^0_{(\bullet)}(0), h^1_{(\bullet)}(-1), \ldots, h^i_{(\bullet)}(-i)$, (0.17)

b) $p^i_{(\bullet)}(0), p^i_{(\bullet)}(-1), \ldots, p^i_{(\bullet)}(-i)$

form a *bounding system for the invariant* $\nu^i_{(\bullet)}$ on the class of all pairs $(X,\mathcal{F}) \in \mathcal{C}$. But it turns out, that the invariants listed in (0.17) do not form a minimal bounding system for the invariant $\nu^i_{(\bullet)}$. In fact, if we fix some $k \in \{0, \ldots, i-1\}$, the $i+1$ invariants listed in (0.17) a) together with the $i-1$ invariants

$p^i_{(\bullet)}(0), \ldots, p^i_{(\bullet)}(-k+1), p^i_{(\bullet)}(-k-1), \ldots, p^{(i)}_{(\bullet)}(-i+1)$ still form a bounding system for the invariant $\nu^i_{(\bullet)}$, (cf. Corollary 4.2). (0.18)

We conclude this from the following finiteness result (cf. Theorem 4.2):

Let $\mathcal{C}'$ be the class of all pairs $(X,\mathcal{F}) \in \mathcal{C}$ for which the $2i$ invariants listed under (0.17) a) and (0.18) take some fixed values. Then, the set of functions

$$\left\{h^i_{X,\mathcal{F}} \lceil : \mathbb{Z}_{\leq -i} \longrightarrow \mathbb{N}_0 | \ (X,\mathcal{F}) \in \mathcal{C}' \ \right\} \tag{0.19}$$

is finite. (For a function $f : \mathbb{D} \longrightarrow \mathbb{M}$ and a set $\mathbb{S} \subseteq \mathbb{D}$, we use $f\lceil : \mathbb{S} \longrightarrow \mathbb{M}$ to denote the restriction of f to $\mathbb{S}$.)

There is much evidence that the invariants listed under (0.17) a) and (0.18) yet do not constitute a minimal bounding system: In [18] it is shown that for projective schemes over an infinite field, the "*full cohomology diagonal*" $(d^{(j)}_{(\bullet)}(-j))_{j\geq 0}$ bounds all cohomological postulation numbers $\nu^i_{(\bullet)}$.

The present paper is written in the same spirit as [7]. So, we first establish bounding results for the cohomological deficiency functions and the cohomological postulation numbers of finitely generated graded modules over noetherian positively graded homogeneous rings $R = R_0 \oplus R_1 \oplus \ldots$ with artinian ground ring R_0 (cf. Theorem 3.1, Theorem 4.1, Corollary 4.1). Then we use the *Serre–Grothendieck correspondence* to translate these results from the language of local cohomology to the language of sheaf cohomology.

In Sec. 1 we give some basic facts on cohomological Hilbert polynomials of finitely generated and graded modules over noetherian positively graded homogeneous rings with artinian ground ring R_0. Using *graded Matlis duality* and *graded local duality* we describe the degrees of these polynomials in terms of certain sets of graded prime ideals. Moreover we give an "associativity formula" for the leading coefficients of these polynomials (cf. Theorem 1.1). In view of the expository character of our paper, we tried to present this subject in much detail, as an introductory treatment seems to lack in the standard literature.

In Sec. 2 we present the main technical tool of the present paper: the method *of admissible sequences of linear forms.* This is a slightly adapted module theoretic version of the method developed by the second author in [20] and is a substitute for the *method of linear systems of hyperplane sections,* which was used in [7]. In fact, this latter method can no longer be used to get satisfactory bounds on the functions $h^i_{X,\mathcal{F}}\lceil_{\mathbb{Z}_{\leq -i}}$ if $i \geq \delta(\mathcal{F})$. We introduce the notion of an admissible sequence of linear forms by means of secondary decompositions of graded artinian modules. The reader who has already some familiarity with the subject will notice, that the same could be achieved on use of *graded local duality.* The main tool in this section is the *"intersection lemma"* Lemma 2.4. This lemma will finally lead to the basic *"rescursive bounding result"* Corollary 2.1.

In Sec. 3 we use Corollary 2.1 to prove the main bounding results mentioned under (0.13) and (0.15).

In the conclusive Sec. 4, we finally shall establish the results mentioned already under (0.18) and (0.19).

At present, not very much seems to be known about cohomological deficiency functions and postulation numbers of arbitrary pairs $(X, \mathcal{F}) \in \mathcal{C}$. Clearly, there is a number of results concerning specific pairs $(X, \mathcal{F})$ (cf. [1, 5] for example). In view of the results presented in this paper (and in [18]) it seems natural to ask, how the functions $\Delta^i_{X,\mathcal{F}}$ and the invariants $\nu^i_{\mathcal{F}}$ depend on the cohomology diagonal $\left(h^i_{X,\mathcal{F}}(-i)\right)_{i\geq 0}$ for an arbitrary pair $(X, \mathcal{F}) \in \mathcal{C}$. The corresponding question also arises concerning the so called *cohomological patterns*

$$P_X(\mathcal{F}) = \left\{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} | h^i_{X,\mathcal{F}}(n) \neq 0\right\}$$

of pairs $(X, \mathcal{F}) \in \mathcal{C}$, (cf. [6]).

As basic references for the present paper we recommend: commutative algebra [11, 19]; local cohomology [9, 14] algebraic geometry [13, 16]; cohomological Hilbert functions [7].

1. Cohomological Hilbert Polynomials and Deficiency Functions

In this section, we present a few basic preliminaries on cohomological Hilbert polynomials and cohomological deficiency functions. So, let $R = \oplus_{n \geq 0} R_n$ be a noetherian positively graded homogeneous ring such that the base ring R_0 is artinian. We first recall some basic notions.

1.1. Notation, Definition and Remark

A) Let $M = \oplus_{n \in \mathbb{Z}} M_n$ be a finitely generated, graded R-module. For $i \in \mathbb{N}_0$ let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to the "irrelevant ideal" $R_+ := \oplus_{n>0} R_n$ of R. Moreover, let $\mathcal{R}^i D_{R_+}(\bullet)$ denote the i-th right derived functor of the R_+-transform functor $D_{R_+}(\bullet) := \operatorname{inj\,lim}_n Hom_R(R^n_+, \bullet)$. Keep in mind, that the R-modules $H^i_{R_+}(M)$ and $\mathcal{R}^i D_{R_+}(M)$ carry a natural grading for all $i \in \mathbb{N}_0$ (cf. [9, (12.3.3), (12.2.10), (12.4.5)]). Moreover we have a natural graded exact sequence (cf. [9, (12.4.2)])

$$0 \longrightarrow H^0_{R_+}(M) \longrightarrow M \longrightarrow D_{R_+}(M) \longrightarrow H^1_{R_+}(M) \longrightarrow 0$$

and natural graded isomorphisms [9, (12.4.5)]

$$\mathcal{R}^i D_{R_+}(M) \cong H^{i+1}_{R_+}(M), \quad (\forall i \in \mathbb{N}).$$

B) Keep the previous notations and hypotheses. By $\ell_{R_0}(N)$ we always denote the length of an R_0-module N. Moreover, for a graded R-module T we use T_n to denote the n-th homogeneous part of T for each $n \in \mathbb{Z}$. Using these notations we set (cf. [9, (15.1.5), (17.1.4)])

$$h^i_M(n) := \ell_{R_0}(H^i_{R_+}(M)_n); \; d^i_M(n) := \ell_{R_0}(\mathcal{R}^i D_{R_+}(M)_n), \quad (\forall i \in \mathbb{N}_0, \forall n \in \mathbb{Z}).$$

Now, the exact sequence and the isomorphisms given at the end of part A) give rise to the equalities

$$(\bullet) \qquad d^0_M(n) = \ell_{R_0}(M_n) - h^0_M(n) + h^1_M(n), \quad (\forall n \in \mathbb{Z});$$
$$(\bullet\bullet) \qquad d^i_M(n) = h^{i+1}_M(n), (\forall n \in \mathbb{Z}, \forall i > 0).$$

Moreover we have (cf. [9, (15.1.5)])

$$h^i_M(n) = 0 \quad \text{for all} \quad n \gg 0, \quad (\forall i \in \mathbb{N}_0),$$

and hence also

$$d^0_M(n) = \ell_{R_0}(M_n), \quad (\forall n \gg 0); \; d^i_M(n) = 0, \quad (\forall n \gg 0, \forall i > 0).$$

C) Next, let $\mathfrak{m}^{(1)}_0, \ldots, \mathfrak{m}^{(r)}_0$ be the different maximal ideals of R_0. For each $j \in \{1, \ldots, r\}$ let $R^{'(j)}$ denote the positively graded homogeneous noetherian ring $R \otimes_{R_0} (R_0)_{\mathfrak{m}^{(j)}_0} = \oplus_{n \geq 0} (R_n)_{\mathfrak{m}^{(j)}_0}$ with artinian local base ring $(R_0)_{\mathfrak{m}^{(j)}_0}$.

Moreover, let $M'^{(j)}$ denote the finitely generated and graded $R'^{(j)}$-module $M \otimes_{R_0} (R_0)_{\mathfrak{m}_0^{(j)}} = \oplus_{n\in\mathbb{Z}}(M_n)_{\mathfrak{m}_0^{(j)}}$. Then we have (cf. [9, (16.2.5)]; [7, (1.25)])

$$h^i_M(n) = \sum_{j=1}^{r} h^i_{M'^{(j)}}(n); \quad d^i_M(n) = \sum_{j=1}^{r} d^i_{M'^{(j)}}(n); \quad (\forall n \in \mathbb{Z}, \forall i \in \mathbb{N}_0). \qquad (*)$$

D) Finally assume that the base ring R_0 is local with maximal ideal $\mathfrak{m}_0$. Let R' be an artinian local flat extension ring of R_0 with maximal ideal $\mathfrak{m}'_0 = \mathfrak{m}_0 R'_0$. Then $R' := R'_0 \otimes_{R_0} R = \oplus_{n\geq 0}(R'_0 \otimes_{R_0} R_n)$ is a positively graded homogeneous noetherian ring with base ring R'_0. Moreover $M' := R' \otimes_R M = R'_0 \otimes_{R_0} M = \oplus_{n\in\mathbb{Z}}(R'_0 \otimes_{R_0} M_n)$ is a finitely generated and graded R'-module. In this situation we have (cf. [7, (2.29)])

$$h^i_{M'}(n) = h^i_M(n); \quad d^i_{M'}(n) = d^i_M(n); \quad (\forall n \in \mathbb{Z}, \forall i \in \mathbb{N}_0). \qquad \bullet$$

Our main goal in this section is to study the so called *cohomological Hilbert polynomials* of a finitely generated graded R-module M and to express their degrees and leading terms by some other "local" invariants. The basic notion needed to define these local invariants is that of cohomological pseudo-support which we shall introduce now.

1.2. Definition and Remark

A) Let $i \in \mathbb{N}_0$ and let M be a finitely generated and graded R-module. We define the i-th *(cohomological) pseudo-support of* M as the set

$$^+\mathrm{Psupp}^i(M) := \{\mathfrak{p} \in \ \mathrm{Proj}\,(R) | \dim(R/\mathfrak{p}) \leq i \ \text{ and } \ H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0\}.$$

Moreover, we define the i-th *(cohomological) pseudo dimension of* M by

$$^+\mathrm{psd}^i(M) := \sup\{\dim(R/\mathfrak{p}) | \mathfrak{p} \in \ ^+\mathrm{Psupp}^i(M)\} - 1$$

using the convention that $\sup \emptyset = -\infty$. Observe that

$$^+\mathrm{psd}^i(M) < i.$$

Finally we set

$$^+\mathrm{Psupp}^i_0(M) := \{\mathfrak{p} \in \ ^+\mathrm{Psupp}^i(M) \mid \dim(R/\mathfrak{p}) = \ ^+\mathrm{psd}^i(M) + 1\}.$$

B) Let $i > \dim(M)$ and let $\mathfrak{p} \in \mathrm{Proj}(R)$ with $\dim(R/\mathfrak{p}) \leq i$. Then $\dim(M_\mathfrak{p}) \leq \dim(M) - \dim(R/\mathfrak{p}) < i - \dim(R/\mathfrak{p})$ and hence Grothendieck's Vanishing Theorem (cf. [9, (6.1.2)]) implies $H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) = 0$. This proves that

$$^+\mathrm{Psupp}^i(M) = \emptyset, \quad \text{if} \quad i > \dim(M).$$

C) Let $\mathfrak{m}_0^{(1)}, \ldots, \mathfrak{m}_0^{(j)}$ be the different maximal ideals of the base ring R_0. As in 1.1 C) we write $R'^{(j)} := R \otimes_{R_0} (R_0)_{\mathfrak{m}_0^{(j)}}$ for $j = 1, \ldots, r$. Let $\eta^{(j)} : R \longrightarrow R'^{(j)}$ denote the natural homomorphism. Then, the induced homomorphism

$$\eta : R \xrightarrow{\cong} \bigoplus_{j=1}^{r} R^{'(j)} \quad (x \longmapsto (\eta^{(1)}(x), \ldots, \eta^{(r)}(x)))$$

is an isomorphism of graded rings and thus induces an isomorphism of schemes

$$\varepsilon : \coprod_{1\leq j\leq r} \operatorname{Proj}(R^{'(j)}) \xrightarrow{\approx} \operatorname{Proj}(R),$$

which sends $\mathfrak{p}^{'(j)} \in \operatorname{Proj}(R^{'(j)})$ to $(\eta^{(j)})^{-1}(\mathfrak{p}^{'(j)}) = \mathfrak{p}^{'(j)} \bigcap R \in \operatorname{Var}(\mathfrak{m}_0^{(j)}R) \bigcap \operatorname{Proj}(R)$. In particular we have

$$(*) \qquad \varepsilon(\operatorname{Proj}(R^{'(j)})) = \operatorname{Var}^+(\mathfrak{m}_0^{(j)}R), \quad (j = 1, \ldots, r),$$

where, for a graded ideal $\mathfrak{a} \subseteq R$, $\operatorname{Var}^+(\mathfrak{a})$ is used to denote the closed subset $\operatorname{Var}(\mathfrak{a}) \cap \operatorname{Proj}(R)$ of $\operatorname{Proj}(R)$. Moreover, for each $\mathfrak{p}^{'(j)} \in \operatorname{Proj}(R^{'(j)})$ we have natural isomorphisms

$$(**) \qquad R^{'(j)}/\mathfrak{p}^{'(j)} \cong R/\varepsilon(\mathfrak{p}^{'(j)}),$$

so that finally

$$(**)' \qquad \dim(R^{'(j)}/\mathfrak{p}^{'(j)}) = \dim(R/\varepsilon(\mathfrak{p}^{'(j)})).$$

D) Now, let M be a finitely generated and graded R-module. Let $j \in \{1, \ldots, r\}$. As in 1.1 C) we write $M^{'(j)}$ for the finitely generated and graded $R'^{(j)}$-module $R^{'(j)} \otimes_R M$. For each $\mathfrak{p}^{'(j)} \in \operatorname{Var}^+(\mathfrak{m}_0^{(j)}R)$ we have a natural isomorphism of $R_{\varepsilon(\mathfrak{p}^{'(j)})}$-modules

$$(\bullet) \qquad (M^{'(j)})_{\mathfrak{p}^{'(j)}} \cong M_{\varepsilon(\mathfrak{p}^{'(j)})},$$

which in the case $M = R$ allows to identify the two local rings $(R^{'(j)})_{\mathfrak{p}^{'(j)}}$ and $R_{\varepsilon(\mathfrak{p}^{'(j)})}$. Doing so, we thus get natural isomorphisms

$$(\bullet\bullet) \qquad H^k_{\mathfrak{p}^{'(j)}R^{'(j)}_{\mathfrak{p}^{'(j)}}}((M^{'(j)})_{\mathfrak{p}^{'(j)}}) \cong H^k_{\varepsilon(\mathfrak{p}^{'(j)})R_{\varepsilon(\mathfrak{p}^{'(j)})}}(M_{\varepsilon(\mathfrak{p}^{'(j)})}), \ (\forall k \in \mathbb{N}_0).$$

By $(*)$ and $(**)'$ we thus get for all $i \in \mathbb{N}_0$

$$(\bullet\bullet\bullet) \qquad \varepsilon({}^+\operatorname{Psupp}^i(M^{'(j)})) = \operatorname{Var}^+(\mathfrak{m}_0^{(j)}R) \cap {}^+\operatorname{Psupp}^i(M)$$

and therefore

$$(\bullet\bullet\bullet)' \qquad {}^+\operatorname{psd}^i(M) = \max\{{}^+\operatorname{psd}^i(M^{'(j)}) \mid j = 1, \ldots, r\}. \qquad \bullet$$

We now consider the special case, in which R_0 is local and bring into the game graded local duality.

1.3. Remark and Notation

A) Assume that the (artinian) base ring R_0 is local with maximal ideal $\mathfrak{m}_0$. Then, by Cohen's Structure Theorem we may write R_0 as a homomorphic image of a complete regular local ring $(R'_0, \mathfrak{m}'_0)$. So, for some polynomial ring $R' := R'_0[\mathbf{x}_0, \ldots, \mathbf{x}_t]$ we have a surjective homogeneous ring homomorphism

$$f : R' \twoheadrightarrow R.$$

We set $d' := \dim(R')$ and we use $\mathfrak{m}$ and $\mathfrak{m}'$ to denote the homogeneous maximal ideals $\mathfrak{m}_0 + R_+$ and $\mathfrak{m}'_0 + R'_+$ of R and R'. Finally, let E denote the $*$-injective envelope of the graded R-module $R/\mathfrak{m}$ (cf. [9, (13.2.1)]). Then, by the *graded local duality theorem*, there is an integer a' such that for each finitely generated graded R-module M and each $i \in \mathbb{N}_0$, there is an isomorphism of graded R-modules (cf. [9, (13.4.3)])

$$\Psi^i : H^i_{R_+}(M) \xrightarrow{\cong} \mathrm{Hom}_R(\mathrm{Ext}^{d'-i}_{R'}(M, R'(a')), E).$$

(We use the standard convention that $\mathrm{Ext}^k_{R'}(*, \bullet) = 0$ for $k < 0$).

B) Keep the above notations and hypotheses and fix a finitely generated and graded R-module M. Then for each $i \in \mathbb{N}_0$ we consider the finitely generated and graded R-module

$$K^i_M := \mathrm{Ext}^{d'-i}_{R'}(M, R'(a')),$$

so that by part A) we have an isomorphism of graded R-modules

$$(*) \qquad \Psi^i : H^i_{R_+}(M) \xrightarrow{\cong} \mathrm{Hom}_R(K^i_M, E).$$

Now, let E_0 be the injective envelope of the R_0-module $R_0/\mathfrak{m}_0$. Then, for each $i \in \mathbb{N}_0$ and each $n \in \mathbb{Z}$, the *graded Matlis duality theorem* gives an isomorphism of R_0-modules

$$(**) \qquad \mathrm{Hom}_R(K^i_M, E)_n \cong \mathrm{Hom}_{R_0}((K^i_M)_{-n}, E_0)$$

(cf. [9, (13.4.5)]). As the Matlis duality functor $\mathrm{Hom}_{R_0}(\bullet, E_0)$ is length-preserving (cf. [9, (10.2.3)]) we thus may use the graded isomorphism Ψ^i of $(*)$ to write

$$(**)' \qquad h^i_M(n) = \ell_{R_0}((K^i_M)_{-n}).$$

Now, let $P_{K^i_M} \in \mathbb{Q}[\mathbf{x}]$ be the Hilbert polynomial of the finitely generated and graded R-module K^i_M, so that $P_{K^i_M}(n) = \ell_{R_0}((K^i_M)_n)$ for all $n \gg 0$. Then $(**)'$ gives

$$(***) \qquad h^i_M(n) = P_{K^i_M}(-n), \quad (\forall n \ll 0, \forall i \in \mathbb{N}_0).$$

C) We keep the above notations and hypotheses and choose $\mathfrak{p} \in \mathrm{Proj}(R)$. Let $\mathfrak{p}' = f^{-1}(\mathfrak{p}) \in \mathrm{Proj}(R')$ the preimage of $\mathfrak{p}$. Then $R'_{\mathfrak{p}'}$ is a local Gorenstein ring such that $\dim(R'_{\mathfrak{p}'}) = d' - \dim(R'/\mathfrak{p}') = d' - \dim(R/\mathfrak{p})$.

Now, let $i \in \mathbb{N}_0$ be such that $\dim(R/\mathfrak{p}) \leq i$ and let $E^{(\mathfrak{p})}$ denote the injective envelope of the $R_\mathfrak{p}$-module $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Then, by local duality we have an isomorphism of $R_\mathfrak{p}$-modules

$$H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \cong \mathrm{Hom}_{R_\mathfrak{p}}(Ext^{d'-i}_{R'_{\mathfrak{p}'}}(M_\mathfrak{p}, R'_{\mathfrak{p}'}), E^{(\mathfrak{p})})$$

(cf. [9, (11.2.6)]). As $R'(a')_{\mathfrak{p}'} \cong R'_{\mathfrak{p}'}$ and as M is finitely generated over R, there is an isomorphism of $R_\mathfrak{p}$-modules $Ext^{d'-i}_{R'_{\mathfrak{p}'}}(M_\mathfrak{p}, R'_{\mathfrak{p}'}) \cong Ext^{d'-i}_{R'}(M, R'(a'))_{\mathfrak{p}'} = (K^i_M)_\mathfrak{p}$. So, we end up with an isomorphism of $R_\mathfrak{p}$-modules

$$(\bullet) \qquad \theta^i_\mathfrak{p} : H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \cong \mathrm{Hom}_{R_\mathfrak{p}}((K^i_M)_\mathfrak{p}, E^{(\mathfrak{p})}).$$

As the Matlis dualtity functor $\mathrm{Hom}_{R_\mathfrak{p}}(\bullet, E^{(\mathfrak{p})})$ transforms non-zero modules to non-zero modules we thus see that

$$(\bullet\bullet) \qquad {}^+\mathrm{Psupp}^i(M) = \mathrm{supp}(K^i_M) \cap \mathrm{Proj}(R), \quad (\forall i \in \mathbb{N}_0).$$

In particular we see:

$(\bullet\bullet)'$ For each $i \in \mathbb{N}_0$, ${}^+\mathrm{Psupp}^i(M)$ is closed in $\mathrm{Proj}(R)$.

D) Keep the above hypotheses and notations, let $i \in \mathbb{N}_0$ and let $\mathfrak{p} \in \mathrm{Proj}(R)$. As the Matlis duality functor $\mathrm{Hom}_{R_\mathfrak{p}}(\bullet, E^{(\mathfrak{p})})$ is length-preserving, we may use the isomorphism $\theta^i_\mathfrak{p}$ of $(\bullet)$ to conclude

$$(\blacksquare) \qquad \ell_{R_\mathfrak{p}}(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) = \ell_{R_\mathfrak{p}}((K^i_M)_\mathfrak{p}).$$

In view of the equality $(\bullet\bullet)$ we get for each $\mathfrak{p} \in \mathrm{Proj}(R)$

$$(\blacksquare\blacksquare) \qquad \mathfrak{p} \in \min {}^+\mathrm{Psupp}^i(M) \Longleftrightarrow 0 < \ \ell_{R_\mathfrak{p}}(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) < \infty,$$

where $\min \mathcal{S}$ is used to denote the set of minimal members of a set $\mathcal{S} \subseteq \mathrm{Spec}(R)$.$\bullet$

1.4. Reminder, Remark and Notation

A) Let $\mathfrak{m}_0^{(1)}, \ldots, \mathfrak{m}_0^{(r)}$ be the different maximal ideals of the base ring R_0 and let M be a finitely generated and graded R-module. Let the positively graded homogeneous ring $R^{'(j)}$ and the finitely generated $R^{'(j)}$-module $M'^{(j)}$ be defined as in 1.1 C). Then each of the base rings $(R^{'(j)})_0 = (R_0)_{\mathfrak{m}_0^{(j)}}$ is artinian and local. So, by the first equality of 1.1 C) $(*)$ and in the notation introduced in 1.3 A), B) we see on use of 1.3 B) $(***)$, that

$$(*) \qquad h^i_M(n) = \sum_{j=1}^{r} P_{K^i_{M'^{(j)}}}(-n); \quad (\forall n \ll 0, \ \ \forall i \in \mathbb{N}_0).$$

In particular we recover once more the well known fact that for each $i \in \mathbb{N}_0$ there is a polynomial $p^i_M \in \mathbb{Q}[\mathbf{x}]$ such that

$$(**) \qquad h^i_M(n) = p^i_M(n), \quad (\forall n \ll 0),$$

the so called i-th *cohomological Hilbert polynomial of* M, (cf. [9, (17.1.9)]). By $(*)$ we may write

$$(**)' \qquad p^i_M(\mathbf{x}) = \sum_{j=1}^{r} P_{K^i_{M'^{(j)}}}(-\mathbf{x}); \quad (\forall i \in \mathbb{N}_0).$$

B) Now, let $T = \oplus_{n\in\mathbb{Z}} T_n$ be a finitely generated and graded R-module and let $P_T \in \mathbb{Q}[\mathbf{x}]$ denote the *Hilbert polynomial of* T, so that $P_T(n) = \ell_{R_0}(T_n)$ for all $n \gg 0$. Let us recall that

$$(\bullet) \qquad \deg(P_T) = \dim(\mathrm{supp}(T) \cap \mathrm{Proj}(R))$$

(cf. [10, (4.15)]). Moreover, if $P_T \neq 0$, let $e_0(T) \in \mathbb{N}$ be the *multiplicity of* T, so that

$(\bullet\bullet)$ $\quad e_0(T)/\deg(P_T)!$ is the leading coefficient of P_T.

Let us also recall the *associativity formula* which, in case $P_T \neq 0$, states that

$$(\bullet\bullet\bullet) \qquad e_0(T) = \sum_{\mathfrak{p}\in \mathrm{supp}_0(T)} \ell_{R_\mathfrak{p}}(T_\mathfrak{p}) e_0(R/\mathfrak{p}),$$

where

$$\mathrm{supp}_0(T) := \{\mathfrak{p} \in \mathrm{supp}(T) \mid \dim(R/\mathfrak{p}) = \dim(T)\}$$

(cf. [10, (4.6.8)]).

C) Finally, for $\mathfrak{p} \in \mathrm{Proj}(R)$ and for $k \in \mathbb{N}_0$ set

$$h^k_{\mathfrak{p}}(M) := \ell_{R_{\mathfrak{p}}}(H^k_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})).$$

As usual, we set $H^k_{\mathfrak{p}R_{\mathfrak{p}}}(\bullet) = 0$ for $k < 0$, so that we are allowed to write $h^k_{\mathfrak{p}}(M) = 0$ for $k < 0$. $\bullet$

Now we are ready to summarize our previous observations in the following result.

Theorem 1.1. *Let M be a finitely generated and graded R-module and let $i \in \mathbb{N}_0$. Then:*
a) ${}^+\mathrm{Psupp}^i(M)$ *is a closed subset of dimension* ${}^+\mathrm{psd}^i(M)$ *in* $\mathrm{Proj}(R)$.
b) $\mathfrak{p} \in \mathrm{Proj}(R)$ *is a minimal member of* ${}^+\mathrm{Psupp}^i(M)$ *if and only if*

$$0 < h^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}}(M) < \infty$$

c) *The polynomial* $p^i_M \in \mathbb{Q}[\mathbf{x}]$ *is of degree* ${}^+\mathrm{psd}^i(M)$.
d) *If* ${}^+\mathrm{psd}^i(M) \geq 0$, *then* p^i_M *has leading coefficient*

$$(-1)^{{}^+\mathrm{psd}^i(M)} e^i_0(M)/{}^+\mathrm{psd}^i(M)!,$$

where

$$e^i_0(M) = \sum_{\mathfrak{p} \in {}^+\mathrm{Psupp}^i_0(M)} h^{i-{}^+\mathrm{psd}^i(M)-1}_{\mathfrak{p}}(M) e_0(R/\mathfrak{p}).$$

Proof. Let $\mathfrak{m}^{(1)}_0, \ldots, \mathfrak{m}^{(r)}_0$ be the different maximal ideals of R_0. For $j \in \{1, \ldots, r\}$ let $R^{'(j)}$ and $M^{'(j)}$ be defined as in 1.1 C). Let $\varepsilon : \prod_{1 \leq j \leq r} \mathrm{Proj}(R^{'(j)}) \longrightarrow \mathrm{Proj}(R)$ be the natural isomorphism of 1.2 C).

"a)": By 1.3 C) $(\bullet\bullet)'$ it follows that ${}^+\mathrm{Psupp}^i(M^{'(j)})$ is a closed subset of $\mathrm{Proj}(R^{'(j)})$ for each $j \in \{1, \ldots, r\}$. But now, by 1.2 D) $(\bullet\bullet\bullet)$, we see immediately, that ${}^+\mathrm{Psupp}^i(M)$ is a closed subset of $\mathrm{Proj}(R)$. So, the statement on the dimension of ${}^+\mathrm{Psupp}^i(M) \subseteq \mathrm{Proj}(R)$ is clear.

"b)": Let $\mathfrak{p} \in \mathrm{Proj}(R)$. Then, there is some index $j \in \{1, \ldots, r\}$ and some prime $\mathfrak{p}^{'(j)} \in \mathrm{Proj}(R^{'(j)})$ with $\varepsilon(\mathfrak{p}^{'(j)}) = \mathfrak{p}$. By 1.2 C) $(**)'$ we have $\dim(R/\mathfrak{p}) = \dim(R^{'(j)}/\mathfrak{p}^{'(j)})$. By 1.2 D) $(\bullet\bullet)$ we may write

$$h^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}}(M) = \ell_{R^{'(j)}_{\mathfrak{p}^{'(j)}}}\big(H^{i-\dim(R^{'(j)}/\mathfrak{p}^{'(j)})}_{\mathfrak{p}^{'(j)}R^{'(j)}_{\mathfrak{p}^{'(j)}}}((M^{'(j)})_{\mathfrak{p}^{'(j)}})\big).$$

So, by 1.3 D) $(\blacksquare\blacksquare)$ we see that $\mathfrak{p}^{'(j)}$ is a minimal member of ${}^+\mathrm{Psupp}^i(M^{'(j)})$ if and only if $0 < h^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}}(M) < \infty$. In view of 1.2 D) $(\bullet\bullet\bullet)$, this proves our claim.

"c)": Fix $j \in \{1, \ldots, r\}$. Using the notation of 1.3 A), B) we see by 1.4 B) $(\bullet)$ and 1.3 C) $(\bullet\bullet)$ that

$$\deg(P_{K^i_{M^{'(j)}}}) = \dim(\mathrm{supp}(K^i_{M^{'(j)}}) \cap \mathrm{Proj}(R^{'(j)})) = \dim({}^+\mathrm{Psupp}^i(M^{'(j)})).$$

So, in view of statement a) of the present theorem, $\deg(P_{K^i_{M'^{(j)}}}) = {}^+\mathrm{psd}^i(M'^{(j)})$. Now we may conclude by 1.2 D) $(\bullet\bullet\bullet)'$ and by 1.4 A) $(**)'$.

"d)": In view of 1.2 D) $(\bullet\bullet\bullet)'$ we may assume that there is some $s \in \{1, \ldots, r\}$ such that ${}^+\mathrm{psd}^i(M'^{(j)}) = {}^+\mathrm{psd}^i(M)$ for all $j \in \{1, \ldots, s\}$ and ${}^+\mathrm{psd}^i(M'^{(j)}) < {}^+\mathrm{psd}^i(M)$ for all $j \in \{s+1, \ldots, r\}$. In the proof of statement "c)" we have learned that $\deg(P_{K^i_{M'^{(j)}}}) = {}^+\mathrm{psd}^i(M'^{(j)})$ for all $j \in \{1, \ldots, r\}$. So, by 1.4 A) $(**)'$ and 1.4 B)$(\bullet\bullet)$ we see that p^i_M has leading coefficient

$$(-1)^{{}^+\mathrm{psd}^i(M)} e^i_0(M)/{}^+\mathrm{psd}^i(M)!$$

with

$$e^i_0(M) = \sum_{j=1}^{s} \sum_{\mathfrak{p}'^{(j)} \in \mathrm{supp}_0(K^i_{M'^{(j)}})} \ell_{R'^{(j)}_{\mathfrak{p}'^{(j)}}}((K^i_{M'^{(j)}})_{\mathfrak{p}'^{(j)}}) e_0(R'^{(j)}/\mathfrak{p}'^{(j)}).$$

By 1.3 C) $(\bullet\bullet)$ it is immediate that for all $j \in \{1, \ldots, s\}$

$$\mathrm{supp}_0(K^i_{M'^{(j)}}) = {}^+\mathrm{Psupp}^i_0(M'^{(j)})$$

By 1.3 D) $(\blacksquare)$ and by 1.2 D) $(\bullet\bullet)$ we also have

$$\ell_{R'^{(j)}_{\mathfrak{p}'^{(j)}}}((K^i_{M'^{(j)}})_{\mathfrak{p}'^{(j)}}) = h^{i-{}^+\mathrm{psd}^i(M)-1}_{\varepsilon(\mathfrak{p}'^{(j)})}(M'^{(j)})$$

for all $j \in \{1,...,s\}$ and all $\mathfrak{p}'^{(j)} \in {}^+\mathrm{Psupp}^i_0(M'^{(j)})$. By 1.2 C) $(**)'$, we know that

$$e_0(R'^{(j)}/\mathfrak{p}'^{(j)}) = e_0(R/\varepsilon(\mathfrak{p}'^{(j)}))$$

for all $j \in \{1, \ldots, s\}$ and all $\mathfrak{p}'^{(j)} \in {}^+\mathrm{Psupp}^i_0(M'^{(j)})$. By 1.2 D) $(\bullet\bullet\bullet)$ and by the fact that ε is a homeomorphism,

$${}^+\mathrm{Psupp}^i_0(M) = \dot{\bigcup_{1\le j\le s}} \varepsilon({}^+\mathrm{Psupp}^i_0(M'^{(j)})).$$

So, altogether we obtain

$$e^i_0(M) = \sum_{\mathfrak{p} \in {}^+\mathrm{Psupp}^i_0(M)} h^{i-{}^+\mathrm{psd}^i(M)-1}_{\mathfrak{p}}(M) e_0(R/\mathfrak{p}). \qquad \blacksquare$$

Remark 1.1. Let M be a finitely generated and graded R-module. Then the *minimum R_+-adjusted depth of* R is defined by

$$\lambda(M) := \inf\{\mathrm{depth}(M_\mathfrak{p}) + \mathrm{height}((\mathfrak{p} + R_+)/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{Var}(R_+)\}$$

(cf. [9, (9.2.2)]). As M is a graded R-module, the right hand side infimum is attained by a graded prime $\mathfrak{p}$ (cf. [9, (13.1.17)]). As R_0 is artinian, $\mathfrak{p} + R_+$ is the unique maximal ideal of R which contains the graded prime $\mathfrak{p}$. So, we have

$(\bullet)$ $$\lambda(M) = \inf\{\mathrm{depth}(M_\mathfrak{p}) + \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Proj}(R)\}.$$

As $\mathrm{depth}(M_\mathfrak{p}) = \min\{k \in \mathbb{N}_0 \mid H^k_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0\}$ we may write

$$\begin{aligned}\lambda(M) &= \inf\{k + \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Proj}(R) \text{ and } k \in \mathbb{N}_0 \text{ with } H^k_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0\} \\ &= \inf\{i \mid \exists \mathfrak{p} \in \mathrm{Proj}(R) : H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0\},\end{aligned}$$

hence

$$(\bullet\bullet) \qquad \lambda(M) = \inf\{i \mid {}^{+}\mathrm{Psupp}^{i}(M) \neq \emptyset\}.$$

So Theorem 1.1 c) gives us

$$(\bullet\bullet\bullet) \qquad \lambda(M) = \inf\{i \mid p_M^i \neq 0\}.$$

Obviously, this last statement also can be deduced from *Grothendieck's Finiteness Theorem* (cf. [9, (9.5.2)]). •

1.5. Notation and Remark

For a finitely generated and graded R-module M and for $i \in \mathbb{N}_0$ we set

$$q_M^i := p_M^{i+1}$$

and call q_M^i the i-th *cohomological Hilbert polynomial of the second kind for* M. By the relations 1.1 B) $(\bullet), (\bullet\bullet)$ it follows immediately

$$(\bullet) \qquad d_M^i(n) = q_M^i(n), \quad (\forall n \ll 0, \forall i \in \mathbb{N}_0).$$

By Theorem 1.1 c) and by the last inequality of 1.2 A)

$$(\bullet\bullet) \qquad \deg(q_M^i) = {}^{+}\mathrm{psd}^{i+1}(M) \leq i.$$

Moreover, by Theorem 1.1 d) we can say that the leading coefficient of q_M^i is given by

$$(\bullet\bullet\bullet) \qquad \frac{(-1)^{{}^{+}\mathrm{psd}^{i+1}(M)}}{{}^{+}\mathrm{psd}^{i+1}(M)!} \sum_{\mathfrak{p} \in {}^{+}\mathrm{Psupp}_0^i(M)} h_{\mathfrak{p}}^{i - {}^{+}\mathrm{psd}^{i+1}(M)}(M) e_0(R/\mathfrak{p}),$$

whenever ${}^{+}\mathrm{psd}^{i+1}(M) \geq 0$. •

For rest of this paper it will be more convenient, to consider the cohomological Hilbert polynomials q_M^i of the second kind instead of the polynomials p_M^i. So, we formulate all further results of this section in terms of the polynomials q_M^i.

Corollary 1.1. *For each finitely generated and graded* R*-module and for each* $i \in \mathbb{N}_0$ *the following statements are equivalent*

i) $\deg(q_M^i) < i$;

ii) $\dim(R/\mathfrak{p}) \neq i + 1$ *for all* $\mathfrak{p} \in \mathrm{Ass}_R(M)$.

Proof. Observe that ${}^{+}\mathrm{psd}^{i+1}(M) = i$ if and only if there is some $\mathfrak{p} \in \mathrm{Proj}(R)$ such that $\dim(R/\mathfrak{p}) = i + 1$ and $H^0_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$. As $H^0_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$ is equivalent to $\mathfrak{p} \in \mathrm{Ass}(M)$ and as $\dim(R/\mathfrak{p}) = 0 \neq i + 1$ whenever $\mathfrak{p} \in \mathrm{Var}(R_+)$, we get our claim by 1.5 $(\bullet\bullet)$. ■

Now we shall introduce the cohomological deficiencies, whose study is the objective of this paper.

1.6. Definition and Remark

A) Let M be a finitely generated and graded R-module and let $i \in \mathbb{N}_0$. We define the i-th *cohomological deficiency function* Δ^i_M of M as follows

$$\Delta^i_M : \mathbb{Z} \longrightarrow \mathbb{Z};\ n \longmapsto \Delta^i_M(n) := d^i_M(n) - q^i_M(n).$$

B) In view of 1.5 (•) we have

$$(*) \qquad \Delta^i_M(n) = 0 \text{ for all } n \ll 0.$$

So, using the standard convention that $\inf \emptyset = +\infty$, we may define the i-th *cohomological postulation number* of M by

$$\nu^i_M := \inf\{m \in \mathbb{Z} \mid \Delta^i_M(m+1) \neq 0\} \in \mathbb{Z} \cup \{+\infty\}.$$

For our convenience we also set $\nu^{-1}_M := \infty$.

C) In view of Remark 1.1 (•••) we have

$$(**) \qquad \lambda(M) - 1 = \inf\{i \mid q^i_M \neq 0\}.$$

As a consequence of this we have in the notation of [7, (2.9)]

$$(***) \qquad \nu^i_M = \operatorname{beg}(\mathcal{R}^i D_{R_+}(M)) \text{ for all } i < \lambda(M) - 1.$$

So, for the *coregularity of M at and below level k* (cf. [7, (2.10)] b)) we get

$$(***)' \qquad \operatorname{coreg}^k(M) = \inf\{\nu^i_M + i - 1 \mid i \leq k\}, \text{ for all } k < \lambda(M) - 1. \qquad \bullet$$

Next, we summarize the above results and observations in sheaf theoretic terms.

1.7. Reminder, Definition and Remark

A) (cf. [7, (2.39 - 2.48)]). We denote by X the projective scheme $\operatorname{Proj}(R)$. For any $i \in \mathbb{N}_0$ and for any coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we introduce the i-th *cohomological Hilbert function of* $\mathcal{F}$

$$h^i_{\mathcal{F}} := h^i_{X,\mathcal{F}} : \mathbb{Z} \longrightarrow \mathbb{N}_0; \quad n \longmapsto h^i_{\mathcal{F}}(n) := \ell_{R_0}(H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n))).$$

Moreover, for each $\mathcal{F}$ as above, consider the *subdepth of* $\mathcal{F}$, which is defined by

$$\delta(\mathcal{F}) := \min\{\operatorname{depth}(\mathcal{F}_x) \mid x \in X, x \quad \text{closed}\}.$$

If M is a finitely generated and graded R-module and if $\widetilde{M}$ denotes the coherent sheaf of $\mathcal{O}_X$-modules induced by M, we have (cf. [9, (20.4.4), (20.4.18)])

$$(*) \qquad h^i_{\widetilde{M}}(n) = d^i_M(n), \quad (\forall i \in \mathbb{N}_0, \forall n \in \mathbb{Z});$$

$$(**) \qquad \delta(\widetilde{M}) = \lambda(M) - 1.$$

B) Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, $\mathcal{F}$ is induced by some finitely generated and graded R-module M. So, by statement $(*)$ of part A) and by 1.5 (•), (••) we see that there is some polynomial $p^i_{\mathcal{F}} \in \mathbb{Q}[\mathbf{x}]$, such that

(•) $$h^i_{\mathcal{F}}(n) = p^i_{\mathcal{F}}(n), \quad (\forall n \ll 0),$$

the so called i-th *cohomological Hilbert polynomial of* $\mathcal{F}$, (cf. [9, (20.4.14)]). For a finitely generated and graded R-module M we may write by statement (∗) of part A)

(••) $$p^i_{\widetilde{M}} = q^i_M = p^{i+1}_M, \quad (\forall i \in \mathbb{N}_0).$$

In view of statement (∗∗) of part A) we see that

(•••) $$\delta(\mathcal{F}) = \min\{i \in \mathbb{N}_0 \mid p^i_{\mathcal{F}} \neq 0\}$$

for each coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules.

C) For a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, for $i \in \mathbb{N}_0$ and for a point $x \in X$ we introduce the notations

$$H^i_x(\mathcal{F}) := H^i_{\mathfrak{m}_{X,x}}(\mathcal{F}_x); \quad h^i_x(\mathcal{F}) := \ell_{\mathcal{O}_{X,x}}(H^i_x(\mathcal{F})).$$

We define the i-th *(cohomological) pseudo support of* $\mathcal{F}$ by

$$\mathrm{Psupp}^i(\mathcal{F}) := \{x \in X \mid \dim(\overline{\{x\}}) \leq i \text{ and } H_x^{i-\dim(\overline{\{x\}})}(\mathcal{F}) \neq 0\}$$

and the i-th *(cohomological) pseudo dimension of* $\mathcal{F}$ by

$$\mathrm{psd}^i(\mathcal{F}) := \sup\{\dim(\overline{\{x\}}) \mid x \in \mathrm{Psupp}^i(\mathcal{F})\}.$$

Finally, we set

$$\mathrm{Psupp}^i_0(\mathcal{F}) := \{x \in \mathrm{Psupp}^i(\mathcal{F}) \mid \dim(\overline{\{x\}}) = \mathrm{psd}^i(\mathcal{F})\}.$$

Let us recall that for $x = \mathfrak{p} \in \mathrm{Proj}(R)$ we have

(■) $$\dim(\overline{\{x\}}) = \dim(R/\mathfrak{p}) - 1;$$

(■)′ $$R_{\mathfrak{p}} \cong \mathcal{O}_{X,x}[\mathbf{x}]_{\mathfrak{m}_{X,x}\mathcal{O}_{X,x}[\mathbf{x}]}, \text{ where } \mathbf{x} \text{ is an indeterminate.}$$

If M is a graded R-module, we may consider $M_{\mathfrak{p}}$ as a module over the right hand side ring of (■)′. Then, we have an isomorphism

$$M_{\mathfrak{p}} \cong \widetilde{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}[\mathbf{x}]_{\mathfrak{m}_{X,x}\mathcal{O}_{X,x}[\mathbf{x}]}.$$

But now, using the flat base chance property of local cohomology, we obtain

(■■) $$H^i_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cong H^i_x(\widetilde{M}) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}[\mathbf{x}]_{\mathfrak{m}_{X,x}\mathcal{O}_{X,x}[\mathbf{x}]}$$

and hence

(■■)′ $$h^i_{\mathfrak{p}}(M) = h^i_x(\widetilde{M}).$$

By (■) and (■■)′ we now may conclude easily, that

(■■■) $$\mathrm{Psupp}^i(\widetilde{M}) = {}^+\mathrm{Psupp}^{i+1}(M), \ \mathrm{psd}^i(\widetilde{M}) = {}^+\mathrm{psd}^{i+1}(M);$$

(■■■)′ $$\mathrm{Psupp}^i_0(\widetilde{M}) = {}^+\mathrm{Psupp}^{i+1}_0(M).$$

D) Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$. We may define the i-th *cohomological deficiency function of* $\mathcal{F}$ by

$$\Delta^i_{\mathcal{F}} := \Delta^i_{X,\mathcal{F}} : \mathbb{Z} \longrightarrow \mathbb{Z};\ n \longmapsto \Delta^i_{\mathcal{F}}(n) := h^i_{\mathcal{F}}(n) - p^i_{\mathcal{F}}(n).$$

In view of statement (•) of part B) we then have

(▲) $$\Delta^i_{\mathcal{F}}(n) = 0 \text{ for all } n \ll 0.$$

This allows us to define the i-th *cohomological postulation number of* $\mathcal{F}$ by

$$\nu^i_{\mathcal{F}} := \nu^i_{X,\mathcal{F}} := \inf\{m \in \mathbb{Z} \mid \Delta^i_{\mathcal{F}}(m+1) \neq 0\} \in \mathbb{Z} \cup \{+\infty\}.$$

Again we may set $\nu^{-1}_{\mathcal{F}} := +\infty$.

If M is a finitely generated and graded R-module it follows from statement $(*)$ of part A) and from (••) of part B) that

(▲▲) $$\Delta^i_{\widetilde{M}} = \Delta^i_M;\ \ \nu^i_{\widetilde{M}} = \nu^i_M, \quad (\forall i \in \mathbb{N}_0).$$

Now, by 1.6 C) $(***)'$, by [7, (2.45)] b) and by statement $(**)$ of part A) we obtain

(▲▲▲) $\operatorname{coreg}^k(\mathcal{F}) = \inf\{\nu^i_{\mathcal{F}} + i - 1 \mid i \leq k\}$, for all $k < \delta(\mathcal{F})$ and for each coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. •

Corollary 1.2. *Let* $X = \operatorname{Proj}(R)$, *let* $\mathcal{F}$ *be a coherent sheaf of* $\mathcal{O}_X$*-modules and let* $i \in \mathbb{N}_0$. *Then*

a) $\operatorname{Psupp}^i(\mathcal{F})$ *is a closed subset of dimension* $\operatorname{psd}^i(\mathcal{F}) \leq i$ *in* X.

b) $x \in X$ *is a generic point of* $\operatorname{Psupp}^i(\mathcal{F})$ *if and only if* $0 < h_x^{i-\dim(\overline{\{x\}})}(\mathcal{F}) < \infty$.

c) $p^i_{\mathcal{F}}$ *is of degree* $\operatorname{psd}^i(\mathcal{F})$.

d) *If* $\operatorname{psd}^i(\mathcal{F}) \geq 0$, *then the leading coefficient of* $p^i_{\mathcal{F}}$ *is*

$$\frac{(-1)^{\operatorname{psd}^i(\mathcal{F})}}{\operatorname{psd}^i(\mathcal{F})!} \sum_{x \in \operatorname{Psupp}^i_0(\mathcal{F})} h_x^{i-\operatorname{psd}^i(\mathcal{F})}(\mathcal{F}) \deg(\overline{\{x\}}).$$

Proof. For all $x = \mathfrak{p} \in \operatorname{Proj}(R)$, $\overline{\{x\}} \subseteq X$ is a closed subscheme of degree $e_0(R/\mathfrak{p})$. So we conclude by Theorem 1.1 a) b), 1.5 (••), (•••) and by statement (••) of 1.7 B). ■

In the case, where R_0 is an algebraically closed field, the four statements of Corollary 1.2 are essentially found in [20, (2.2.3)] (cf. also [4, (2.4)]).

2. Admissible Linear Forms

In this section we present a few preliminary results, which are needed to bound the cohomological deficiency functions and the cohomological postulation numbers introduced in 1.6. Again, we assume that $R = \oplus_{n \geq 0} R_n$ is a noetherian positively graded homogeneous ring with artinian base ring R_0.

2.1. Convention and Remark

A) For this whole section, we assume that our base ring R_0 has *infinite residue fields.* More precisely, we always write $\mathfrak{m}_0^{(1)}, \ldots, \mathfrak{m}_0^{(r)}$ for the different maximal ideals of R_0 and assume that the residue fiels $R_0/\mathfrak{m}_0^{(j)}$ are infinite for $j = 1, \ldots, r$.

B) Let $j \in \{1, \ldots, r\}$. As earlier in this paper, we write $R'^{(j)}$ for the positively graded noetherian homogeneous ring $(R_0)_{\mathfrak{m}_0^{(j)}} \otimes_{R_0} R$ with local artinian base ring $(R_0)_{\mathfrak{m}_0^{(j)}}$. We write $\eta^{(j)}$ for the natural homomorphism of graded R_0-algebras $R \to R'^{(j)}$ and keep in mind, that we have an isomorphism of graded R_0-algebras

$$\eta : R \xrightarrow{\cong} \bigoplus_{j=1}^{r} R'^{(j)}; \ \big(x \mapsto \big(\eta^{(1)}(x), \ldots, \eta^{(r)}(x)\big)\big).$$

C) If M' is a graded R-module, we write again $M'^{(j)}$ for the graded $R'^{(j)}$-module $(R_0)_{\mathfrak{m}_0^{(j)}} \otimes_{R_0} M = R'^{(j)} \otimes_R M$. If $\mathfrak{a} \subseteq R$ is a graded ideal, we may write $\mathfrak{a}'^{(j)} = \mathfrak{a} R'^{(j)}$ for $j = 1, \ldots, r$. If $\mathfrak{p} \subseteq R$ is a graded prime ideal, there is a unique index $j(\mathfrak{p}) \in \{1, \ldots, r\}$ with $\mathfrak{p} \cap R_0 = \mathfrak{m}_0^{(j(\mathfrak{p}))}$. So, we have $\mathfrak{p}'^{(j)} = R'^{(j)}$ for $j \neq j(\mathfrak{p})$, whereas $\mathfrak{p}'^{(j(\mathfrak{p}))} \subsetneqq R'^{(j(\mathfrak{p}))}$ is a proper prime. Moreover $\mathfrak{p}$ is essential if and only if $\mathfrak{p}'^{(j(\mathfrak{p}))}$ is, and finally $\mathfrak{p} = \big(\eta^{(j)}\big)^{-1}\big(\mathfrak{p}'^{(j(\mathfrak{p}))}\big)$.

D) Now, let $\mathcal{S} \subseteq \mathrm{Proj}(R)$ be a finite set of essential graded primes. For any $j \in \{1, \ldots, r\}$ let $\mathcal{S}_j := \{\mathfrak{p} \in \mathcal{S} \mid j(\mathfrak{p}) = j\}$. Then $\mathcal{S} = \bigcup_{1 \leq j \leq r} \mathcal{S}_j$ and $\{\mathfrak{p}'^{(j)} \mid \mathfrak{p} \in \mathcal{S}_j\}$ is a finite set of essential graded primes of $R'^{(j)}$. As the base ring $(R_0)_{\mathfrak{m}_0^{(j)}}$ of $R'^{(j)}$ is local with infinite residue field, there is some linear form $f^{(j)} \in \big(R'^{(j)}\big)_1 \setminus \bigcup_{\mathfrak{p} \in \mathcal{S}_j} \mathfrak{p}'^{(j)}$, (cf. [10, (1.5.12)]). By the isomorphism η of part B) we thus may define the linear form $f := \eta^{-1}(f^{(1)}, \ldots, f^{(r)}) \in R$. For each $j \in \{1, \ldots, r\}$ we obtain $\eta^{(j)}(f) = f^{(j)} \notin \bigcup_{\mathfrak{p} \in \mathcal{S}_j} \mathfrak{p}'^{(j)}$, hence $f \notin \bigcup_{\mathfrak{p} \in \mathcal{S}_j} \mathfrak{p}$ and thus $f \in R_1 \setminus \bigcup_{\mathfrak{p} \in \mathcal{S}} \mathfrak{p}$.

E) A graded R-module $S = \oplus_{n \in \mathbb{Z}} S_n$ is called **secondary* (or *graded-secondary*) if for each homogeneous element $x \in R$ the multiplication map $x : S \to S$ is either surjective or nilpotent, (cf. [21, (2.1) (ii)]). In this case $\mathfrak{p} := \sqrt{0 :_R S}$ is a homogeneous prime of R (cf. [21, (2.4)]) and we say that S is $\mathfrak{p}$-**secondary.* If S is artinian and $\mathfrak{p}$-*secondary, S is $\mathfrak{p}$-secondary in the usual sense. To see this, it suffices to show that the multiplication map $x : S \to S$ is surjective for each (not necessarily homogeneous) element $x \in R \setminus \mathfrak{p}$. Indeed, let $x_i \in R_i$ be the i-th homogeneous part of x, for all $i \in \mathbb{N}_0$ and let $r = \min\{i \in \mathbb{N}_0 \mid x_i \notin \mathfrak{p}\}$. Raising x to a sufficiently large power we may assume at once, that $x_i = 0$ for all $i \leq r$. Now, let $S_{\geq h} := \oplus_{n \geq h} S_n$. As $S_{\geq h} = 0$ for all $h \gg 0$ and as $x_r : S \to S$ is surjective it follows easily by descending induction on h, that $x S_{\geq h-r} = S_{\geq h}$. This proves the requested equality $xS = S$.

F) Now, let $T = \oplus_{n \in \mathbb{Z}} T_n$ be an artinian and graded R-module. Then, by [21 (2.4)], the module T has a minimal **secondary representation* (or a *minimal graded-secondary representation*). More precisely, T may be expressed as a finite sum $T = S^{(1)} + \cdots + S^{(p)}$ where each $S^{(i)}$ is a *secondary submodule of T, with $T \neq \sum_{i \neq k} S^{(i)}$ for all $k \in \{1, \ldots, p\}$ and where the graded primes $\mathfrak{p}^{(i)} :=$

$\sqrt{0:_R S^{(i)}}$, $(i = 1, \ldots, p)$, are pairwise different. By part E), the module $S^{(i)}$ is $\mathfrak{p}^{(i)}$-secondary and therefore $T = S^{(1)} + \cdots + S^{(p)}$ is a minimal secondary representation of T and $\{\mathfrak{p}^{(1)}, \ldots, \mathfrak{p}^{(p)}\}$ is precisely the set $\mathrm{Att}_R(T)$ of primes attached to T, (cf. [17]). •

2.2. Remark and Definition

A) Let M be a finitely generated and graded R-module. Then, for each $i \in \mathbb{N}_0$, the local cohomology module $H^i_{R_+}(M)$ is an artinian R-module (cf. [9 (3.1.4)]). So, the set $\mathrm{Att}_R\big(H^i_{R_+}(M)\big)$ of prime ideals in R which are attached to $H^i_{R_+}(M)$, is finite and consists entirely of graded primes (cf. (2.1) F)). If R_0 is local, using the notation of 1.3 B) we also may write (cf. [9, (10.2.20)])

$$\mathrm{Att}_R(H^i_{R_+}(M)) = \mathrm{Ass}_R(K^i_M).$$

As $H^i_{R_+}(M) = 0$ for all $i > \dim(R)$ and as $\mathrm{Att}_R(0) = \emptyset$, the set of essential graded primes

$$\mathcal{P}(M) := \big[\mathrm{Ass}_R(M) \cup \bigcup_{i>0} \mathrm{Att}_R\big(H^i_{R_+}(M)\big)\big] \setminus \mathrm{Var}(R_+)$$

is finite.

B) We keep the previous notations and hypotheses and set

$$\mathcal{U}(M) := R_1 \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}(M)} \mathfrak{p}.$$

So, for an arbitrary finite collection of finitely generated graded R- modules $M^{(1)}, \ldots, M^{(p)}$, we have by 2.1 D)

$$\bigcap_{i=1}^{p} \mathcal{U}(M^{(i)}) \neq \emptyset.$$

C) Let M still be a finitely generated and graded R-module. A sequence of linear forms $f_1, \ldots, f_r \in R_1$ is said to be *M-admissible*, if

$$f_i \in \bigcap_{j=1}^{i-1} \mathcal{U}\big(M/\sum_{k=1}^{j} f_k M\big) \quad \text{for} \quad i = 1, \ldots, r\ .$$

From part B) it follows immediately by induction, that any M-admissible sequence $f_1, \ldots, f_s \in R_1$ can be extended to an M-admissible sequence $f_1, \ldots, f_{s+1} \in R_1$. This shows, that M-admissible sequences $f_1, \ldots, f_r \in R_1$ of arbitrary length $r \in \mathbb{N}$ exist.

A linear form $f \in R_1$ is said to be *M-admissible*, if it constitutes an M-admissible sequence of length 1, i.e. precisely if $f \in \mathcal{U}(M)$.

D) Let $f_1, \ldots, f_r \in R_1$ be an M-admissible sequence of linear forms. Then, for any $s \in \{0, \ldots, r\}$ the sequence $f_{s+1}, \ldots, f_r$ is $\big(M/\sum_{k=1}^{s} f_k M\big)$-admissible. Moreover, f_t is $\big(M/\sum_{k=1}^{s} f_k M\big)$-admissible for all $t \in \{s+1, \ldots, r\}$.

E) Let $n \in \mathbb{Z}$. Then, for each $i \in \mathbb{N}_0$ we have $H^i_{R_+}\big(M(n)\big) = \big(H^i_{R_+}(M)\big)(n)$ (cf. [9, (13.1.9)]). This shows immediately, that $\mathcal{P}\big(M(n)\big) = \mathcal{P}(M)$ and hence $\mathcal{U}(M(n)) = \mathcal{U}(M)$. As for each homogeneous element $f \in R$ we have $(M/fM)(n) = M(n)/fM(n)$, it follows at once that a sequence $f_1, \ldots, f_r \in R_1$ of linear forms is M-admissible if and only if it is $M(n)$-admissible.

F) Let $T \subseteq M$ be an arbitrary graded R_+-torsion submodule. Then

$$\mathrm{Ass}_R(M/T)\backslash \mathrm{Var}(R_+) = \mathrm{Ass}_R(M) \setminus \mathrm{Var}(R_+) = \mathrm{Ass}_R\big(M/\Gamma_{R_+}(M)\big)$$

and for each $i \in \mathbb{N}$ there is a graded isomorphism $H^i_{R_+}(M/T) \cong H^i_{R_+}(M)$. This shows, that $\mathcal{P}(M/T) = \mathcal{P}(M)$ and hence that $\mathcal{U}(M/T) = \mathcal{U}(M)$. A repeated application of this shows, that a sequence $f_1, \ldots, f_r \in R_1$ is M/T-admissible if and only if it is M-admissible.

G) If $f \in R_1$ is M-admissible, it is avoided by all members of $\mathrm{Ass}_R\big(M/\Gamma_{R_+}(M)\big)$, so that f is $M/\Gamma_{R_+}(M)$-regular. So, if we apply [7 (2.20)] to the linear form f and the graded R-module $M/\Gamma_{R_+}(M)$ and keep in mind, that for $i \geq 0$ we have natural isomorphisms of graded R-modules $\mathcal{R}^i D_{R_+}\big(M/\Gamma_{R_+}(M)\big) \cong \mathcal{R}^i D_{R_+}(M)$ and $\mathcal{R}^i D_{R_+}\big(\big(M/\Gamma_{R_+}(M)\big)/f\big(M/\Gamma_{R_+}(M)\big)\big) \cong \mathcal{R}^i D_{R_+}(M/fM)$ we get exact sequences of R_0-modules

$$\begin{array}{llllll}
0 & \longrightarrow & \mathcal{R}^0 D_{R_+}(M)_{n-1} & \xrightarrow{f} & \mathcal{R}^0 D_{R_+}(M)_n & \xrightarrow{\varrho^0_n} & \mathcal{R}^0 D_{R_+}(M/fM)_n \\
 & \xrightarrow{\delta^0_n} & \mathcal{R}^1 D_{R_+}(M)_{n-1} & \ldots & \ldots & \xrightarrow{\varrho^{i-1}_n} & \mathcal{R}^{i-1} D_{R_+}(M/fM)_n \\
 & \xrightarrow{\delta^{i-1}_n} & \mathcal{R}^i D_{R_+}(M)_{n-1} & \xrightarrow{f} & \mathcal{R}^i D_{R_+}(M)_n & \xrightarrow{\varrho^i_n} & \mathcal{R}^i D_{R_+}(M/fM)_n \\
 & \xrightarrow{\delta^i_n} & \mathcal{R}^{i+1} D_{R_+}(M)_{n-1} & \xrightarrow{f} & \mathcal{R}^{i+1} D_{R_+}(M)_n & \ldots, &
\end{array}$$

in which the maps $\varrho^i_n := \mathcal{R}^i D_{R_+}(\pi)_n$ are induced by the canonical homomorphism $M \xrightarrow{\pi} M/fM$ and in which the maps δ^i_n come from the connecting homomorphisms in the right derived sequence of D_{R_+} associated to the sequence

$$0 \to \big(M/\Gamma_{R_+}(M)\big)(-1) \xrightarrow{f} M/\Gamma_{R_+}(M) \to \big(M/\Gamma_{R_+}(M)\big)/f\big(M/\Gamma_{R_+}(M)\big) \to 0 \,.$$

•

Lemma 2.1. *Let M be a finitely generated graded R-module. Then, there is a $\tau \in \mathbb{Z} \cup \{\infty\}$ such that the multiplication map*

$$H^j_{R_+}(M)_m \xrightarrow{f} H^j_{R_+}(M)_{m+1}$$

is surjective for all $j \in \mathbb{N}_0$ and all $m \leq \tau$ and all M-admissible linear forms $f \in R_1$.

Proof. Fix $j \in \{0, \ldots, \dim(M)\}$. As $H^j_{R_+}(M)$ is an artinian and graded R-module (cf. (2.2) A)), it admits a minimal secondary representation $H^j_{R_+}(M) =$

$S^{(1)} + \cdots + S^{(p)}$ in which each of the secondary submodules $S^{(i)} \subseteq H^j_{R_+}(M)$ is graded (cf. 2.1 F)). Consider the graded primes $\mathfrak{p}^{(i)} = \sqrt{0 :_R S^{(i)}}$ for $i \in \{1, \dots, p\}$ so that

$$\operatorname{Att}_R\big(H^j_{R_+}(M)\big) = \big\{\mathfrak{p}^{(1)}, \dots, \mathfrak{p}^{(p)}\big\} .$$

Assume first, that $\operatorname{Att}_R\big(H^j_{R_+}(M)\big) \cap \operatorname{Var}(R_+) \neq \emptyset$. After reordering the modules $S^{(k)}$ we find an index $s \in \{1, \dots, p-1\}$ such that $\mathfrak{p}^{(k)} \in \operatorname{Var}(R_+)$ if and only if $k \in \{s+1, \dots, p\}$. Then, for each $k \in \{s+1, \dots, p\}$, $S^{(k)}$ is annihilated by some power of R_+, so that there is some $\tau_{j,k} \in \mathbb{Z}$ such that $(S^{(k)})_{m+1} = 0$ for all $m \leq \tau_{j,k}$. With $\tau_j := \min\{\tau_{j,k} \mid s < k \leq p\}$ we thus get $\big(S^{(s+1)} + \cdots + S^{(p)}\big)_{m+1} = 0$ for all $m \leq \tau_j$. Therefore $H^j_{R_+}(M)_{m+1} = (S^{(1)})_{m+1} + \cdots + (S^{(s)})_{m+1}$ for all $m \leq \tau_j$.

Now, let $f \in R_1$ be M-admissible. Then $f \notin \mathfrak{p}^{(1)} \cup \cdots \cup \mathfrak{p}^{(s)}$ implies, that $fS^{(i)} = S^{(i)}$ and hence that $f(S^{(i)})_m = (S^{(i)})_{m+1}$ for all $i \in \{1, \dots, s\}$ and all $m \in \mathbb{Z}$.

Choosing $m \leq \tau_j$, we thus get $f\big(H^j_{R_+}(M)_m\big) = H^j_{R_+}(M)_{m+1}$ so that $f : H^j_{R_+}(M)_m \to H^j_{R_+}(M)_{m+1}$ becomes in fact surjective for all $m \leq \tau_j$.

If $\operatorname{Att}_R\big(H^j_{R_+}(M)\big) \cap \operatorname{Var}(R_+) = \emptyset$ we have $fS^{(i)} = S^{(i)}$ for all $i \in \{1, \dots, p\}$ and for each M- admissible linear form $f \in R_1$. So, we may conclude, that $f : H^j_{R_+}(M)_m \to H^j_{R_+}(M)_{m+1}$ is even surjective for all $m \in \mathbb{Z}$.

As $H^i_{R_+}(M) = 0$ for $i > \dim(M)$, this proves our claim. ∎

Lemma 2.2. *Let M be a finitely generated graded R-module. Then, there is a $\sigma \in \mathbb{Z}$ such that the multiplication map*

$$\mathcal{R}^i D_{R_+}(M)_m \xrightarrow{f} \mathcal{R}^i D_{R_+}(M)_{m+1}$$

is surjective for all $i \in \mathbb{N}_0$, all $m \leq \sigma$ and all M-admissible linear forms $f \in R_1$.

Proof. Let $f \in R_1$ be M-admissible. In view of 2.2 G), the above map is injective for all $m \in \mathbb{Z}$ if $i = 0$. As the R_0-modules $\mathcal{R}^0 D_{R_+}(M)_m$ are of finite length, there is a $m_0 \in \mathbb{Z}$ such that the map $f : \mathcal{R}^0 D_{R_+}(M)_m \to \mathcal{R}^0 D_{R_+}(M)_{m+1}$ is an isomorphism for all $m \leq m_0$ and for each M-admissible linear form $f \in R_1$. Moreover by Lemma 2.1 there is an $m_1 \in \mathbb{Z}$ such that for each M-admissible linear form $f \in R_1$ the multiplication map $H^j_{R_+}(M)_m \xrightarrow{f} H^j_{R_+}(M)_{m+1}$ is surjective for all $m \leq m_1$ and all $j \geq 2$. Now, we may conclude, as we have natural graded isomorphisms $\mathcal{R}^i D_{R_+}(M) \cong H^{i+1}_{R_+}(M)$ for all $i \geq 1$. ∎

2.3. Notation and Remark

A) Let M be a finitely generated and graded R-module and let $f \in R_1$ be M-admissible. Let $i \in \mathbb{N}_0$. We set

$$\sigma^i_{M,f} := \inf\big\{m \in \mathbb{Z} \mid \mathcal{R}^i D_{R_+}(M)_m \xrightarrow{f} \mathcal{R}^i D_{R_+}(M)_{m+1} \quad \text{is not surjective}\big\} - 1,$$

where "inf" is formed in $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ according to our standard conventions. By Lemma 2.2 we know that $\inf\big\{\sigma^i_{M,f} \mid i \in \mathbb{N}_0,\ f \in \mathcal{U}(M)\big\} > -\infty$. In particular $\sigma^i_{M,f} > -\infty$ for all $f \in \mathcal{U}(M)$ and all $i \in \mathbb{N}_0$. We further set $\sigma^{-1}_{M,f} := \infty$.

B) Let M and f be as in part A). Then, the sequences 2.2 G) give rise to isomorphisms

$$f : \mathcal{R}^0 D_{R_+}(M)_m \xrightarrow{\cong} \mathcal{R}^0 D_{R_+}(M)_{m+1} \quad (\forall\, m \leq \sigma^0_{M,f})$$

and to short exact sequences

$$0 \to \mathcal{R}^{i-1} D_{R_+}(M/fM)_{m+1} \xrightarrow{\delta^{i-1}_{m+1}} \mathcal{R}^i D_{R_+}(M)_m \xrightarrow{f} \mathcal{R}^i D_{R_+}(M)_{m+1} \to 0$$

$$\big(\forall\, m \leq \min\{\sigma^{i-1}_{M,f},\ \sigma^i_{M,f}\}, \quad \forall\, i \in \mathbb{N}\big).$$

So we get

$$d^0_M(m) = d^0_M(m+1), \quad (\forall\, m \leq \sigma^0_{M,f});$$

$$d^i_M(m) - d^i_M(m+1) = d^{i-1}_{M/fM}(m+1), \ \ \big(\forall\, i \in \mathbb{N},\ \forall\, m \leq \min\{\sigma^{i-1}_{M,f}, \sigma^i_{M,f}\}\big).$$

As an easy consequence of these latter equalities we get

$$q^i_M(\mathbf{x}-1) - q^i_M(\mathbf{x}) \ = \ q^{i-1}_{M/fM}(\mathbf{x}), \quad (\forall i \in \mathbb{N}) \ ,$$

where $q^i_M,\ q^{i-1}_{M/fM} \in \mathbb{Q}[\mathbf{x}]$ are the cohomological Hilbert polynomials defined in 1.5.

C) Let M and f be as above. By 2.2 G), the maps $f : \mathcal{R}^0 D_{R_+}(M)_m \longrightarrow \mathcal{R}^0 D_{R_+}(M)_{m+1}$ are injective for all $m \in \mathbb{Z}$. Therefore $d^0_M(m) \leq d^0_M(m+1)$ for all $m \in \mathbb{Z}$. As q^0_M is constant (cf. 1.5 (••)), we thus get $\Delta^0_M(n+1) \geq \Delta^0_M(n) \geq 0$ for all $n \in \mathbb{Z}$. •

The main interest of the present section is to study the behaviour of the cohomological deficiency functions $n \mapsto \Delta^i_M(n)$ in the range $n \leq \nu^{i-1}_{M/fM}$, where $f \in R_1$ is an M-admissible linear form. We now start doing this by proving an auxiliary result. First, we introduce one more notation. For $i \in \mathbb{N}_0$, for $m \in \mathbb{Z}$, for a finitely generated graded R- module M and for a linear form $f \in R_1$, we set

$$\gamma^i_{M,f,m} := \ell_{R_0}\Big(\mathcal{R}^i D_{R_+}(M)_{m+1} / f\big(\mathcal{R}^i D_{R_+}(M)\big)_m\Big), \tag{2.4}$$

so that $\gamma^i_{M,f,m}$ denotes the length of the cokernel of the multiplication map

$$\mathcal{R}^i D_{R_+}(M)_m \xrightarrow{f} \mathcal{R}^i D_{R_+}(M)_{m+1}.$$

For the sake of completeness we also convene that $\gamma^{-1}_{M,f,m} = 0$. Using this notation we have

Lemma 2.3. *Let M be a finitely generated graded R-module, let $f \in R_1$ be M-admissible, let $i \in \mathbb{N}$ and let $m \in \mathbb{Z}$. Then*

$$\Delta^i_M(m) = \Delta^i_M(m+1) + \Delta^{i-1}_{M/fM}(m+1) - \gamma^i_{M,f,m} - \gamma^{i-1}_{M,f,m}.$$

Proof. By 2.2 G) we have an exact sequence

$$\mathcal{R}^{i-1}D_{R_+}(M)_m \xrightarrow{f^{(i-1)}} \mathcal{R}^{i-1}D_{R_+}(M)_{m+1} \xrightarrow{\varrho^{i-1}_{m+1}} \mathcal{R}^{i-1}D_{R_+}(M/fM)_{m+1}$$
$$\xrightarrow{\delta^{i-1}_{m+1}} \mathcal{R}^{i}D_{R_+}(M)_m \xrightarrow{f^{(i)}} \mathcal{R}^{i}D_{R_+}(M)_{m+1}$$

in which $f^{(i-1)}$ and $f^{(i)}$ denote the corresponding multiplication maps by f. So, we get

$$\begin{aligned}
\gamma^i_{M,f,m} &= \ell_{R_0}\big(\text{ coker } (f^{(i)})\big) = d^i_M(m+1) - \ell_{R_0}\big(\text{im } (f^{(i)})\big)\\
&= d^i_M(m+1) - d^i_M(m) + \ell_{R_0}\big(\text{ker } (f^{(i)})\big)\\
&= d^i_M(m+1) - d^i_M(m) + \ell_{R_0}\big(\text{im } (\delta^{i-1}_{m+1})\big)\\
&= d^i_M(m+1) - d^i_M(m) + d^{i-1}_{M/fM}(m+1) - \ell_{R_0}\big(\text{im } (\varrho^{i-1}_{m+1})\big)\\
&= d^i_M(m+1) - d^i_M(m) + d^{i-1}_{M/fM}(m+1) - d^{i-1}_M(m+1) + \ell_{R_0}\big(\text{ker } (\varrho^{i-1}_{m+1})\big)\\
&= d^i_M(m+1) - d^i_M(m) + d^{i-1}_{M/fM}(m+1) - d^{i-1}_M(m+1) + \ell_{R_0}\big(\text{im } (f^{(i-1)})\big)\\
&= d^i_M(m+1) - d^i_M(m) + d^{i-1}_{M/fM}(m+1) - \ell_{R_0}\big(\text{ coker } (f^{(i-1)})\big)\\
&= d^i_M(m+1) - d^i_M(m) + d^{i-1}_{M/fM}(m+1) - \gamma^{i-1}_{M,f,m}
\end{aligned}$$

and hence

$$d^i_M(m) = d^i_M(m+1) + d^{i-1}_{M/fM}(m+1) - \gamma^i_{M,f,m} - \gamma^{i-1}_{M,f,m}.$$

By 2.3 B) we also have

$$q^i_M(m) = q^i_M(m+1) + q^{i-1}_{M/fM}(m+1).$$

By subtracting the last equality from the preceding one, we get our claim. ■

Remark 2.1. If we set $d^{-1}_M \equiv 0$, we get $q^{-1}_M \equiv 0$ and $\Delta^{-1}_M \equiv 0$. Observe that by 1.5 (••) the polynomial q^0_M is constant. So, with the above convention it is clear, that the equality stated in Lemma 2.3 also holds for $i = 0$. •

2.4. Notation and Remark

A) Let M be a finitely generated graded R-module, let $i \in \mathbb{N}$ and let $f_1, \dots, f_i \in R_1$ be an M-admissible sequence of linear forms. We put

$$M_{\langle s \rangle} := M/\sum_{j=1}^{s} f_j M, \quad (s \in \{0, \dots, i-1\}).$$

For $l \in \{0, \dots, i\}$ and $k \in \{-1, 0, \dots, i-l\}$ we define

$$T^{(k)}_{M,l}(f_1, \dots, f_i) := \inf\big\{\sigma^{k-j-1}_{M_{\langle l+j \rangle}, f_{l+j+1}} - j - 1 \mid 0 \le j < k\big\}$$

where the infimum is formed in $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$ according to our usual convention and where $\sigma^{k-j-1}_{M_{\langle l+j \rangle},\ f_{l+j+1}}$ is defined according to 2.3 A).

B) Keep the previous notations and hypotheses. By 2.3 A) we then have $T^{(k)}_{M,l}(f_1, \dots, f_i) > -\infty$. Moreover, $T^{(k)}_{M,l}(f_1, \dots, f_i)$ does only depend on the linear forms $f_1, \dots, f_{\max\{1,l+k\}}$, so that

$$(*) \qquad T^{(k)}_{M,l}\ (f_1, \dots, f_i) = T^{(k)}_{M,l}\ (f_1, \dots, f_{\max\{1,l+k\}}).$$

Finally, in view of (2.2) D) we may write

$$(**) \qquad T^{(k)}_{M,l}\ (f_1, \dots, f_i)\ =\ T^{(k)}_{M_{\langle l \rangle},0}\ (f_{l+1}, \dots, f_i)\ \text{ if } l < i.$$

C) By our definition we have

$$T^{(-1)}_{M,l}\ (f_1, \dots, f_i) = T^{(0)}_{M,l}\ (f_1, \dots, f_i) = \infty,$$
$$T^{(1)}_{M,l}\ (f_1, \dots, f_i) = \sigma^0_{M_{\langle l \rangle},\ f_{l+1}} - 1 \quad \text{if} \quad l < i.$$

As $\sigma^{-1}_{M_{\langle l+k \rangle}, f_{l+k+1}} = +\infty$, we have

$$(\bullet) \quad T^{(k)}_{M,l}\,(f_1, \dots, f_i) = \min\big\{\sigma^{k-j-1}_{M_{\langle l+j \rangle}, f_{l+j+1}} - j - 1 \ \big|\ 0 \le j \le k\big\} \quad \text{for} \quad k < i-l$$

and moreover

$$(\bullet\bullet) \quad T^{(k)}_{M,l}(f_1, \dots, f_i) = \min\big\{\sigma^{k-1}_{M_{\langle l \rangle}, f_{l+1}}, T^{(k-1)}_{M,(l+1)}\,\big(f_1, \dots, f_i\big)\big\} - 1 \quad \text{whenever} \quad 0 < k \le i-l.$$

•

The next result provides us with the main technical tool of the present section. It is a substitute for the two results [5, (3.1), (3.2)], which made available the "method of linear systems of hyperplane sections" for proving the bounds in Secs. 4 and 5 of [7]. So, the result to come allows to replace the latter method by the "method of admissible sequences of linear forms". A certain interplay between these two methods is used in a rudimentary form in [8]. A modified version of the announced result is given in [20, (3.2.5)].

Lemma 2.4. *Let M be a finitely generated and graded R-module, let $i \in \mathbb{N}_0$ and let $f_1, \dots, f_{i+1} \in R_1$ be an M-admissible sequence of linear forms. Then, in the notation of* 2.4 A) *we have*

$$\bigcap_{s=1}^{i+1} \ker\big[\mathcal{R}^i D_{R_+}(M)_m \xrightarrow{f_s} \mathcal{R}^i D_{R_+}(M)_{m+1}\big] = 0$$

whenever $\quad m \le T^{(i)}_{M,0}\,(f_1, \dots, f_{i+1})$.

Proof. (Induction on i). Let $i = 0$. Then, by the sequences 2.2 G), the multiplication maps $f_1 : \mathcal{R}^0 D_{R_+}(M)_m \to \mathcal{R}^0 D_{R_+}(M)_{m+1}$ are injective for all $m \in \mathbb{Z}$. This gives our claim if $i = 0$.

Let $i \ge 1$. By 2.2 D) we know that $f_2, \dots, f_{i+1}$ is an admissible sequence with respect to $M/f_1M = M_{\langle 1 \rangle}$. So, by induction on i and as

$$T^{(i-1)}_{M_{\langle 1 \rangle},0}(f_2, \dots, f_{i+1}) = T^{(i-1)}_{M,1}\,(f_1, \dots, f_{i+1})$$

(cf. 2.4 B) $(**)$), we see that

$$(*)\qquad \bigcap_{t=2}^{i+1} \ker\big[\mathcal{R}^{i-1}D_{R_+}(M/f_1M)_{m+1} \xrightarrow{f_t} \mathcal{R}^{i-1}D_{R_+}(M/f_1M)_{m+2}\big] = 0,$$

$$\big(\forall\, m \leq T^{(i-1)}_{M,1}\big(f_1,\dots,f_{i+1}\big) - 1\big).$$

Now, fix $t \in \{2,\dots,i+1\}$ and consider the following commutative diagram of graded R-modules

$$\begin{array}{ccccc}
(M/f_1M)(1) & \xrightarrow{f_t} & (M/f_1M)(2) & \longrightarrow & \big(M/(f_1,f_t)M\big)(2) \\
\uparrow & & \uparrow & & \uparrow \\
M(1) & \xrightarrow{f_t} & M(2) & \longrightarrow & (M/f_tM)(2) \\
\uparrow f_1 & & \uparrow f_1 & & \uparrow f_1 \\
M & \xrightarrow{f_t} & M(1) & \longrightarrow & (M/f_tM)(1)
\end{array}$$

By 2.2 D), E) we know that f_1 and f_t are M- and $M(1)$-admissible and that f_t is $(M/f_1M)(1)$-admissible. So, for each $m \in \mathbb{Z}$, 2.2 G) yields the following commutative diagram with exact rows and columns, in which $\delta_{[j]}$ $(j = 1, t)$, δ' and $\overline{\delta}_{[t]}$ arise as connecting homomorphisms and in which $\mathcal{R}^{i-2}D_{R_+} \equiv 0$ for $i = 1$.

$$\begin{array}{ccccc}
 & & \mathcal{R}^{i}D_{R_+}(M)_{m+1} & & \\
 & & \uparrow f_1^{(i)} := f_1 & & \\
\mathcal{R}^{i-1}D_{R_+}(M/f_tM)_{m+1} & \xrightarrow{\delta_{[t]}} & \mathcal{R}^{i}D_{R_+}(M)_m & \xrightarrow{f_t^{(i)} := f_t} & \mathcal{R}^{i}D_{R_+}(M)_{m+1} \\
 & & \uparrow \delta_{[1]} & & \uparrow \delta' \\
\mathcal{R}^{i-2}D_{R_+}(M/(f_1,f_t)M)_{m+2} & \xrightarrow{\overline{\delta}_{[t]}} & \mathcal{R}^{i-1}D_{R_+}(M/f_1M)_{m+1} & \xrightarrow{f_t} & \mathcal{R}^{i-1}D_{R_+}(M/f_1M)_{m+2} \\
 & & \uparrow & & \uparrow \\
 & & \mathcal{R}^{i-1}D_{R_+}(M)_{m+1} & & \mathcal{R}^{i-1}D_{R_+}(M)_{m+2} \\
 & & \uparrow f_1 & & \uparrow f_1 \\
 & & \mathcal{R}^{i-1}D_{R_+}(M)_m & & \mathcal{R}^{i-1}D_{R_+}(M)_{m+1}
\end{array}$$

Now, let $m \leq \sigma^{i-1}_{M,f_1} - 1$ so that the two lowest vertical maps of the above diagram are surjective (cf. 2.3 A)). Then $\delta_{[1]}$ and δ' become injective. As δ' is injective, we get

$$\operatorname{im}(\delta_{[1]}\cdot\overline{\delta}_{[t]}) \;=\; \ker(f_1^{(i)})\cap\ker(f_t^{(i)}),\ \big(\forall\, m \leq \sigma^{i-1}_{M,f_1}-1,\ \forall\, t \in \{2,\dots,i+1\}\big)\ (**)$$

As $\delta_{[1]}$ is injective, we also have

$$\bigcap_{t=2}^{i+1}\operatorname{im}\big(\delta_{[1]}\cdot\overline{\delta}_{[t]}\big) = \delta_{[1]}\Big(\bigcap_{t=2}^{i+1}\operatorname{im}(\overline{\delta}_{[t]})\Big),\quad \big(\forall\, m \leq \sigma^{i-1}_{M,f_1} - 1\big).$$

Let $m \leq T^{(i)}_{M,0}(f_1,\dots,f_{i+1})$. Then $m \leq T^{(i-1)}_{M,1}(f_1,\dots,f_{i+1}) - 1$ and $m \leq \sigma^{i-1}_{M,f_1} - 1$ (cf. 2.4 C) $(\bullet\bullet)$). As

$$\mathrm{im}(\overline{\delta}_{[t]}) = \ker\big[\mathcal{R}^{i-1}D_{R_+}(M/f_1M)_{m+1} \xrightarrow{f_t} \mathcal{R}^{i-1}D_{R_+}(M/f_1M)_{m+2}\big]$$

we now get from $(*)$ that $\bigcap_{t=2}^{i+1} \mathrm{im}(\delta_{[1]} \cdot \overline{\delta}_{[t]}) = 0$ for all $m \leq T^{(i)}_{M,0}(f_1, \ldots, f_{i+1})$. By $(**)$ it thus follows that

$$\bigcap_{s=1}^{i+1} \ker\big[\mathcal{R}^{i}D_{R_+}(M)_{m} \xrightarrow{f_s} \mathcal{R}^{i}D_{R_+}(M)_{m+1}\big] = 0,$$

if $m \leq T^{(i)}_{M,0}(f_1, \ldots, f_{i+1})$. ■

Now, we are ready to present the main result of this section. In order to formulate it, we introduce another notation:

2.5. Notation. Let M be a finitely generated graded R-module, let $i \in \mathbb{N}_0$ and let $f_1, \ldots, f_{i+1} \in R_1$ be an M- admissible sequence of linear forms. We introduce the intervals

$\mathbb{I}^{(i)}_M(f_1, \ldots, f_{i+1}) := \big\{m \in \mathbb{Z} \mid \sigma^i_{M,f_1} < m < \min\big\{T^{(i)}_{M,1}(f_1, \ldots, f_{i+1}),\ \nu^{i-1}_{M/f_1M}\big\}\big\}$

and

$$\mathbb{J}^{(i)}_M(f_1, \ldots, f_{i+1}) := \big\{m \in \mathbb{Z} \mid \sigma^{i-1}_{M,f_1} < m < \min\big\{T^{(i-1)}_{M,1}(f_1, \ldots, f_{i+1}),\ \nu^{i-1}_{M/f_1M}\big\}\big\},$$

where σ^{i-1}_{M,f_1} is defined according to 2.3 A), ν^{i-1}_{M/f_1M} is defined according to 1.6 B) and where $T^{(k)}_{M,1}(f_1, \ldots, f_{i+1})$ is defined according to 2.4 A). •

Now, the announced main result, which must be compared to [20, (3.2.4)], can be stated as below. For the definition of the invariants $\Delta^i_M(\bullet)$, $\nu^j_\bullet$ see 1.6 A), B), for the definition of the invariants σ^j_{M,f_1} see 2.3 A)).

Proposition 2.1. *Let M be a finitely generated and graded R-module. Let $i \in \mathbb{N}_0$ and let $f_1, \ldots, f_{i+1} \in R_1$ be an M-admissible sequence of linear forms. Then*

a) $0 \leq \Delta^i_M(m) \leq \Delta^i_M(m+1)$ *for all* $m \leq \nu^{i-1}_{M/f_1M} - 1$;

b) $0 \leq \Delta^i_M(m) \leq \Delta^i_M(m+1) - 1$ *for all* $m \in \mathbb{I}^{(i)}_M(f_1, \ldots, f_{i+1}) \cup \mathbb{J}^{(i)}_M(f_1, \ldots, f_{i+1})$;

c) $0 \leq \Delta^i_M(m) \leq \Delta^i_M(m+1) - 2$ *for all* $m \in \mathbb{I}^{(i)}_M(f_1, \ldots, f_{i+1}) \cap \mathbb{J}^{(i)}_M(f_1, \ldots, f_{i+1})$;

d) $\Delta^i_M(m) = 0$ *for all* $m \leq \min\big\{\sigma^{i-1}_{M,f_1} + 1, \sigma^i_{M,f_1} + 1, \nu^{i-1}_{M/f_1M}\big\}$.

Proof. "a)": The case $i = 0$ is clear by 2.3 C). So let $i > 0$. Let $m \leq \nu^{i-1}_{M/f_1M} - 1$. Then $\Delta^i_{M/f_1M}(m+1) = 0$. As $\gamma^i_{M,f,m}$ and $\gamma^{i-1}_{M,f,m}$ are ≥ 0, Lemma 2.3 gives $\Delta^i_M(m) \leq \Delta^i_M(m+1)$. As this inequality holds for all $m \leq \nu^{i-1}_{M/f_1M} - 1$ and as $\Delta^i_M(m) = 0$ for all $m \ll 0$, we obtain $0 \leq \Delta^i_M(m)$ for all $m \leq \nu^{i-1}_{M/f_1M}$.

"b), c)": By 2.2 C) we find a linear form $f_{i+2} \in R_1$ such that $f_1, \ldots, f_{i+1}, f_{i+2}$ is an M-admissible sequence.

Let $s \in \{2, \ldots, i+2\}$. Then, as in the proof of Lemma 2.4, the commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{f_1} & M(1) & \longrightarrow & M_{\langle 1 \rangle}(1) \\ f_s \downarrow & & f_s \downarrow & & f_s \downarrow \\ M(1) & \xrightarrow{f_1} & M(2) & \longrightarrow & M_{\langle 1 \rangle}(2) \end{array}$$

(in which $M_{\langle 1 \rangle} := M/f_1M$) together with 2.2 D), E), G) gives rise to the following commutative diagram with exact rows

$$(\bullet) \qquad \begin{array}{ccccc} \mathcal{R}^i D_{R_+}(M)_m & \xrightarrow{f_1} & \mathcal{R}^i D_{R_+}(M)_{m+1} & \xrightarrow{\varrho^i_{m+1}} & \mathcal{R}^i D_{R_+}(M_{\langle 1 \rangle})_{m+1} \\ f_s \downarrow & & f_s \downarrow & & f_s \downarrow \\ \mathcal{R}^i D_{R_+}(M)_{m+1} & \xrightarrow{f_1} & \mathcal{R}^i D_{R_+}(M)_{m+2} & \xrightarrow{\varrho^i_{m+2}} & \mathcal{R}^i D_{R_+}(M_{\langle 1 \rangle})_{m+2} \end{array}$$

for each $m \in \mathbb{Z}$ and each $s \in \{2, \ldots, i+2\}$. If we apply Lemma 2.4 to the $M_{\langle 1 \rangle}$-admissible sequence $f_2, \ldots, f_{i+2}$ (cf. (2.2) D)), we see that

$$(\bullet\bullet) \qquad \bigcap_{s=2}^{i+2} \ker \Big[\mathcal{R}^i D_{R_+}(M_{\langle 1 \rangle})_{m+1} \xrightarrow{f_s} \mathcal{R}^i D_{R_+}(M_{\langle 1 \rangle})_{m+2} \Big] = 0$$

whenever $m < T^{(i)}_{M_{\langle 1 \rangle},0}(f_2, \ldots, f_{i+2})$, thus whenever $m < T^{(i)}_{M,1}(f_1, \ldots, f_{i+1})$ (cf. 2.4 B) $(**)$ and $(*)$). So, let $m < T^{(i)}_{M,1}(f_1, \ldots, f_{i+1})$ and assume that the map $\mathcal{R}^i D_{R_+}(M)_m \xrightarrow{f_1} \mathcal{R}^i D_{R_+}(M)_{m+1}$ is not surjective, so that in the diagram $(\bullet)$ we have $\operatorname{im}(\varrho^i_{m+1}) \neq 0$. Then by $(\bullet\bullet)$, there is some $s \in \{2, \ldots, i+2\}$ with $f_s(\operatorname{im}(\varrho^i_{m+1})) \neq 0$. From this we see that the map ϱ^i_{m+2} in the diagram $(\bullet)$ is not zero. So, finally the map $\mathcal{R}^i D_{R_+}(M)_{m+1} \xrightarrow{f_1} \mathcal{R}^i D_{R_+}(M)_{m+2}$ is not surjective, too. As the map $\mathcal{R}^i D_{R_+}(M)_m \xrightarrow{f_1} \mathcal{R}^i D_{R_+}(M)_{m+1}$ is not surjective for $m = \sigma^i_{M,f_1} + 1$, we thus get inductively that $\mathcal{R}^i D_{R_+}(M)_m \xrightarrow{f_1} \mathcal{R}^i D_{R_+}(M)_{m+1}$ is not surjective for $\sigma^i_{M,f_1} < m < T^{(i)}_{M,1}(f_1, \ldots, f_{i+1})$. So, in the notation of (2.4) we obtain

$$\gamma^i_{M,f_1,m} \ \geq \ 1 \quad \text{for} \quad \sigma^i_{M,f_1} < m < T^{(i)}_{M,1}(f_1, \ldots, f_{i+1}).$$

In the same way we can show that

$$\gamma^{i-1}_{M,f_1,m} \ \geq 1 \quad \text{for} \quad \sigma^{i-1}_{M,f_1} < m < T^{(i-1)}_{M,1}(f_1, \ldots, f_{i+1}).$$

Therefore we can say

$$m \in \mathbb{I}^{(i)}_M(f_1, \ldots, f_{i+1}) \ \Rightarrow \ \gamma^i_{M,f_1,m} \geq 1 \ ,$$
$$m \in \mathbb{J}^{(i)}_M(f_1, \ldots, f_{i+1}) \ \Rightarrow \ \gamma^{i-1}_{M,f_1,m} \geq 1.$$

If $m \in \mathbb{I}^{(i)}_M(f_1, \ldots, f_{i+1})$ or $m \in \mathbb{J}^{(i)}_M(f_1, \ldots, f_{i+1})$ we also have $m < \nu^{i-1}_{M/f_1M}$, thus $\Delta^{i-1}_{M/f_1M}(m+1) = 0$. In view of Lemma 2.3, this proves claims b) and c).

"d)": Let $m < \min\Big\{\sigma^{i-1}_{M,f_1} + 1, \ \sigma^i_{M,f_1} + 1, \ \nu^{i-1}_{M/f_1M}\Big\}$. Then

$$\gamma^{i-1}_{M,f_1,m} \ = \ \gamma^i_{M,f_1,m} = 0 \quad \text{and} \quad \Delta^{i-1}_{M/f_1M}(m+1) = 0.$$

So, by Lemma 2.3 we have $\Delta^i_M(m) = \Delta^i_M(m+1)$. As $\Delta^i_M(m) = 0$ for all $m \ll 0$, this proves our claim. ■

Proposition 2.1 is the crucial step in the direction of the goal we are heading for in the next section of this paper: to give *a priori bounds of extended Severi type*. Observe that Proposition 2.1 gives a first insight into the behaviour of the deficiency function $n \mapsto \Delta^i_M(n)$ in the range $n \leq \nu^{i-1}_{M/fM}$, where $f \in R_1$ is an M-admissible linear form.

The last part of the present section is devoted to draw a consequence of Proposition 2.1 which will furnish the main ingredient to finally deduce the mentioned bounds of extended Severi type. We namely shall prove an "intermediate" bounding result for cohomological deficiency functions similar to what is shown in [20, (4.1.1)].

Lemma 2.5. *Let M be a finitely generated and graded R-module. Let $i \in \mathbb{N}_0$ and let $f_1, \ldots, f_{i+1} \in R_1$ be an M-admissible sequence of linear forms. Then in the notation of* 2.4 A), 1.6 B) *and* 2.3 A)

$$\nu^i_M \geq \min\left\{\sigma^{i-j-1}_{M_{\langle j\rangle},f_{j+1}} + 1,\ \sigma^{i-j}_{M_{\langle j\rangle},f_{j+1}} + 1 \mid 0 \leq j \leq i\right\}, \text{ where } M_{\langle j\rangle} := M/\sum_{l=1}^{j} f_l M.$$

Proof. (Induction on i). If we apply Proposition 2.1 d) with $i = 0$ and keep in mind that $\nu^{-1}_{M/f_1M} = \infty$ we see that $\Delta^0_M(m) = 0$ for all $m \leq \min\{\sigma^{-1}_{M,f_1} + 1, \sigma^0_{M,f_1} + 1\}$, so that

$$\nu^0_M \geq \min\{\sigma^{-1}_{M,f_1} + 1, \sigma^0_{M,f_1} + 1\} = \min\{\sigma^{0-1}_{M_{\langle 0\rangle},f_1} + 1,\ \sigma^{0-0}_{M_{\langle 0\rangle},f_1} + 1\}.$$

So, let $i > 0$. Then by Proposition 2.1 d) we have

$$\begin{aligned} \nu^i_M &\geq \min\left\{\sigma^{i-1}_{M,f_1} + 1,\ \sigma^i_{M,f_1} + 1,\ \nu^{i-1}_{M/f_1M}\right\} \\ &= \min\left\{\sigma^{i-0-1}_{M_{\langle 0\rangle},f_1} + 1,\ \sigma^{i-0}_{M_{\langle 0\rangle},f_1} + 1,\ \nu^{i-1}_{M_{\langle 1\rangle}}\right\}. \end{aligned}$$

If we apply the hypothesis of induction to the graded R-module $M_{\langle 1\rangle}$ and the $M_{\langle 1\rangle}$-admissible sequence $f_2, \ldots, f_{i+1} \in R_1$ (cf. 2.2 D)) and observe that $M_{\langle 1\rangle} \,/\, \sum_{l=2}^{k} f_l M_{\langle 1\rangle} = M_{\langle k\rangle}$ for $k = 1, \ldots, i$, we get

$$\nu^{i-1}_{M_{\langle 1\rangle}} \geq \min\left\{\sigma^{i-k-1}_{M_{\langle k\rangle},\, f_{k+1}} + 1,\ \sigma^{i-k}_{M_{\langle k\rangle},f_{k+1}} + 1 \mid 1 \leq k \leq i\right\}.$$

Altogether, this proves our claim. ■

Lemma 2.6. *Let M be a finitely generated and graded R-module, let $i \in \mathbb{N}_0$ and let $f \in R_1$ be M-admissible. Then*

$$\min\left\{\sigma^{i-1}_{M,f},\ \sigma^i_{M,f}\right\} \geq \min\left\{\nu^i_M,\ \nu^{i-1}_{M/fM}\right\} - 1.$$

Proof. For each $m \leq \min\{\nu^i_M, \nu^{i-1}_{M/fM}\} - 1$ we have $\Delta^i_M(m) = \Delta^i_M(m+1) = \Delta^{i-1}_{M/fM}(m+1) = 0$ and hence $\gamma^i_{M,f,m} + \gamma^{i-1}_{M,f,m} = 0$, (cf. Lemma 2.3, Remark 2.1). In view of the Definition 2.4 this proves our claim. ■

Now, we are ready to prove the previously announced bounding result for our deficiency functions.

Corollary 2.1. *Let M be a finitely generated and graded R-module. Let $i \in \mathbb{N}_0$ and let $f_1, \ldots, f_{i+1} \in R_1$ be an M-admissible sequence of linear forms. Let $\tau \in \mathbb{Z}$ and let $b \in \mathbb{N}_0$ with $\tau \leq \min\{T^{(i)}_{M,1}(f_1, \ldots, f_{i+1}),\ T^{(i-1)}_{M,1}(f_1, \ldots, f_{\max\{1,i\}})\}$ and $\Delta^i_M(\tau) \leq b$. Then*
a) $0 \leq \Delta^i_M(n) \leq \max\{0, b+n-\tau\}$ *for all* $n \leq \tau$ *;*
b) $\tau - b \leq \nu^i_M$ *;*
c) $\tau - b \leq \min\{T^{(i+1)}_{M,0}(f_1, \ldots, f_{i+1}),\ T^{(i)}_{M,0}(f_1, \ldots, f_{\max\{1,i\}})\} + 2.$

Proof. "a)": First, let $i > 0$. As $(M/f_1M)/\sum_{s=1}^{j} f_{s+1}(M/f_1M) = M_{\langle j+1\rangle}$ for all $j \in \{0, \ldots, i-1\}$, we can use Lemma 2.5, applied to the module M/f_1M and the M/f_1M-admissible sequence $f_2, \ldots, f_{i+1}$, in order to conclude that

$$\begin{aligned}
\nu^{i-1}_{M/f_1M} &\geq \min\{\sigma^{i-1-j-1}_{M_{\langle j+1\rangle}, f_{j+2}} + 1,\ \sigma^{i-1-j}_{M_{\langle j+1\rangle}, f_{j+2}} + 1 \mid 0 \leq j \leq i-1\} \\
&\geq \min\{\sigma^{(i-1)-j-1}_{M_{\langle 1+j\rangle}, f_{1+j+1}} - j - 1,\ \sigma^{i-j-1}_{M_{\langle 1+j\rangle}, f_{1+j+1}} - j - 1 \mid 0 \leq j < i\} + 2 \\
&= \min\{T^{(i-1)}_{M,1}(f_1, \ldots, f_{i+1}),\ T^{(i)}_{M,1}(f_1, \ldots, f_{i+1})\} + 2 \\
&= \min\{T^{(i-1)}_{M,1}(f_1, \ldots, f_{\max\{1,i\}}),\ T^{(i)}_{M,1}(f_1, \ldots, f_{i+1})\} + 2,
\end{aligned}$$

where the last two equalities follow from 2.4 C) ($\bullet$) and 2.4 B) ($*$) respectively. This shows that $\nu^{i-1}_{M/f_1M} \geq \tau + 2$ and hence

$$\begin{aligned}
\tau &\leq \min\{T^{(i)}_{M,1}(f_1, \ldots, f_{i+1}),\ \nu^{i-1}_{M/f_1M}\}, \\
\tau &\leq \min\{T^{(i-1)}_{M,1}(f_1, \ldots, f_{\max\{1,i\}}),\ \nu^{i-1}_{M/f_1M}\}.
\end{aligned}$$

If $i = 0$, we have the same inequalities, simply as $\nu^{-1}_{M/f_1M} = \infty$ and by the first statement of 2.4 C). We set $s = \min\{\sigma^{i-1}_{M,f_1} + 1,\ \sigma^i_{M,f_1} + 1,\ \nu^{i-1}_{M/f_1M}\}$. For $s \leq m < \tau$ we thus have (cf. 2.5)

$$m \in \mathbb{I}^{(i)}_M(f_1, \ldots, f_{i+1}) \cup \mathbb{J}^{(i)}_M(f_1, \ldots, f_{i+1})$$

and hence $0 \leq \Delta^i_M(m) \leq \Delta^i_M(m+1) - 1$, (cf. Proposition 2.1 b)). So, we get $0 \leq \Delta^i_M(n) \leq b + n - \tau$ for $s+1 \leq n \leq \tau$. By Proposition 2.1 d) we have $\Delta^i_M(n) = 0$ for all $n \leq s$. This proves claim a).

"b)": By statement a) we have $\Delta^i_M(n) = 0$ for all $n \leq \tau - b$. This proves our claim.

"c)": In the proof of statement a) we have seen that $\nu^{i-1}_{M/f_1M} \geq \tau + 2$. By statement b) we have $\nu^i_M \geq \tau - b$. So, by Lemma 2.6 we get

$$\min\{\sigma^{i-1}_{M,f_1},\ \sigma^i_{M,f_1}\} \ \geq\ \min\{\nu^i_M,\ \nu^{i-1}_{M/f_1M}\} - 1 \geq \tau - b - 1,$$

hence

$$\tau - b - 1 \ \leq\ \sigma^i_{M,f_1} \quad \text{and} \quad \tau - b - 1 \ \leq\ \sigma^{i-1}_{M,f_1}.$$

By our hypotheses

$$\tau - b - 1 \ \leq\ T^{(i)}_{M,1}(f_1, \ldots, f_{i+1}) - 1 \quad \text{and} \quad \tau - b - 1 \leq T^{(i-1)}_{M,1}(f_1, \ldots, f_{\max\{1,i\}}).$$

By statement (••) of 2.4 C) we thus obtain

$$\tau - b - 2 \ \leq\ T^{(i+1)}_{M,0}(f_1, \ldots, f_{i+1})$$

and, if $i > 0$

$$\tau - b - 2 \ \leq\ T^{(i)}_{M,0}(f_1, \ldots, f_i).$$

By the first statement of 2.4 C), the latter inequality also holds for $i = 0$. This proves statement c). ■

3. A Priori Bounds of Extended Severi Type

The goal of this section is to establish the bounds on cohomological deficiency functions which were already mentioned in the introduction. We first will introduce the occuring bounding functions. More precisely, we shall define functions

$$E^{(i)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}^{i+1} \times \mathbb{Z}_{\leq -i} \longrightarrow \mathbb{Z}, \tag{3.1}$$

$$G^{(i)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}^{i+1} \times \mathbb{Z}_{\leq -i} \longrightarrow \mathbb{Z}, \tag{3.2}$$

$$F^{(i)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}^{i+1} \longrightarrow \mathbb{Z}_{\leq -2i-1} \tag{3.3}$$

for each $i \in \mathbb{N}_0$. We do this by induction on i.

Assume first, that $i = 0$. For $e_0 \in \mathbb{N}_0$ and $b_0 \in \mathbb{Z}$ we set

$$E^{(0)}(e_0; b_0; n) := \max\{0, e_0 - b_0 + n\}, (\forall n \in \mathbb{Z}_{\leq 0}) \tag{3.4}$$

$$F^{(0)}(e_0; b_0) := \min\{0, b_0 - e_0\} - 1. \tag{3.5}$$

Assume now that $i > 0$ and that $E^{(j)}$ and $F^{(j)}$ have already been defined for all $j \in \mathbb{N}_0$ with $j < i$. Let $\underline{e} = (e_0, \ldots, e_i) \in \mathbb{N}_0^{i+1}$, $\underline{b} = (b_0, \ldots, b_i) \in \mathbb{Z}^{i+1}$.

We consider the i-tuples

a) $\underline{e}' := (e_0 + e_1, e_1 + e_2, \ldots, e_{i-1} + e_i) \in \mathbb{N}_0^i$; (3.6)

b) $\underline{b}^* := (b_1 - b_0, b_2 - b_1, \ldots, b_i - b_{i-1}) \in \mathbb{Z}^i$.

Then we set

$$t = (\underline{e}, \underline{b}) := F^{(i-1)}(\underline{e}'; \underline{b}^*) - 1$$

$$u = (\underline{e}, \underline{b}) := e_i - b_i$$

and define

$$E^{(i)}(\underline{e};\underline{b};n) := \begin{cases} \max\{0, u + \sum_{n+1\leq m\leq -i} E^{(i-1)}(\underline{e}';\underline{b}^*;m)\} & \text{if } t \leq n \leq -i \\ \max\{0, E^{(i)}(\underline{e};\underline{b};t) + n - t\} & \text{if } n < t \end{cases} \tag{3.7}$$

$$F^{(i)}(\underline{e};\,\underline{b}) := t - E^{(i)}(\underline{e};\underline{b};t) - 1. \tag{3.8}$$

Finally we set

$$G^{(i)}(\underline{e};\,\underline{b};n) := \begin{cases} 0, & \text{if } i = 0 \\ -\sum_{t<m\leq n} E^{(i-1)}(\underline{e}';\,\underline{b}^*;\,m)\ , & \text{if } i > 0. \end{cases} \tag{3.9}$$

Remark 3.1.

A) It is immediate from the above definitions that $E^{(i)}(\underline{e},\underline{b};n) \geq 0$ for all $n \leq -i$ and $E^{(i)}(\underline{e};\,\underline{b};\,n) = 0$ for all $n \leq F^{(i)}(\underline{e};\,\underline{b}) + 1$. Moreover, if $i > 0$, then $G^{(i)}(\underline{e};\,\underline{b};\,n) \leq 0$ for all $n \leq -i$, with equality for all $n \leq F^{(i-1)}(\underline{e}';\,\underline{b}^*) + 1$. Finally, with $\underline{0} := (0, \ldots, 0) \in \mathbb{N}_0^{i+1}$ we have

$$E^{(i)}\,(\underline{0};\,\underline{0};\,n) = G^{(i)}(\underline{0};\,\underline{0};\,n) = 0 \quad \text{for all } n \leq -i \quad \text{and } F^{(i)}(\underline{0};\,\underline{0}) = -2i - 1.$$

B) Let $\underline{e} = (e_0, \ldots, e_i)$, $\underline{a} = (a_0, \ldots, a_i) \in \mathbb{N}_0^{i+1}$. We write $\underline{e} \geq \underline{a}$ if $e_j \geq a_j$ for all $j \in \{0, \ldots, i\}$. If $\underline{e} \geq \underline{a}$ and using the above notations we have $u(\underline{e},\underline{b}) \geq u(\underline{a},\underline{b})$. If in addition $i > 0$, we have $\underline{e}' \geq \underline{a}'$, where $\bullet' : \mathbb{N}_0^{i+1} \to \mathbb{N}_0^i$ is defined according to (3.6) a). Using these observations we easily get by induction on i: If $\underline{e} \geq \underline{a}$, then

$$E^{(i)}(\underline{e};\,\underline{b};\,n) \geq E^{(i)}(\underline{a};\,\underline{b};\,n),\ G^{(i)}(\underline{e};\,\underline{b};\,n) \leq G^{(i)}(\underline{a};\,\underline{b};\,n), (\forall n \leq -i)$$

and $F^{(i)}(\underline{e};\underline{b}) \leq F^{(i)}(\underline{a};\underline{b})$.

C) Let $i, l \in \mathbb{N}_0$ with $i < l$. Then, it follows easily by induction on i, that

$$\begin{aligned} B_{(l)}^{(i)}(\underline{e};n) &\leq E^{(i)}(\underline{e};\,\underline{0};n) \quad \text{for all} \quad n \leq -i \\ C_{(l)}^{(i)}(\underline{e}) &\geq F^{(i)}(\underline{e};\,\underline{0}) + 1 \end{aligned}$$

where $B_{(l)}^{(i)}$ and $C_{(l)}^{(i)}$ are the bounding functions defined in [7, (5.1) - (5.4)]. •

Before we start to establish the announced bounds on our cohomological deficiency functions, we introduce a few notations and make some preparatory remarks. Let $R = \oplus_{n\geq 0} R_n$ be a positively graded homogeneous noetherian ring with artinian base ring R_0, let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. In this situation we introduce the notation

$$\underline{d}_M^{(i)} := \left(d_M^0(0), d_M^1(-1), \ldots, d_M^i(-i)\right) \in \mathbb{N}_0^{i+1}; \tag{3.10}$$

$$\underline{q}_M^{(i)} := (q_M^i(0), q_M^i(-1), \ldots, q_M^i(-i)) \in \mathbb{Z}^{i+1}. \tag{3.11}$$

where d_M^j and q_M^j are defined according to (1.1) B) and (1.5) respectively.

If $i \in \mathbb{N}$ we also may consider the i-tuples (cf. (3.6))

$$\big(\underline{d}_M^{(i)}\big)' \;:=\; \Big(d_M^0(0){+}d_M^1(-1),\, d_M^1(-1){+}d_M^2(-2), \dots ,\, d_M^{i-1}(-i{+}1){+}d_M^i(-i)\Big) \in \mathbb{N}_0^i$$

and

$$\left(\underline{q}_M^{(i)}\right)^* = (q_M^i(-1) - q_M^i(0), q_M^i(-2) - q_M^i(-1), \dots, q_M^i(-i) - q_M^i(-i+1)) \in \mathbb{Z}^i$$

Remark 3.2. A) Let $R = \oplus_{n \geq 0} R_n$ and M be as above but assume in addition that R_0 has infinite residue fields (cf. 2.1 A)). Let $i \in \mathbb{N}$ and let $f \in R_1$ be M-admissible. Then, the exact sequences 2.2 G) show that

$$d^j_{M/fM}(-j) \;\leq\; d^j_M(-j) + d^{j+1}_M(-j-1)$$

for all $j \in \mathbb{N}_0$. In the notation used in Remark 3.1 B), this means that

$$\underline{d}^{(i-1)}_{M/fM} \;\leq\; (\underline{d}^{(i)}_M)', \quad (\forall\, i \in \mathbb{N}).$$

B) By 2.3 B) we also have $q^j_{M/fM}(-l) = q^{j+1}_M(-l-1) - q^{j+1}_M(-l)$ for all $j \in \mathbb{N}_0$ and all $l \in \mathbb{Z}$. This shows that

$$\underline{q}^{(i-1)}_{M/fM} \;=\; (\underline{q}^{(i)}_M)^*, \quad (\forall\, i \in \mathbb{N}).$$

C) In view of Remark 3.1 B) we get from part A) and B) the estimates

$$E^{(i-1)}\big(\underline{d}^{(i-1)}_{M/fM};\; \underline{q}^{(i-1)}_{M/fM};\, m\big) \;\leq\; E^{(i-1)}\big(\big(\underline{d}^{(i)}_M\big)';\; \big(\underline{q}^{(i)}_M\big)^*; m\big), \quad (\forall\, m \leq -i+1)$$

and

$$F^{(i-1)}\big(\underline{d}^{(i-1)}_{M/fM};\; \underline{q}^{(i-1)}_{M/fM}\big) \;\geq\; F^{(i-1)}\big(\big(\underline{d}^{(i)}_M\big)';\; \big(\underline{q}^{(i)}_M\big)^*\big)$$

whenever $i \in \mathbb{N}$.

D) Keep the above notations and hypotheses and choose some $r \in \mathbb{Z}$. By Lemma 2.3

$$\Delta^i_M(m) \leq \Delta^i_M(m+1) + \Delta^{i-1}_{M/fM}\;(m+1)$$

for all $m \in \mathbb{Z}$. For all $n \leq r$ we thus get

$$\Delta^i_M(n) \;\leq\; \Delta^i_M(r) + \sum_{n+1 \leq m \leq r} \Delta^{i-1}_{M/fM}(m). \qquad \bullet$$

Now, we are ready to prove the crucial bounding result of this section. In its statement b) we use the notational convention

$$F^{(-1)}\big(\big(\underline{d}^{(0)}_M\big)';\; \big(\underline{q}^{(0)}_M\big)^*\big) \;:=\; 0\;.$$

Proposition 3.1. *Let $R = \oplus_{n \geq 0} R_n$ be a positively graded noetherian homogeneous ring such that R_0 is artinian with infinite residue fields. Let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Then*
a) $G^{(i)}\big(\underline{d}^{(i)}_M;\; \underline{q}^{(i)}_M;\, n\big) \;\leq\; \Delta^i_M(n) \;\leq\; E^{(i)}\;\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\; n\big)$ *for all* $n \leq -i$.

b) $\Delta^i_M(n) \geq 0$ *for all* $n \leq F^{(i-1)}\big((\underline{d}^{(i)}_M)'; (\underline{q}^{(i)}_M)^*\big) + 1.$
c) $\nu^i_M \geq F^{(i)}\big(\underline{d}^{(i)}_M; \underline{q}^{(i)}_M\big) + 1.$
d) *If* $f_1, \dots, f_{i+1} \in R_1$ *is an* M*-admissible sequence of linear forms, then*

$$\min\Big\{T^{(i+1)}_{M,0}(f_1, \dots, f_{i+1}),\ T^{(i)}_{M,0}(f_1, \dots, f_{\max\{1,i\}})\Big\} \ \geq\ F^{(i)}\big(\underline{d}^{(i)}_M; \underline{q}^{(i)}_M\big) - 1\ .$$

Proof. We fix an M-admissible sequence of linear forms $f_1, \dots, f_{i+1} \in R_1$ and proceed by induction on i.
Assume first, that $i = 0$. Then, by 2.4 C) we have $\min\{T^{(0)}_{M,1}(f_1),\ T^{(-1)}_{M,1}(f_1)\} = \infty$. As $\Delta^0_M(0) = d^0_M(0) - q^0_M(0) \geq 0$ (cf. 2.3 C)) we thus may apply Corollary 2.1 with $\tau = 0$ and $b = d^0_M(0) - q^0_M(0)$ and obtain in view of (3.4), (3.5) and (3.9):

$$\begin{aligned} G^{(0)}\big(\underline{d}^{(0)}_M; \underline{q}^{(0)}_M; n\big) = 0 \leq \Delta^0_M(n) \leq{}& \max\{0, d^0_M(0) - q^0_M(0) + n\} \\ ={}& E^{(0)}\big(\underline{d}^{(0)}_M; \underline{q}^{(0)}_M; n\big) \ \text{ for all } \ n \in \mathbb{Z}_{\leq 0}, \\ \nu^0_M \geq q^0_M(0) - d^0_M(0) \geq{}& F^{(0)}\big(\underline{d}^{(0)}_M;\ \underline{q}^{(0)}_M\big) + 1 \end{aligned}$$

and finally

$$\min\{T^{(1)}_{M,0}(f_1), T^{(0)}_{M,0}(f_1)\} \geq q^0_M(0) - d^0_M(0) - 2 \geq F^{(0)}\big(\underline{d}^{(0)}_M;\ \underline{q}^{(0)}_M\big) - 1.$$

This proves our claim if $i = 0$.
So, let $i > 0$. We set $M_{\langle 1\rangle} := M/f_1M$ and

$$\begin{aligned} t &:= F^{(i-1)}\big((\underline{d}^{(i)}_M)'; (\underline{q}^{(i)}_M)^*\big) - 1, \\ u &:= u\big(\underline{d}^{(i)}_M, \underline{q}^{(i)}_M\big) = d^i_M(-i) - q^i_M(-i) = \Delta^i_M(-i). \end{aligned}$$

If we apply induction to the R-module $M_{\langle 1\rangle}$ and the $M_{\langle 1\rangle}$-admissible sequence $f_2, \dots, f_{i+1} \in R_1$ and keep in mind the inequalities. Remark of 3.2 C) and the equalities $(*)$ and $(**)$ of 2.4 B), we get

$(*)$ $\quad \Delta^{i-1}_{M_{\langle 1\rangle}}(m) \leq E^{(i-1)}\big((\underline{d}^{(i)}_M)';\ (\underline{q}^{(i)}_M)^*;\ m\big), \quad (\forall\, m \leq -i+1),$

$(**)$ $\min\{T^{(i)}_{M,1}(f_1, \dots, f_{i+1}),\ T^{(i-1)}_{M,1}(f_1, \dots, f_{i+1})\} \geq F^{(i-1)}\big((\underline{d}^{(i)}_M)';\ (\underline{q}^{(i)}_M)^*\big) - 1.$

If we apply the estimate of Remark 3.2 D) with $r = -i$, we get from $(*)$

$$\Delta^i_M(n) \leq u + \sum_{n+1 \leq m \leq -i} E^{(i-1)}\big((\underline{d}^{(i)}_M)';\ (\underline{q}^{(i)}_M)^*; m\big) \quad \text{for all} \quad n \leq -i\ .$$

In view of the definition this means that

$(\blacktriangle)$ $\qquad \Delta^i_M(n)\ \leq\ E^{(i)}\big(\underline{d}^{(i)}_M;\ \underline{q}^{(i)}_M;\ n\big) \quad$ if $\quad t \leq n \leq -i$

In particular $\Delta^i_M(t) \leq E^{(i)}\big(\underline{d}^{(i)}_M;\ \underline{q}^{(i)}_M;\ t\big)$. By $(**)$ we also have

$$\min\{T^{(i)}_{M,1}(f_1,\dots,f_{i+1}),\ T^{(i-1)}_{M,1}(f_1,\dots,f_{i+1})\} \ \geq t.$$

So, we may apply Corollary 2.1 with $\tau = t$ and with $b = E^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, t\big)$ in order to get

$$(\bullet) \qquad 0 \leq \Delta^i_M(n) \leq \max\Big\{0, E^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, t\big) + n - t\Big\} \quad \text{for all} \quad n \leq t,$$

$$(\bullet\bullet) \qquad \nu^i_M \geq t - E^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, t\big),$$

$$(\bullet\bullet\bullet) \quad \min\Big\{T^{(i+1)}_{M,0}(f_1,\dots,f_{i+1}),\ T^{(i)}_{M,0}(f_1,\dots,f_i)\Big\} \geq t - E^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, t\big) - 2.$$

By (3.7) we obtain from $(\bullet)$ that

$$(\blacktriangle\blacktriangle) \qquad 0 \leq \Delta^i_M(n) \leq E^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, n\big) \quad \text{for all} \quad n \leq t.$$

In view of $(\blacktriangle)$ this already proves the inequalities

$$\Delta^i_M(n) \leq E^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, n\big) \quad \text{for all} \quad n \leq -i\ .$$

If we choose $r \in \{t,\dots,-i\}$, apply Remark 3.2 D) with $n = t$ and keep in mind $(*)$ and the definition, we get

$$\begin{aligned}\Delta^i_M(t) &\leq \Delta^i_M(r) + \sum_{t<m\leq r} \Delta^{i-1}_{M_{\langle 1\rangle}}(m) \\ &\leq \Delta^i_M(r) + \sum_{t<m\leq r} E^{(i-1)}\big((\underline{d}^{(i)}_M)';\, (\underline{q}^{(i)}_M)^*;\, m\big) \\ &= \Delta^i_M(r) - G^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, r\big).\end{aligned}$$

By $(\blacktriangle\blacktriangle)$ we know that $\Delta^i_M(t) \geq 0$. So, we see that $G^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, r\big) \leq \Delta^i_M(r)$ for all $r \in \{t,\dots,-i\}$. By Remark 3.1 A) we have $G^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, n\big) = 0$ for all $n \leq t$. In view of $(\blacktriangle\blacktriangle)$ we now get $G^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, n\big) \leq \Delta^i_M(n)$ for all $n \leq -i$. So, statements a) and b) are shown completely.

By (3.8) we have $F^{(i)}(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M) \leq t - E^{(i)}(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, t) - 1$. So, the inequalities $(\bullet\bullet)$ and $(\bullet\bullet\bullet)$ prove statements c) and d). ■

Now, we are ready to prove the main result of the present section:

Theorem 3.1. *Let $R = \oplus_{n\geq 0} R_n$ be a positively graded noetherian homogeneous ring such that R_0 is artinian. Let M be a finitely generated graded R-module. Then, for each $i \in \mathbb{N}_0$*

a) $G^{(i)}\big(\underline{d}^{(i)}_M\ \underline{q}^{(i)}_M;\, n\big) \leq \Delta^i_M(n) \leq E^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M;\, n\big)$ *for all* $n \leq -i$.

b) $\Delta^i_M(n) \geq 0$ *for all* $n \leq F^{(i-1)}\big((\underline{d}^{(i)}_M)';\ (\underline{q}^{(i)}_M)^*\big) + 1$.

c) $\Delta^i_M(n) = 0$ *for all* $n \leq F^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M\big) + 1$*. Hence* $\nu^i_M \geq F^{(i)}\big(\underline{d}^{(i)}_M;\, \underline{q}^{(i)}_M\big) + 1$.

Proof. Let $\mathbf{x}$ be an indeterminate. Let $\mathfrak{m}^{(1)}_0,\dots,\mathfrak{m}^{(r)}_0$ be the different maximal ideals of R_0. Let $S = R_0[\mathbf{x}] \ \setminus \bigcup_{j=1}^{r} \mathfrak{m}^{(j)}_0 R_0[\mathbf{x}]$ and consider the positively graded noetherian homogeneous ring $R' := R'_0 \otimes_{R_0} R$ with artinian base ring $R'_0 := S^{-1}R_0[\mathbf{x}]$. Moreover, let M' be the finitely generated and graded R'-module $R'_0 \otimes_{R_0} M = R' \otimes_R M$. As $R'_+ = R_+ \cdot R'$ the observations of [9, (15.2.2) (ii), (vi)] give rise to natural isomorphisms of R'_0-modules $\mathcal{R}^i D_{R'_+}(M')_n \cong R'_0 \otimes_{R_0} \mathcal{R}^i D_{R_+}(M)_n$, which show that $d^i_{M'}(n) = d^i_M(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. In particular, we get $q^i_{M'} = q^i_M$ for all $i \in \mathbb{N}_0$ and hence

$$\underline{d}^{(i)}_{M'} = \underline{d}^{(i)}_M \quad \text{and} \quad \underline{q}^{(i)}_{M'} = \underline{q}^{(i)}_M \quad \text{for all} \quad i \in \mathbb{N}_0.$$

This allows us to replace R and M respectively by R' and M'. As R' has infinite residue fields, we may conclude by Proposition 3.1. ■

Now, let X be a projective scheme over the artinian ring R_0 and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $i \in \mathbb{N}_0$. Then, using the notation of 1.7 A), B) we set

$$\underline{h}^{(i)}_{\mathcal{F}} := \left(h^0_{\mathcal{F}}(0),\ h^1_{\mathcal{F}}(-1), \dots, h^i_{\mathcal{F}}(-i)\right) \in \mathbb{N}_0^{i+1}; \tag{3.12}$$

$$\underline{p}^{(i)}_{\mathcal{F}} := \left(p^i_{\mathcal{F}}(0), p^i_{\mathcal{F}}(-1), \dots, p^i_{\mathcal{F}}(-i)\right) \in \mathbb{Z}^{i+1}. \tag{3.13}$$

Using this notation we get from Theorem 3.1, and from 1.7 A) $(*)$, D) $(\blacktriangle\blacktriangle)$:

Corollary 3.1. *Let X be a projective scheme over the artinian ring R_0, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$. Then*
a) $G^{(i)}\big(\underline{h}^{(i)}_{\mathcal{F}}; \underline{p}^{(i)}_{\mathcal{F}}; n\big) \leq \Delta^i_{\mathcal{F}}(n) \leq E^{(i)}\big(\underline{h}^{(i)}_{\mathcal{F}};\ \underline{p}^{(i)}_{\mathcal{F}};\ n\big)$ *for all* $n \leq -i$.
b) $\Delta^i_{\mathcal{F}}(n) \geq 0$ *for all* $n \leq F^{(i-1)}\big((\underline{h}^{(i)}_{\mathcal{F}})';\ (\underline{p}^{(i)}_{\mathcal{F}})^*\big) - 1$.
c) $\Delta^i_{\mathcal{F}}(n) = 0$ *for all* $n \leq F^{(i)}\big(\underline{h}^{(i)}_{\mathcal{F}};\ \underline{p}^{(i)}_{\mathcal{F}}\big) + 1$, *hence* $\nu^i_{\mathcal{F}} \geq F^{(i)}\big(\underline{h}^{(i)}_{\mathcal{F}};\ \underline{p}^{(i)}_{\mathcal{F}}\big) + 1$. ■

Remark 3.3.
A) For $i \in \mathbb{N}_0$ let $\underline{0} \in \mathbb{N}_0^{n+1}$ be defined as in Remark 3.1 A). Let R and M be as in Theorem 3.1. Then, by 1.6 C) $(**)$ we have $q^j_M = 0$ and $\Delta^j_M = d^j_M$ for all $j < \lambda(M) - 1$. So, statements a) and b) of Theorem 3.1 give the estimates

$$d^i_M(n) \leq E^{(i)}\big(\underline{d}^{(i)}_M;\ \underline{0}; n\big) \quad \text{for all} \quad n \leq -i$$
$$d^i_M(n) = 0, \quad \text{for all} \quad n \leq F^{(i)}\big(\underline{d}^{(i)}_M;\ \underline{0}\big) + 1,$$

whenever $i < \lambda(M) - 1$.

This is the same kind of bounds as given in [7, Theorem 5.2]. A closer examination of the definitions of the bounding functions $B^{(i)}_{(l)}, C^{(i)}_{(l)}$ and $E^{(i)}, F^{(i)}$ shows that in general the values of $E^{(i)}\big(\underline{d}^{(i)}_M;\ \underline{0}; n\big)$ will be much larger than those of $B^{(i)}_{(l)}\big(\underline{d}^{(i)}_M; n\big)$ and that in general $F^{(i)}\big(\underline{d}^{(i)}_M, \underline{0}\big) \ll C^{(i)}_{(l)}\big(\underline{d}^{(i)}_M\big)$. So, the bounds we get from Theorem 3.1 in the range $i < \lambda(M) - 1$ are in general much weaker than those furnished by [7, Theorem 5.2]. The reason for this is that the Severi bounds of section 5 of [7] are deduced by means of the method of "linear systems of linear forms" which gives sharper estimates than the method available in the case of extended Severi bounds which bases on Lemma 2.4.

B) If we compare the bounds of Corollary 3.1 in the range $i < \delta(\mathcal{F})$ with those given by [7, Corollary 5.3] the same comment as in part A) applies. Nevertheless, the comparison of Theorem 3.1 with [7, Theorem 5.2] (resp. of Corollary 3.1 with [7, Corollary 5.3] shows that - up to the sharpness of the occuring estimates - the results of the present section naturally extend those given in Sec. 5 of [7]. For this reason, we call the bounds given in Theorem 3.1 and Corollary 3.1 bounds of *extended Severi type.* •

As in [7, Remark 6, Remark 10], we now want to replace the bounds established above by weaker but simpler estimates. To do so, we first introduce some notation. So, let $i \in \mathbb{N}_0$ and let $\underline{e} = (e_0, \ldots, e_i) \in \mathbb{N}_0^{i+1}, \underline{b} = (b_0, \ldots, b_i) \in \mathbb{Z}^{i+1}$. We write

$$S = S^{(i)}(\underline{e}; \underline{b}) := \sum_{j=0}^{i} \binom{i}{j} (e_j + \mid b_j \mid). \tag{3.14}$$

Our first aim is to establish some inequalities, which relate the function

$$S^{(i)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}^{i+1} \longrightarrow \mathbb{N}_0 \ ; \ ((\underline{e}, \underline{b}) \longmapsto S^{(i)}(\underline{e}; \underline{b}))$$

to our previous bounding functions $E^{(i)}, F^{(i)}$ and $G^{(i)}$ of (3.1), (3.2), (3.3).

Remark 3.4.

A) Keep all previous notations. Our first aim is to prove the inequalities

$$(\bullet) \qquad \max\{E^{(i)}(\underline{e}; \underline{b}; n) \bigm| n \leq -i\} < \frac{1}{2}(2(1+S))^{2^i}$$

$$(\bullet\bullet) \qquad F^{(i)}(\underline{e}; \underline{b}) > -(2(1+S))^{2^i}.$$

We do this by induction on i. For $i = 0$, (3.4) and (3.5) give indeed $E^{(0)}(e_0; b_0; n) \leq e_0 + \mid b_0 \mid < S + 1$ for all $n < 0$ and $F^{(0)}(e_0; b_0) \geq -e_0 - \mid b_0 \mid -1 = -S - 1$ and hence our claim.

So, let $i > 0$. If $S = 0$, we have $\underline{e} = \underline{b} = \underline{0}$ and hence our claim is clear as $E^{(2)}(\underline{0}; \underline{0}; 0) = 0 < (1/2)2^{2^i}$ for all $n \in \mathbb{Z}_{\leq -i}$ and $F^{(i)}(\underline{0}; \underline{0}) = -2i - 1 > -2^{2^i}$ (cf. Remark 3.1 A)). We thus may assume that $S > 0$. On use of the Pascal formulas we get

$$S^{(i-1)}(\underline{e}'; \underline{b}^*) \leq S^{(i)}(\underline{e}; \underline{b}) = S.$$

So, by induction we have $E^{(i-1)}(\underline{e}'; \underline{b}^*; m) < \frac{1}{2}(2(1+S))^{2^{i-1}}$ for all $m \leq -i+1$, and $F^{(i-1)}(\underline{e}'; \underline{b}^*) > -(2(1+S))^{2^{i-1}}$. Let $t := F^{(i-1)}(\underline{e}'; \underline{b}^*) - 1$. As $E^{(i-1)}(\underline{e}'; \underline{b}^*; m)$

$= 0$ for all $m \leq t{+}2$ (cf. Remark 3.1 A)) we may use (3.7) to write for all $n \leq -i$:

$$\begin{aligned}
E^{(i)}(\underline{e};\underline{b};n) &\leq e_i + \mid b_i \mid + \sum_{t+3\leq m\leq -i} E^{(i-1)}(\underline{e}';\underline{b}^*;m)\\
&< S + \sum_{t+3\leq m\leq -i} \frac{1}{2}(2(1+S))^{2^{i-1}}\\
&< 1 + S + \big(-F^{(i)}(\underline{e}';\underline{b}^*) - 1\big)\frac{1}{2}\big(2(1+S)\big)^{2^{i-1}}\\
&< 1 + S + ((2(1+S))^{2^{i-1}} - 1)\frac{1}{2}(2(1+S))^{2^{i-1}}\\
&\leq \frac{1}{2}(2(1+S))^{2^i}.
\end{aligned}$$

This proves statement (•).

By (3.8) we also may write $F^{(i)}(\underline{e};\underline{b}) = t - E^{(i)}(\underline{e};\underline{b};t) - 1 > -(2(1+S))^{2^{i-1}} - (1/2)(2(1+S))^{2^i} - 2$. As $S \geq 1$, we have $(2(1+S))^{2^{i-1}} + 2 \leq (1/2)(2(1+S))^{2^i}$ and hence obtain the inequality (••).

B) In the above proof we have seen that

(■) $\quad F^{(i-1)}(\underline{e}';\underline{b}^*) > -(2(1+S))^{2^{i-1}}, (\forall i \in \mathbb{N}).$

So, by (3.9) it follows easily

(■■) $\quad \min\{G^{(i)}(\underline{e};\underline{b};n) \mid n \leq -i\} > -\frac{1}{2}(2(1+S))^{2^i}, (\forall i \in \mathbb{N}_0).$ •

Now, we are ready to prove the following bounding result.

Proposition 3.2. *Let $R = \oplus_{n\geq 0} R_n$ be a positively graded homogeneous noetherian ring with artinian base ring R_0. Let M be a finitely generated and graded R-module. Let $i \in \mathbb{N}_0$ and let*

$$S_M^{(i)} := S^{(i)}(\underline{d}_M^{(i)}, \underline{q}_M^{(i)}) = \sum_{j=0}^{i} \binom{i}{j} (d_M^j(-j) + |q_M^i(-j)|).$$

Then

a) $|\Delta_M^i(n)| < \frac{1}{2}(2(1+S_M^{(i)}))^{2^i}$ *for all* $n \leq -i$;

b) $\Delta_M^i(n) \geq 0$ *for all* $n \leq -(2(1+S_M^{(i)}))^{2^{i-1}} + 2$;

c) $\Delta_M^i(n) = 0$ *for all* $n \leq -(2(1+S_M^{(i)}))^{2^i} + 2$, *thus* $\nu_M^i \geq -(2(1+S_M^{(i)}))^{2^i} + 2$.

Proof. Clear from Theorem 3.1 in view of the inequalities of Remark 3.4 A) (•), (••) and B) (■), (■■). ■

Proposition 3.3. *Let X be a projective scheme over an artinian ring R_0, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$. Let*

$$S_{\mathcal{F}}^{(i)} := S_{\mathcal{F}}^{(i)}(\underline{h}_{\mathcal{F}}^{(i)}, \underline{p}_{\mathcal{F}}^{(i)}) = \sum_{j=0}^{i} \binom{i}{j} (h_{\mathcal{F}}^j(-j) + |p_{\mathcal{F}}^i(-j)|).$$

Then

a) $\left|\Delta^i_{\mathcal{F}}(n)\right| < \frac{1}{2}(2(1+S^{(i)}_{\mathcal{F}}))^{2^i}$ *for all* $n \leq -i$;

b) $\Delta^i_{\mathcal{F}}(n) \geq 0$ *for all* $n \leq -(2(1+S^{(i)}_{\mathcal{F}}))^{2^{i-1}} + 2$;

c) $\Delta^i_{\mathcal{F}}(n) = 0$ *for all* $n \leq -(2(1+S^{(i)}_{\mathcal{F}}))^{2^i} + 2$, *thus* $\nu^i_{\mathcal{F}} \geq -(2(1+S^{(i)}_{\mathcal{F}}))^{2^i} + 2$.

Proof. Clear from Theorem 3.1 by means of the inequalities of Remark 3.4 A) $(\bullet), (\bullet\bullet)$ and B) $(\blacksquare), (\blacksquare)$. ∎

4. Bounding Systems

Here again, let $R = \oplus_{n\geq 0} R_n$ be a noetherian positively graded homogeneous ring with artinian base ring R_0. For each $i \in \mathbb{N}_0$ and each finitely generated graded R-module M, Theorem 3.1 c) shows that the i-th cohomological postulation number of M (cf. 1.6 B)) satisfies the inequality

$$\nu^i_M \geq F^{(i)}\big(\underline{d}^{(i)}_M;\ \underline{q}^{(i)}_M\big) + 1, \tag{4.1}$$

where $\underline{d}^{(i)}_M$ and $\underline{q}^{(i)}_M$ are defined respectively according to (3.10), (3.11) and where $F^{(i)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}^{i+1} \to \mathbb{Z}_{\leq -2i-1}$ is the bounding function defined by (3.8). This tells us, that ν^i_M has a lower bound, which depends only on the $2(i+1)$ numbers.

$$d^0_M(0), d^1_M(-1), \ldots, d^i_M(-i);\quad q^i_M(0), q^i_M(-1), \ldots, q^i_M(-i).$$

So, in the terminology of [7] (Def. 4.8), the $2(i+1)$ invariants

$$d^0_{(\bullet)}(0), d^1_{(\bullet)}(-1), \ldots, d^i_{(\bullet)}(-i);\quad q^i_{(\bullet)}(0), q^i_{(\bullet)}(-1), \ldots, q^i_{(\bullet)}(-i) \tag{4.2}$$

form a lower bounding system for the invariant $\nu^i_{(\bullet)}$ on the class of all finitely generated and graded R-modules. Our first aim is to show, that we may delete certain of the invariants $q^i_{(\bullet)}(-k)$ in (4.2) and still get a lower bounding system for $\nu^i_{(\bullet)}$. We start our presentation with a few auxiliary results on the arithmetic of cohomological Hilbert functions and cohomological Hilbert polynomials.

Lemma 4.1. *Let M be a finitely generated and graded R-module, let $f \in R_1$ be an M-admissible linear form and let $i \in \mathbb{N}$, $k \in \mathbb{N}_0$. Then*

a) $\displaystyle\sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}\, q^{i+k}_M(-j) \;=\; -\sum_{l=0}^{i-1}(-1)^{i+l}\binom{i-1}{l}\, q^{i+k-1}_{M/fM}(-l)$;

b) $\displaystyle\sum_{j=0}^{i}\binom{i}{j}\, d^{j+k}_M(-j) \;\geq\; \sum_{l=0}^{i-1}\binom{i-1}{l}\, d^{l+k}_{M/fM}(-l)$;

c) $\displaystyle d^i_M(n) \;\leq\; d^i_M(-i) \;+ \sum_{n<m\leq -i} d^{i-1}_{M/fM}(m)\ ,\quad \forall\, n \leq -i\ .$

Proof. "a)": By 2.3 B) we have $q^{i+k-1}_{M/fM}(-l) = q^{i+k}_M(-l-1) - q^{i+k}_M(-l)$ for all $l \in \mathbb{Z}$. So, the right hand side of the equation stated in a) takes the form

$$-\sum_{l=0}^{i-1}(-1)^{i+l}\binom{i-1}{l}\left[q_M^{i+k}(-l-1)-q_M^{i+k}(-l)\right]=$$

$$(-1)^i q_M^{i+k}(0)+\sum_{j=1}^{i-1}\left[(-1)^{i+j}\binom{i-1}{j-1}+(-1)^{i+j}\binom{i-1}{j}\right]q_M^{i+k}(-j)+q_M^{i+k}(-i)$$

and hence coincides with the left hand side of the equation stated in a).

"b)": The sequences 2.2 G) show that $d_{M/fM}^{l+k}(-l)\leq d_M^{l+k+1}(-l-1)+d_M^{l+k}(-l)$ for all $l\in\mathbb{N}_0$. So, the right hand side of the inequality stated in b) is not bigger than

$$\sum_{l=0}^{i-1}\binom{i-1}{l}\left[d_M^{l+k+1}(-l-1)+d_M^{l+k}(-l)\right]=$$

$$d_M^k(0)+\sum_{j=1}^{i-1}\left[\binom{i-1}{j-1}+\binom{i-1}{j}\right]d_M^{j+k}(-j)+d_M^{i+k}(-i).$$

But this is precisely the left hand side of the stated inequality.

"c)": The sequences (2.2) G) show that $d_M^i(n)\leq d_M^i(n+1)+d_{M/fM}^{i-1}(n+1)$ for all $n\in\mathbb{Z}$. This immediately proves the stated inequality. ■

As a first consequence we now get (with the convention $\binom{-1}{0}:=1$ in statement b))

Lemma 4.2. *Let M be a finitely generated and graded R-module and let $i\in\mathbb{N}_0$. Then*

a) $0\leq\sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}q_M^i(-j)\leq\sum_{j=0}^{i}\binom{i}{j}d_M^j(-j)$;

b) $d_M^i(n)\leq\sum_{j=0}^{i}\binom{-n-j-1}{i-j}\left[\sum_{l=0}^{i-j}\binom{i-j}{l}d_M^{i-l}(l-i)\right],\quad\forall n\leq -i.$

Proof. As in the proof of Theorem 3.1 we may assume that R_0 has infinite residue fields and hence that there is an M-admissible linear form $f\in R_1$ (cf. 2.2 C)). We prove both statements by induction on i.

"a)": By 1.5 (••) and 2.3 C), we have $0\leq q_M^0(0)\leq d_M^0(0)$. This proves the case $i=0$.

So, let $i>0$. By induction we have

$$0\leq\sum_{l=0}^{i-1}(-1)^{i+l-1}\binom{i-1}{l}q_{M/fM}^{i-1}(-l)\ \leq\ \sum_{l=0}^{i-1}\binom{i-1}{l}d_{M/fM}^{l}(-l).$$

Now, the stated inequalities are obvious by Lemma 4.1 a) b), applied with $k = 0$.

"b)": By 2.3 C) we have $d^0_M(n) \leq d^0_M(0)$ for all $n \leq 0$. This proves the case $i = 0$. So, let $i > 0$. By Lemma 4.1 c), and by induction we have

$$d^i_M(n) \leq d^i_M(-i) + \sum_{n<m\leq -i} \left\{ \sum_{j=0}^{i-1} \binom{-m-j-1}{i-1-j} \Big[\sum_{l=0}^{i-1-j} \binom{i-1-j}{l} d^{i-1-l}_{M/fM}(l-i+1) \Big] \right\}$$

for all $n \leq -i$. The inequality of Lemma 4.1 b) allows to write

$$\sum_{l=0}^{i-1-j} \binom{i-1-j}{l} d^{i-1-l}_{M/fM}(l-i+1) =$$
$$\sum_{l=0}^{i-j-1} \binom{i-j-1}{i-j-1-l} d^{(i-j-1-l)+j}_{M/fM}\big(-(i-j-1-l)-j\big) =$$
$$\sum_{h=0}^{i-j-1} \binom{i-j-1}{h} d^{h+j}_{M(-j)/fM(-j)}(h) \leq \sum_{p=0}^{i-j} \binom{i-j}{p} d^{p+j}_{M(-j)}(-p) =$$
$$\sum_{p=0}^{i-j} \binom{i-j}{i-j-p} d^{i-(i-j-p)}_M\big((i-j-p)-i\big) = \sum_{l=0}^{i-j} \binom{i-j}{l} d^{i-l}_M(l-i).$$

Therefore we get our claim as $\sum_{n<m\leq -i} \binom{-m-j-1}{i-1-j} = \binom{-n-j-1}{i-j}$ for all $j \in \{0, \ldots, i-1\}$. ∎

Lemma 4.3. *Let M be a finitely generated and graded R-module. Then*:
a) $0 \leq q^0_M(0) \leq d^0_M(0)$;
b) $-d^0_M(0) \leq q^1_M(0)$;
c) $q^1_M(0) \leq \big(d^0_M(0) + d^1_M(-1)\big)^2 - d^0_M(0)$;
d) $-d^0_M(0) \leq q^1_M(-1) \leq \big(d^0_M(0) + d^1_M(-1)\big)^2 + d^1_M(-1)$.

Proof. As usual, we may assume that R_0 has infinite residue fields, so that R_1 contains M-admissible linear forms.

"a)": Obvious if we apply Lemma 4.2 a) with $i = 0$.
"b)": We use the notation of [7, (2.34), (2.36)] and set $L := \Gamma_{\mathfrak{a}^{[2]}}(M)$ and $N := M^{[2]} = M/L$. Then, by [7, (2.37)] we have $\dim(R/\mathfrak{p}) > 2$ for all $\mathfrak{p} \in \operatorname{Ass}_R(N)$. So by Corollary 1.1 we see that $q^0_N \equiv 0$ and that q^1_N is constant. As $q^1_N = d^1_N(n)$ for all $n \ll 0$, we have $q^1_N \geq 0$. Moreover $\dim(L) \leq 2$, so that $d^2_L(n) = h^3_L(n) = 0$ for all for all $n \in \mathbb{Z}$ (cf. 1.1 B) (••) and [7, (2.32)]). So, if we apply $\mathcal{R}^* D_{R_+}$ to the graded short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ we get the relations $d^0_L(0) \leq d^0_M(0)$ and $d^1_M(n) = d^1_L(n) + d^1_N(n)$, $(\forall n \ll 0)$.

As a first consequence of this, we obtain

$$q^1_M(\mathbf{x}) = q^1_L(\mathbf{x}) + q^1_N.$$

As $\dim(L) \leq 2$, the Hilbert polynomial $P_L(\mathbf{x})$ of L takes the form $P_L(\mathbf{x}) = d\mathbf{x} + d_L^0(0) - d_L^1(0)$ with an appropriately choosen integer $d \in \mathbb{N}_0$ ([9, (17.1.4), (17.1.7)]). As $P_L = q_L^0 - q_L^1$ (cf. loc. cit) and as q_L^0 is a non-negative constant we thus obtain

$$q_M^1(\mathbf{x}) = -d\mathbf{x} + d_L^1(0) - d_L^0(0) + q_L^0 + q_N^1,$$

hence $q_M^1(0) = d_L^1(0) - d_L^0(0) + q_L^0 + q_N^1 \geq -d_L^0(0)$. As $d_L^0(0) \leq d_M^0(0)$, this proves our claim.

"c)": By Lemma 4.2 b) we have

$$(*) \qquad d_M^1(n) \leq -n(d_M^1(-1) + d_M^0(0)) - d_M^0(0), (\forall n \leq -1).$$

Let $f \in R_1$ be M-admissible. By 2.2 G) we have $d_{M/fM}^0(0) \leq d_M^1(-1) + d_M^0(0)$. As $q_{M/fM}^0(0) \geq 0$, Proposition 3.1 c) and (3.5) show that

$$\begin{aligned} \nu_{M/fM}^0 &\geq F^{(0)}\big(d_{M/fM}^0(0); q_{M/fM}^0(0)\big) + 1 \\ &= \min\{0, q_{M/fM}^0(0) - d_{M/fM}^0(0)\} \geq -d_{M/fM}^0(0) \end{aligned}$$

and hence

$$\nu_{M/fM}^0 \geq -d_M^1(-1) - d_M^0(0).$$

According to Proposition 2.1 a) we have $q_M^1(n) \leq d_M^1(n)$ for all $n \leq \nu_{M/fM}^0$. If $d_M^1(-1)$ and $d_M^0(0)$ are not both 0, we now may apply the estimate $(*)$ with $n = -d_M^1(-1) - d_M^0(0)$ and thus obtain $q_M^1(-d_M^1(-1) - d_M^0(0)) \leq \big(d_M^1(-1) + d_M^0(0)\big)^2 - d_M^0(0)$. As $q_M^1(\mathbf{x})$ is a polynomial of degree ≤ 1 with non-negative values for $\mathbf{x} \ll 0$, we have $q_M^1(0) \leq q_M^1(-d_M^1(-1) - d_M^0(0))$ and hence obtain the requested inequality for $q_M^1(0)$ in this case.

If $d_M^1(-1) = d_M^0(0) = 0$, the estimate $(*)$ shows that $q_M^1 \equiv 0$ and so gives our claim in any case.

"d)": If we apply Lemma 4.2 a) with $i = 1$, we get

$$q_M^1(0) \leq q_M^1(-1) \leq q_M^1(0) + d_M^0(0) + d_M^1(-1).$$

In view of b) and c) we thus get our claim. ∎

Now, the previous lemma allows us to prove

Lemma 4.4. *Let M be a finitely generated and graded R-module. Let $i \in \mathbb{N}$. Then*

$$-\sum_{j=0}^{i-1} \binom{i-1}{j} d_M^j(-j) \leq -\sum_{j=0}^{i-1} (-1)^{i+j} \binom{i-1}{j} q_M^i(-j) \leq \left[\sum_{j=0}^{i} \binom{i}{j} d_M^j(-j)\right]^2.$$

Proof. As usual we may assume that R_0 has infinite residue fields so that R_1 contains M-admissible linear forms.

We proceed by induction on i. For $i = 1$, the first inequality is clear from Lemma 4.3 b), whereas the second inequality follows from Lemma 4.3 c). So, let $i > 1$. Let $f \in R_1$ be M-admissible. By induction we have

$$-\sum_{l=0}^{i-2} \binom{i-2}{l} d^l_{M/fM}(-l) \leq -\sum_{l=0}^{i-2} (-1)^{i-1+l} \binom{i-2}{l} q^{i-1}_{M/fM}(-l)$$
$$\leq \left[\sum_{l=0}^{i-1} \binom{i-1}{l} d^l_{M/fM}(-l) \right]^2 .$$

Now, we get our claim if we apply Lemma 4.1 a) with $k = 1$ and with $i - 1$ instead of i and if we apply Lemma 4.1 b) with $k = 0$. ■

Now, we are ready to prove the first main result of this chapter. We convene that $\mathbb{Z}^0 = \emptyset$.

Notation. Let $\mathcal{D}$ denote the class of all pairs (R, M) in which $R = \oplus_{n \geq 0} R_n$ is a noetherian positively graded homogeneous ring with artinian base ring R_0 and in which $M = \oplus_{n \in \mathbb{Z}} M_n$ is a finitely generated and graded R-module. •

Theorem 4.1. *Let $i \in \mathbb{N}_0$, let $\underline{d} = (d^0, d^1, \ldots, d^i) \in \mathbb{N}_0^{i+1}$, let $\underline{q} = (q^0, q^1, \ldots, q^{i-1}) \in \mathbb{Z}^i$ and let $k \in \{0, 1, \ldots, i-1\}$. Moreover, let $\mathcal{D}' = \mathcal{D}^{\widehat{k}}_{\underline{d},\underline{q}}$ be the class of all pairs $(R, M) \in \mathcal{D}$ for which*

$$d^j_M(-j) = d^j \text{ for } j = 0, 1, \ldots, i \text{ and}$$
$$q^i_M(-j) = q^j \text{ for all } j \in \{0, \ldots, i-1\} \setminus \{k\}.$$

Then, the set of functions

$$\{d^i_M \lceil : \mathbb{Z}_{\leq -i} \longrightarrow \mathbb{N}_0 \mid (R, M) \in \mathcal{D}'\}$$

is finite.

Proof. If $(R, M) \in \mathcal{D}$ and $k \in \{0, 1, \ldots, i-1\}$, Lemma 4.4 gives

$$-\sum_{j=0}^{i-1} \binom{i-1}{j} d^j \leq -(-1)^{i+k} \binom{i-1}{k} q^i_M(-k) + \sum_{\substack{0 \leq j \leq i-1 \\ j \neq k}} (-1)^{i+j} \binom{i-1}{j} q^j$$
$$\leq \left[\sum_{j=0}^{i} \binom{i}{j} d^j \right]^2 .$$

This shows, in the notation of (3.11), that the set $\{\underline{q}^{(i)}_M \mid (R, M) \in \mathcal{D}'\}$ is finite. In the notation of (3.10) we also have $\underline{d}^{(i)}_M = \underline{d}$ for all $(R, M) \in \mathcal{D}'$ so that the set $\{\nu^i_M \mid (R, M) \in \mathcal{D}'\}$ becomes finite by Theorem 3.1 c). Therefore, we may define

$$\nu := \min\{\nu^i_M \mid (R, M) \in \mathcal{D}'\}.$$

As $q^i_M \in \mathbb{Q}[\mathbf{x}]$ is of degree $\leq i$ (cf. Corollary 1.1), $q^i_M$is determined by $\underline{q}^{(i)}_M = (q^i_M(0), \ldots,$
$q^i_M(-i))$. So, the finiteness of the set $\{\underline{q}^{(i)}_M | (R, M) \in \mathcal{D}'\}$ shows, that the set of polynomials $\{q^i_M \mid (R, M) \in \mathcal{D}'\}$ is finite. By our choice of ν we have $d^i_M(n) = q^i_M(n)$ for all $n \leq \nu$. Therefore, the set of functions

$$\{d^i_M\lceil : \mathbb{Z}_{\leq \nu} \longrightarrow \mathbb{N}_0 \mid (R, M) \in \mathcal{D}'\}$$

is finite, too.

By Lemma 4.2 b) we also have

$$0 \leq d^i_M(n) \leq \sum_{j=0}^{i} \binom{n-j-1}{i-j} \sum_{l=0}^{i-j} \binom{i-j}{l} d^{i-l}$$

for all $(R, M) \in \mathcal{D}'$ and all $n \leq -i$. Therefore, the set

$$\{d^i_M(n) \mid \nu < n \leq -i; (R, M) \in \mathcal{D}'\}$$

is finite. This proves our claim. ∎

As an easy consequence of this result, we immediately get the following improvement of the statement made at the beginning of this section, (cf. (4.2)). Again, we fix a noetherian positively graded homogeneous ring $R = \oplus_{n \geq 0} R_n$ with an artinian base ring R_0.

Corollary 4.1. *Let $i \in \mathbb{N}_0$ and let $k \in \{0, \ldots, i-1\}$. Then, the $2i$ invariants*

$$d^0_{(\bullet)}(0), d^1_{(\bullet)}(-1), \ldots, d^i_{(\bullet)}(-i);$$
$$q^{\,i}_{(\bullet)}(0), \ldots, q^i_{(\bullet)}(-k+1), q^i_{(\bullet)}(-k-1), \ldots, q^i_{(\bullet)}(-i+1)$$

form a lower bounding system for the invariant $\nu^i_{(\bullet)}$ on the class of finitely generated and graded R-modules.

Proof. As ν^i_M is determined by d^i_M for each finitely generated and graded R-module, this follows from Theorem 4.1. ∎

Now, let us reformulate the previous main result in geometric terms. To do so, we first recall some notation.

4.1. Notation and Remark. Let $\mathcal{C}$ denote the class of all pairs $(X, \mathcal{F})$, where X is a projective scheme over some artinian base ring R_0 and where $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules. So, in the notation introduced in (4.7) we have

$$\mathcal{C} = \{(\mathrm{Proj}(R), \widetilde{M}) \mid (R, M) \in \mathcal{D}\} \qquad \bullet$$

Now, on use of the Serre-Grothendieck Correspondence (cf. [9, (20.4.4)]) we obtain from Theorem 4.1:

Theorem 4.2. *Let $i \in \mathbb{N}_0$, let $\underline{d} = (d^0, d^1, \dots, d^i) \in \mathbb{N}_0^{i+1}$, let $\underline{q} = (q^0, q^1, \dots, q^{i-1}) \in \mathbb{Z}^i$ and let $k \in \{0, 1, \dots, i-1\}$.*
Moreover, let $\mathcal{C}' = \mathcal{C}^{\hat{k}}_{\underline{d},\underline{q}}$ be the class of all pairs $(X, \mathcal{F}) \in \mathcal{C}$ for which

$$h^j_{\mathcal{F}}(-j) = d^j \text{ for } j = 0, 1, \dots, i \quad \text{and}$$
$$p^i_{\mathcal{F}}(-j) = q^j \text{ for all } j \in \{0, \dots, i-1\} \setminus \{k\}.$$

Then, the set of functions

$$\{h^i_{\mathcal{F}}\lceil : \mathbb{Z}_{\leq -i} \longrightarrow \mathbb{N}_0 \mid (X, \mathcal{F}) \in \mathcal{C}'\}$$

is finite.

Proof. In the notation of Theorem 4.1 we have $\mathcal{C}' = \{(\mathrm{Proj}(R), \widetilde{M}) \big| (R, M) \in \mathcal{D}'\}$, (cf. 1.7 A) $(*)$ and B) $(\bullet\bullet)$). Now, another use of 1.7 B) $(\bullet\bullet)$ gives our claim by Theorem 4.1. ■

Now, let X be a projective scheme over an artinian ring R_0. For a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and for $i \in \mathbb{N}_0$ let $\nu^i_{\mathcal{F}}$ denote the i-th cohomological postulation number of $\mathcal{F}$ (cf. 1.7 D)). Then, Corollary 4.1 may be translated into geometric terms as follows.

Corollary 4.2. *Let $i \in \mathbb{N}_0$ and let $k \in \{0, \dots, i-1\}$. Then, the $2i$ invariants*

$$h^0_{(\bullet)}(0), h^1_{(\bullet)}(-1), \dots, h^i_{(\bullet)}(-i);$$
$$p^i_{(\bullet)}(0), \dots, p^i_{(\bullet)}(-k+1), p^i_{(\bullet)}(-k-1), \dots, p^i_{(\bullet)}(-i+1)$$

form a lower bounding system for the invariant $\nu^i_{(\bullet)}$ on the class of all coherent sheaves of $\mathcal{O}_X$-modules.

Proof. Immediate from Corollary 4.1 by 1.7 A) $(*)$ and B) $(\bullet\bullet)$. ■

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