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Pointed Planar Geodesic Submanifolds in Euclidean Space*

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Abstract. We study the pointed planar geodesic immersion of a submanifold into a Euclidean space and determine topological structure of the pointed planar geodesic submanifolds with maximum codimension. From the typically obtained property we can easily get the classification of planar geodesic immersion of submanifolds into a Euclidean space.

1. Introduction

The behavior of geodesics on a Riemannian manifold gives many valuable informations in studying the geometric structure of the manifold. For example, if every geodesic of a submanifold in a Euclidean space is a planar curve as a space curve in the ambient Euclidean space, then the submanifold can be determined as either one of compact rank one symmetric spaces or a Euclidean plane [12, 15]. Such a kind of submanifold in a Euclidean space is called planar geodesic. If every geodesic of the submanifold in an ambient Riemannian manifold has constant Frenet curvatures independent of the choice of geodesic, the submanifold is called a helical submanifold in the ambient manifold [16, 17, 18]. If the ambient manifold is a space form, then the helical submanifold turns out to be either a Blaschke manifold or a totally geodesic submanifold [16, 17]. It was also treated differently by viewing geodesics as normal sections if the ambient manifold is Euclidean [4]. Such studies provided a partial solution to the Blaschke conjec-

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ture [1] that every Blaschke manifold is isometric to one of compact rank one symmetric spaces.

On the other hand, the author studied the so-called pointed helical submanifolds of an Euclidean space, i.e., every geodesic of the submanifold passing through a point has constant curvatures independent of the choice of direction [9, 11]. He also studied submanifolds of a Euclidean space with a point through which every geodesic is planar and proved that the submanifolds are pointed Blaschke manifolds or every geodesic through the point does not have cut points [10]. In 1998, Fueki [6] extended the notion of pointed helical submanifolds to pseudo-Riemannian version. For the Blaschke manifolds and pointed Blaschke manifolds, see [1].

In the present paper we study more about the pointed planar geodesic submanifolds of a Euclidean space and their topological behavior. We completely determine the isometric immersion around such a point with maximum codimension. We can easily have the classification theorem of planar geodesic submanifolds in Euclidean spaces by using the symmetry property of pointed planar geodesic submanifolds.

2. Preliminaries

Let $x : M \to E^m$ be an isometric immersion of n-dimensional Riemannian manifold M into a Euclidean m-space E^m. Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean metric tensor of E^m. Then M has the induced metric from that of E^m that is denoted by the same notation $\langle \cdot, \cdot \rangle$ unless we have confusion. We denote by $\tilde{\nabla}$ the Levi-Civita connection on E^m and ∇ the induced connection on M. Then, we have the Gauss equation $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$, where X and Y denote vector fields on M and σ is the second fundamental form. The equation of Weingarten is also given by $\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$, where A_ξ is the Weingarten map associated with a normal vector field ξ to M and $\nabla^\perp$ the normal connection in the normal bundle $T^\perp M$. As is well known, the Weingarten map A_ξ and the second fundamental form σ are related by $\langle A_\xi X, Y \rangle = \langle \sigma(X, Y), \xi \rangle$ for all vector fields X and Y on M and ξ normal to M. We now define the covariant derivative of σ on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^\perp M$ of M as

$$(\overline{\nabla}_X \sigma)(Y, Z) = \nabla^\perp_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

for all vector fields X, Y and Z on M. We denote $(\overline{\nabla}_X \sigma)(Y, Z)$ by $(\overline{\nabla}\sigma)(X, Y, Z)$ which is a tensor field of type (1.3). Let R be the curvature tensor of M. The Gauss equation is then given by

$$\langle R(X, Y)Z, W \rangle = \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(Y, W), \sigma(X, Z) \rangle.$$

We also obtain the Codazzi equation

$$(\overline{\nabla}\sigma)(X, Y, Z) - (\overline{\nabla}\sigma)(Y, X, Z) = 0.$$

The submanifold M in a Euclidean space E^m is said to be *isotropic* at $p \in M$ if the normal curvature of curves passing through p is independent of the choice of the curve, that is, $\langle\sigma(t,t),\sigma(t,t)\rangle$ does not depend on the choice of the unit vector t tangent to M at p. By B. O'Neill [13], M is isotropic at p if and only if $\langle\sigma(t,t),\sigma(t,t^{\perp})\rangle = 0$ for all unit vectors t and $t^{\perp}$ perpendicular to t. For a point $p \in M$ and a unit vector t tangent to M at p, the vector t and the normal space $T_p^{\perp}M$ of M at p form an $(m-n+1)$-dimensional affine space $E(p;t)$ in E^m through p. The intersection of M with $E(p;t)$ gives rise to a curve in a neighborhood of p which is called the normal section at p in the direction t. A submanifold M in a Euclidean space E^m is said to have *geodesic normal sections* if all the geodesics of M are normal sections [4].

We now introduce the following

Lemma 2.1. [10, 11] *Let M be an n-dimensional submanifold of a Euclidean space E^m. If γ is a planar geodesic of M, then γ is a normal section of M along γ.*

A submanifold M of a Euclidean space E^m with a point o is called *pointed planar geodesic* if every geodesic through o is a plane curve regarded as a curve in E^m. We also call M *pointed helical at o* if every geodesic through o has constant Frenet curvatures that are independent of the choice of geodesic through o.

We now suppose that M is pointed planar geodesic at o. Without loss of generality we may assume that the point o is the origin of E^m. Let γ be a geodesic of M passing through o and let γ be parametrized by the arc length s. Let $\gamma(0) = o$. Since γ is a geodesic of M, we have

$$\gamma'(s) = T, \gamma''(s) = \sigma(T,T), \gamma''(s) = -A_{\sigma(T,T)}T + (\bar{\nabla}\sigma)(T,T,T).$$

Since γ is a normal section of M at o according to Lemma 2.1, $A_{\sigma(t,t)}t \wedge t = 0$, where $t = T(0)$. In other words,

$$\langle\sigma(t,t),\sigma(t,t^{\perp})\rangle = 0,$$

where $t^{\perp}$ is a unit vector tangent to M at o perpendicular to t. According to O'Neill [13] M is isotropic at o. Thus we have

Proposition 2.2. *Let M be a pointed planar geodesic submanifold at o in E^m. Then M is isotropic at o.*

Since every geodesic through o is a plane curve, we may represent the immersion $x : M \longrightarrow E^m$ locally on a neighborhood U of o in terms of the geodesic polar coordinate system $(s, \theta_1, \theta_2 \cdots, \theta_{n-1})$ as

$$\begin{aligned} x(s,\theta_1,\cdots,\theta_{n-1}) &= h(s,\theta_1,...,\theta_{n-1})e(\theta_1,...,\theta_{n-1}) \\ &\quad + k(s,\theta_1,...,\theta_{n-1})N(\theta_1,...,\theta_{n-1}), \end{aligned} \tag{1}$$

where $e(\theta_1,...,\theta_{n-1})$ is a unit tangent vector to M at o, h and k some smooth functions satisfying $h(0,\theta_1,...,\theta_{n-1}) = k(0,\theta_1,...,\theta_{n-1}) = 0$ and $N(\theta_1,...,\theta_{n-1})$ a unit vector normal to M at o depending on $\theta_1,...,\theta_{n-1}$. Then, it is obvious that $\frac{\partial}{\partial\theta_i}e(\theta_1,...,\theta_{n-1})$ is tangent to M and $\frac{\partial}{\partial\theta_i}N(\theta_1,...,\theta_{n-1})$ normal to M at o

for all $i = 1, 2, \cdots, n-1$. We then have orthogonal vector fields tangent to M defined on U:

$$\begin{aligned} x_*(\frac{\partial}{\partial s}) &= \frac{\partial h}{\partial s}(s, \theta_1, ..., \theta_{n-1}) e(\theta_1, ..., \theta_{n-1}) \\ &\quad + \frac{\partial k}{\partial s}(s, \theta_1, \cdots, \theta_{n-1}) N(\theta_1, ..., \theta_{n-1}), \end{aligned} \tag{2}$$

$$\begin{aligned} x_*(\frac{\partial}{\partial \theta_i}) &= \frac{\partial h}{\partial \theta_i}(s, \theta_1, ..., \theta_{n-1}) e(\theta_1, ..., \theta_{n-1}) \\ &\quad + h(s, \theta_1, ..., \theta_{n-1}) \frac{\partial e}{\partial \theta_i}(\theta_1, ..., \theta_{n-1}) \\ &\quad + \frac{\partial k}{\partial \theta_i}(s, \theta_1, ..., \theta_{n-1}) N(\theta_1, ..., \theta_{n-1}) \\ &\quad + k(s, \theta_1, ..., \theta_{n-1}) \frac{\partial N}{\partial \theta_i}(\theta_1, ..., \theta_{n-1}) \end{aligned} \tag{3}$$

for $i = 1, 2, \cdots, n-1$, where $x_*(\frac{\partial}{\partial s})(0, \theta_1, ..., \theta_{n-1}) = e(\theta_1, ..., \theta_{n-1})$. Taking the covariant differentiation of $x_*(\frac{\partial}{\partial s})$ given by (2) along $x(s, \theta_1, ..., \theta_{n-1})$ for fixed $\theta_1, ..., \theta_{n-1}$, we have

$$\overline{\nabla}_{x_*(\frac{\partial}{\partial s})} x_*(\frac{\partial}{\partial s}) = \frac{\partial^2 h}{\partial s^2} e(\theta_1, ..., \theta_{n-1}) + \frac{\partial^2 k}{\partial s^2} N(\theta_1, ..., \theta_{n-1}) \tag{4}$$

which is normal to M since $x(s, \theta_1, ..., \theta_{n-1})$ is a geodesic for each $\theta_1, ..., \theta_{n-1}$. From (4) we can find the curvature

$$\kappa(s, \theta_1, ..., \theta_{n-1}) = \Big((\frac{\partial^2 h}{\partial s^2})^2 + (\frac{\partial^2 k}{\partial s^2})^2 \Big)^{\frac{1}{2}} \tag{5}$$

for each $\theta_1, ..., \theta_{n-1}$.

Lemma 2.3. *Let M be a pointed planar geodesic submanifold at o in a Euclidean space. Then the curvature of all the geodesics passing through o is independent of the choice of geodesics and the functions h and k depend only on s.*

Proof. Let γ be a geodesic through o. For some $\theta_1, \theta_2, \cdots, \theta_{n-1}$, $\gamma(s) = x(s, \theta_1, ..., \theta_{n-1})$. The curvature of γ is denoted by κ. Let p be a point in the image of γ, say $\gamma(s_0) = p$, and $v \in T_pM$ a nonzero vector orthogonal to $T(p) = x_*(\partial/\partial s)(s_0)$. Extend v to V about p in such a way that $\langle V, T\rangle = 0$ and V is parallel along γ. If we apply Lemma 2.1 and the Codazzi equation, we have

$$\begin{aligned} \frac{1}{2} v\kappa^2 &= \frac{1}{2} v \langle \sigma(T,T), \sigma(T,T)\rangle \\ &= \langle \nabla^{\perp}_v \sigma(T,T), \sigma(T(p), T(p))\rangle \\ &= \langle (\bar{\nabla}\sigma)(v, T(p), T(p)) + 2\langle \sigma(\nabla_v T, T(p)), \sigma(T(p), T(p))\rangle \\ &= \langle \bar{\nabla}\sigma)(V, T, T), \sigma(T,T)\rangle_{s=s_0} \\ &= \langle (\bar{\nabla}_T \sigma)(V, T), \sigma(T,T)\rangle_{s=s_0} \\ &= \langle \nabla^{\perp}_T \sigma(V,T), \sigma(T,T)\rangle_{s=s_0} \\ &= T(p)\langle \sigma(V,T), \sigma(T,T)\rangle = 0. \end{aligned}$$

Since p is an arbitrary point in the image of γ and v is an arbitrary vector orthogonal to $T(p)$, the curvature depends only on arc length s but not on the direction $\theta_1, ..., \theta_{n-1}$.

Since $\langle x_*(\frac{\partial}{\partial s}), x_*(\frac{\partial}{\partial s})\rangle = 1$ for each $(s, \theta_1, ..., \theta_{n-1})$, we may put

$$\frac{\partial h}{\partial s}(s,\theta_1,...,\theta_{n-1})=\cos\varphi(s,\theta_1,...,\theta_{n-1}), \frac{\partial k}{\partial s}(s,\theta_1,...,\theta_{n-1})=\sin\varphi(s,\theta_1,...,\theta_{n-1})$$

for some function φ satisfying $\varphi(0, \theta_1, ..., \theta_{n-1}) = 0$. From this and (5), we have

$$\frac{\partial \varphi}{\partial s}(s,\theta_1,...,\theta_{n-1}) = \kappa(s,\theta_1,...,\theta_{n-1}).$$

Since the curvature of geodesic through o is independent of the direction, so is φ and hence the functions h and k depend only on s. ∎

Using the lemmas above, we have

Proposition 2.4. *Let M be an n-dimensional pointed planar geodesic submanifold in a Euclidean space E^m. Then M is locally a warped product of a plane curve and an $(n-1)$-dimensional unit sphere.*

Corollary 2.5. [4] *Let M be a planar geodesic submanifold in E^m. Then the curvature of geodesic is constant independent of the choice of geodesic.*

For later use, we recall the notion of geodesic ball. A *geodesic ball* of radius r of a Riemannian manifold $\tilde{M}$ is defined by $\{\exp_o(su) | u \in U_o\widetilde{M}, 0 \le s \le r\}$, where $\exp_o$ is the exponential map at o and $U_o\tilde{M}$ is the unit tangent space at o. A *geodesic sphere* is a hypersurface of $\tilde{M}$ defined by $\{\exp_o(ru) | u \in U_o\tilde{M}\}$.

3. Pointed Planar Geodesic Submanifolds of a Euclidean Space

Let M be an n-dimensional pointed planar geodesic submanifold at o in an m-dimensional Euclidean space E^m. Let $(\mathrm{Im}\sigma)_p$ be the first normal space at p defined by $(\mathrm{Im}\sigma)_p = \mathrm{Span}\{\sigma(X,Y) | X, Y \in T_pM\}$. As is described in Sec. 2, the isometric immersion $x : M \to E^m$ can be written in the form given in (1) in Sec. 2. Since $e(\theta_1, ..., \theta_{n-1}) = x_*\big(\frac{\partial}{\partial s}\big)(0, \theta_1, ..., \theta_{n-1}\big)$ is a unit tangent vector to M at o, it may be expressed in the following way

$$e(\theta_1,...,\theta_{n-1}) = \cos\theta_1 E_1 + \sin\theta_1\cos\theta_2 E_2 + \cdots + \Big(\prod_{j=1}^{i-1}\sin\theta_j\Big)\cos\theta_i E_i + \cdots + \Big(\prod_{k=1}^{n-1}\sin\theta_k\Big)E_n \qquad (6)$$

for a suitable orthonormal basis $\{E_1, \ldots, E_n\}$ for T_oM and $\theta_1, \ldots, \theta_{n-2} \in [0, \pi]$, $\theta_{n-1} \in [-\pi, \pi]$.

Choosing a geodesic γ passing through o and taking the covariant differentiation of the tangent vector field γ_* along γ, we have

$$
\begin{aligned}
(\tilde{\nabla}_{\gamma_*}\gamma_*)_{s=0} &= (\sigma(\gamma_*, \gamma_*))_{s=0} \\
&= \sigma(e(\theta_1, ..., \theta_{n-1}), e(\theta_1, ..., \theta_{n-1})) \qquad (7)\\
&= \cos^2\theta_1\sigma(E_1, E_1) + \sin^2\theta_1\cos^2\theta_2\sigma(E_2, E_2) + \cdots + \\
&\prod_{i=1}^{n-1}\sin\theta_i\sigma(E_n, E_n) + 2\sum_{1\le i<j}\prod_{k=1}^{i-1}\prod_{t=1}^{j-1}\sin\theta_k\sin\theta_t\cos\theta_i\cos\theta_j\sigma(E_i, E_j).
\end{aligned}
$$

By following the proofs in [6, 9] and modifying them slightly we have

Proposition 3.1. *Let M be a pointed planar geodesic at o in E^m. If the first normal space $(\mathrm{Im}\sigma)_o$ at o is 1-dimensional, then o is an umbilic point. Furthermore, if the dimension of M is not smaller than 2, the first normal space $(\mathrm{Im}\sigma)_o$ cannot be of dimension 2.*

By using Lemma 2.1 and Proposition 3.1 we have

Proposition 3.2. [5, 8] *Let M be a pointed planar geodesic hypersurface at o in a Euclidean space E^{n+1}. Then, M is a generalized surface of revolution.*

If we take account of Lemma 2.3, we obtain

Proposition 3.3. *Let M be a complete pointed planar geodesic at o in E^m. If there is a simple geodesic through o, then M is a pointed helical submanifold of order 2.*

As a consequence, we have

Corollary 3.4. *Let M be a complete pointed planar geodesic at o in E^m. If two geodesics through o have a common cut point and if there is a simple geodesic through o, then M is a standard sphere.*

Theorem 3.5. *Let M be an n-dimensional complete pointed planar geodesic at o in E^m. Then M is symmetric with respect to o. In particular, if M is compact, then M is a geodesic ball diffeomorphic to a sphere S^n or a real projective space RP^n.*

Proof. If we define a geodesic symmetry S_o satisfying $S_o\exp_o = \exp_o S$, where S is a map of T_oM onto T_oM defined by $S(X) = -X$, (1) and Lemma 2.3 yield that M is symmetric with respect to o.

We now consider the case that M is compact. Suppose some two geodesics have a common cut point. Let L be the cut value of such two geodesics. Then for some $\theta_1, ..., \theta_{n-1}$ and $\theta'_1, \theta'_2, \cdots, \theta'_{n-1}$, we have

$$x(L, \theta_1, \dots, \theta_{n-1}) = x(L, \theta'_1, \theta'_2, \dots, \theta'_{n-1}).$$

Considering (1), (6) and (7), we have $h(L, \theta_1, ..., \theta_{n-1}) = k(L, \theta_1, ..., \theta_{n-1}) = 0$ if $N(\theta_1, ..., \theta_{n-1}) \wedge N(\theta'_1, \theta'_2, \cdots, \theta'_{n-1}) \neq 0$. Therefore, $\dim(\mathrm{Im}\sigma_o) = 1$, that is,

o is an umbilic point. Using (1), Lemma 2.3, (6) and (7), M is diffeomorphic to a sphere.

Suppose there is no common cut point for geodesics through o. By the above argument we still have a common cut value L for every geodesic through o. Therefore, M is a SC^o_{2L}-manifold whose geodesics through o are simple closed with period $2L$. Since the vector fields $x_*(\partial/\partial\theta_i)$ are Jacobi fields along geodesics through o, we have the index 0. According to Bott-Samelson [2, 14], M is diffeomorphic to RP^n. ∎

As a direct consquence of the above theorem and Corollary 2.5, we get

Corollary 3.6. [12, 15] *Let M be a complete planar geodesic submanifold in a Euclidean space E^m. Then, M is a plane or one of compact rank one symmetric spaces.*

Combining the expressions (1), (4), (6) and (7), we also have

Theorem 3.7. *Let M be an n-dimensional complete pointed planar geodesic submanifold of an m-dimensional Euclidean space E^m. Then, M lies in at most N-dimensional Euclidean space $E^N \subset E^m$ where $N = n(n+3)/2$.*

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