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On the Solvability in a Closed Form of a Class of Singular Integral Equations

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Abstract. This paper deals with the solvability of singular integral equations of the form

$$\sum_{k=1}^{n} a_k(t)\varphi(\epsilon_k t) + \sum_{k=1}^{n} \frac{1}{\pi i} \int_{\Gamma} \frac{\tau^{n-1-k} t^k}{\tau^n - t^n} m_k(\tau, t)\varphi(\tau) d\tau = f(t), \tag{1}$$

where Γ is the unit circle on the complex plane, $\epsilon_1 = \exp(2\pi i/n), \epsilon_k = \epsilon_1^k$. We indicate that the equation (1) is exactly a singular integral equation of Cauchy type with a certain shift. The method in this paper is to reduce the equation (1) to the well-known Riemann boundary value problems and describe solutions in a closed form.

1. Introduction

Let Γ be a closed Liapunov curve on the complex plane $\mathbb{C}$. It is well-known that the singular integral operator of Cauchy type in spaces $L_p(\Gamma), H^\mu(\Gamma)$ $(0 < \mu < 1, 1 < p < \infty)$

$$(S\varphi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau$$

has the property $S^2 = I$, where I is the identity operator on the spaces above (see [1]). The Noetherian theory of singular integral equations of Cauchy type

$$a\varphi + bS\varphi + K\varphi = f \tag{2}$$

started with the works of Noether and Carlemann in 1921, and was later developed by many mathematicians (see [1–3, 6, 9] and the references therein). One reason for the great interest in this theory is an effective relation between Riemann boundary value problems of analytic functions and equations of the form (2). The operator S plays an important role in the theory of Riemann boundary value problems, since this operator is like simple-layer and double-layer potentials in Neumann and Dirichlet problems. In [4], Mau presented the integral operator

$$(S_{n,k}\varphi)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{\tau^{n-1-k}t^k}{\tau^n - t^n}\varphi(\tau)d\tau, \tag{3}$$

where n, k are nonnegative integers, $0 \le k \le n-1$, Γ the unit circle on the complex plane. Moreover, the author of [4] proved that if $n \neq 1$ then $S^3_{n,k} = S_{n,k}$ (note that $S_{1,0} \equiv S$) and investigated the equation induced by a certain operator $S_{n,k}$ of the form

$$a(t)\varphi(t) + b(t)(S_{n,k}\varphi)(t) + (N_{n,k}\varphi)(t) = f(t) \tag{4}$$

under the assumption $a(t) \neq 0$ on Γ. In the same way of integral equations of Cauchy type, $(a\varphi + bS_{n,k}\varphi)$ is called the singular part and $N_{n,k}\varphi$ is called the regular part of the equation (4). In [8], the second author studied the solvability of equation (4) in the case when the function $a(t)$ has isolated zero-points on Γ. In [5], we studied the solvability of the equations induced by a class of operators S_{n_j,k_j} of the form

$$a(t)\varphi(t) + \Big(\big(\prod_{j=1}^{m} b_{n_j,k_j}(t)S_{n_j,k_j}\big)\varphi\Big)(t) = f(t).$$

In this paper, we deal with the conditions of the solvability in a closed form of the equation (1) by means of Riemann boundary value problems. The method of our investigation is to use basic characterizations of algebraic operators (the Carlemann shift W and the Cauchy singular integral operator S). We divide the space $X = H^\mu(\Gamma)$ into subspaces, and then solve the given equation in a closed form by combining all solutions of a system of equations without shifts.

2. Representation of Equation (1) in the Form of Operator Equation

Let $\Gamma = \{t \in \mathbb{C} : |t| = 1\}$ be the unit circle on the complex plane $\mathbb{C}$ and let $D^+ = \{t \in \mathbb{C} : |t| < 1\}$; $D^- = \{t \in \mathbb{C} : |t| > 1\}$. Consider the following operators in $X := H^\mu(\Gamma), 0 < \mu < 1:$

$$\begin{aligned}
(S\varphi)(t) &= \frac{1}{\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t}d\tau, \qquad (5)\\
(S_k\varphi)(t) &= \frac{1}{\pi i}\int\limits_{\Gamma} \frac{\tau^{n-1-k}t^k}{\tau^n - t^n}\varphi(\tau)d\tau,\\
(M_k\varphi)(t) &= \frac{1}{\pi i}\int\limits_{\Gamma} \frac{\tau^{n-1-k}t^k}{\tau^n - t^n}m_k(\tau, t)\varphi(\tau)d\tau,\\
(W\varphi)(t) &= \varphi(\epsilon_1 t), \quad \epsilon_1 = \exp(2\pi i/n),
\end{aligned}$$

where $n, k \in \mathbb{N}, n > 1, 1 \le k \le n-1$ and $m_k(\tau, t)$ are functions satisfying the Holder condition in both variables $(\tau, t) \in \Gamma \times \Gamma$. Write

$$\begin{aligned} P &= \frac{1}{2}(I+S), \quad Q = \frac{1}{2}(I-S), \\ X^+ &= P(X), \quad X^- = Q(X), \quad X_k = P_k(X), \\ P_j &= \frac{1}{n}\sum_{\nu=1}^{n} \epsilon_j^{n-1-\nu} W^{\nu+1}, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{6}$$

where $\epsilon_j = \epsilon_1^j,\ j \in \mathbb{Z}$. Then we have (see [4, 7])

$$\begin{aligned} I &= P + Q, \ \ P^2 = P, \\ Q^2 &= Q, \ \ PQ = QP = 0, \\ I &= \sum_{j=1}^{n} P_j, \quad P_k P_j = \delta_{kj} P_j, \\ W^k &= \sum_{j=1}^{n} \epsilon_j^k P_j, \quad k = 1, 2, \ldots, n, \\ X &= X^+ \oplus X^-, \qquad X = \oplus_{j=1}^n X_j, \\ SW &= WS, \quad S_k = SP_k = P_k S, \quad WS_k = S_k W, \end{aligned} \tag{7}$$

where δ_{kj} is Kronecker's symbol.

Lemma 1. *Let $m_k(t) := m_k(t,t) \neq 0$ on Γ. Suppose that $m_k(\tau, t)$ are invariant with respect to the rotation operator W, i.e. $m_k(\tau, t) = m_k(\epsilon_1 \tau, t) = m_k(\tau, \epsilon_1 t)$ for $k = 1, 2, \ldots, n$. Then*

$$\begin{aligned} N_k P_j &= P_j N_k, \\ M_k &= m_k(S_k + N_k P_k) = m_k(S + N_k) P_k, \end{aligned}$$

where $k, j = 1, 2, \ldots, n$, and

$$\begin{aligned} (m_k \varphi)(t) &= m_k(t)\varphi(t), \\ n_k(\tau, t) &= \frac{1}{m_k(t)} \frac{m_k(\tau, t) - m_k(t,t)}{\tau - t}, \\ (N_k \varphi)(t) &= \frac{1}{\pi i} \int_\Gamma n_k(\tau, t)\varphi(\tau) d\tau. \end{aligned} \tag{8}$$

Proof. The assumption on $m_k(\tau, t)$ implies equalities

$$\begin{aligned} (WN_k\varphi)(t) &= \frac{1}{\pi i}\int_\Gamma n_k(\tau, \epsilon_1 t)\varphi(\tau) d\tau \\ &= \frac{1}{\pi i}\int_\Gamma \frac{1}{m_k(t,t)} \frac{m_k(\tau, t) - m_k(t,t)}{\tau - \epsilon_1 t} \varphi(\tau) d\tau \\ &= \frac{1}{\pi i}\int_\Gamma n_k(\tau, t)\varphi(\epsilon_1 \tau) d\tau = (N_k W \varphi)(t). \end{aligned}$$

Hence, from (6) it follows that $N_kP_j = P_jN_k$.
The remaining equalities immediately follow from (7) and Lagrange's interpolation formula (see [4]). The lemma is proved. ■

Now we can represent the equation (1) in the form

$$\sum_{j=1}^{n} a_j^*(t)(P_j\varphi)(t) + \sum_{j=1}^{n} m_j(t)\big((S+N_j)P_j\varphi\big)(t) = f(t), \tag{9}$$

where

$$a_j^*(t) = \sum_{k=1}^{n} \epsilon_j^k a_k(t),$$

and S, P_j, N_j are defined by (5),(6) and (8).

3. Reduction of Equation (9) to a System of Singular Integral Equations of Cauchy Type

We recall the following lemmas

Lemma 2. (see [4]) *Let $K(\tau,t)$ be a function satisfying Holder's condition in each variable $\tau, t \in \Gamma$. Let $K(\tau,t)$ admit an analytic continuation onto D^+ in each variable τ, t and let $K(\tau,t)$ be invariant with respect to the rotation operator W, i.e. $K(\tau,t) = K(\epsilon_1\tau,t) = K(\tau,\epsilon_1 t)$. Suppose that the function $(\tau-t)^{-1}[K(\tau,t)-K(t,t)]$ is continuous in $(\tau,t)\in\Gamma\times\Gamma$. Then*

1. $\Phi^+(t) := \frac{1}{\pi i}\int_\Gamma \frac{\tau^{n-1-k}t^k}{\tau^n - t^n}K(\tau,t)\varphi(\tau)d\tau \in X^+$ *for every* $\varphi \in X$.
2. $\Phi^+(t) \equiv 0$ *for every* $\varphi \in X^+$.

In the sequel, for every function $a \in X$ we sometimes write $(K_a\varphi)(t) = a(t)\varphi(t)$.

Lemma 3. (see [5]) *Let $a \in X$ be fixed. Then for any $k, j \in \{1,2,\dots,n\}$ there exists $b \in X$ such that $K_bX \subset X_k$ and $P_kK_aP_j = K_bP_j$.*

Lemma 4. *Let $a \in X$. Then for any $k, j \in \{1,2,\dots,n\}$ we have*

$$P_kK_{a_{kj}} = K_{a_{kj}}P_j,$$

where $a_{kj}(t)$ are defined as follows

$$a_{kj}(t) = \frac{1}{n}\sum_{\nu=1}^{n} \epsilon_{\nu+1}^{j-k} a(\epsilon_{\nu+1}t). \tag{10}$$

Proof. It is easy to see (for any ν) that

$$a_{kj}(t) = \frac{1}{n}\sum_{\mu=1}^{n} \epsilon_{\mu+1}^{j-k}\epsilon_{\nu+1}^{j-k} a(\epsilon_{\nu+1}\epsilon_{\mu+1}t).$$

Hence, for an arbitrary $\varphi \in X$ we have

$$\begin{aligned}
(P_k K_{a_{kj}}\varphi)(t) &= \frac{1}{n}\sum_{\nu=1}^{n}\epsilon_k^{n-1-\nu}W^{\nu+1}\left[a_{kj}(t)\varphi(t)\right] \\
&= \left(\frac{1}{n}\sum_{\nu=1}^{n}\epsilon_k^{n-1-\nu}W^{\nu+1}\right)\left[\Big(\frac{1}{n}\sum_{\mu=1}^{n}\epsilon_{\mu+1}^{j-k}a(\epsilon_{\mu+1}t)\Big)\varphi(t)\right] \\
&= \frac{1}{n}\sum_{\nu=1}^{n}\left[\frac{1}{n}\sum_{\mu=1}^{n}\epsilon_{\mu+1}^{j-k}\epsilon_{\nu+1}^{j-k}a(\epsilon_{\nu+1}\epsilon_{\mu+1}t)\right]\epsilon_{\nu+1}^{k-j}\epsilon_k^{n-1-\nu}\big(W^{\nu+1}\varphi\big)(t) \\
&= \frac{1}{n}\sum_{\nu=1}^{n}a_{kj}(t)\epsilon_{\nu+1}^{k-j}\epsilon_k^{n-1-\nu}W^{\nu+1}\varphi(t) \\
&= a_{kj}(t)\left(\left[\frac{1}{n}\sum_{\nu=1}^{n}\epsilon_j^{n-1-\nu}W^{\nu+1}\right]\varphi\right)(t) \\
&= a_{kj}(t)(P_j\varphi)(t) = (K_{a_{kj}}P_j\varphi)(t),
\end{aligned}$$

which proves the lemma.

3.1. At first, we deal with the equation (9) in the case $a_k(t)\equiv 0$ on Γ, $k = 1, 2, ..., n$.

$$\sum_{j=1}^{n} m_j(t)\big((S+N_j)P_j\varphi\big)(t) = f(t). \tag{11}$$

Theorem 1. *Suppose that every $n_j(\tau,t)$ $(j = 1,2,\ldots,n)$ admits an analytic continuation onto D^+ in both variables τ, t and $n_j(\tau,t) = n_j(\epsilon_1\tau,t) = n_j(\tau,\epsilon_1 t)$.*

a) *If $\varphi \in X$ is a solution of (11) then $(P_1\varphi, P_2\varphi,\ldots,P_n\varphi)$ is a solution of the following system*

$$\sum_{j=1}^{n} m_{kj}(S+N_j)\varphi_j = P_k f,\quad k = 1,2,\ldots,n \tag{12}$$

where for any $k, j = 1,2,\ldots,n$, the function $m_{kj} := a_{kj}$ is defined as (10) with $a := m_j$; i.e.,

$$m_{kj}(t) = \frac{1}{n}\sum_{\nu=1}^{n}\epsilon_{\nu+1}^{j-k}m_j(\epsilon_{\nu+1}t).$$

b) *If $(\varphi_1,\varphi_2,\ldots,\varphi_n)$ is a solution of (12) then*

$$\varphi = \sum_{k=1}^{n} P_k\varphi_k$$

is a solution of (11).

Proof.
a) Let φ be a solution of (11). Applying operators P_k to both sides of equations (11), we get

$$\sum_{j=1}^{n} P_k\left[m_j(S+N_j)P_j\varphi\right] = P_k f,\quad k = 1,2,\ldots,n.$$

According to Lemmas 1, 2, 3, 4, the last system can be written as follows

$$\sum_{j=1}^{n} m_{kj}(S+N_j)P_j\varphi = P_k f, \quad k=1,2,\dots,n.$$

Thus, $(P_1\varphi, P_2\varphi, \dots, P_n\varphi)$ is a solution of (12).
b) Suppose that $(\varphi_1, \varphi_2, \dots, \varphi_n)$ is a solution of (12). Applying operators $P_k, k = 1,2,\dots,n$, to both sides of equations (12), we get

$$\sum_{j=1}^{n} P_k\left[m_{kj}(S+N_j)\varphi_j\right] = P_k f, \quad k=1,2,\dots,n.$$

From Lemmas 1, 4 it follows that

$$\sum_{j=1}^{n} m_{kj}(S+N_j)P_j\varphi_j = P_k f, \;\; k=1,2,\dots,n.$$

Hence, we have

$$\begin{aligned} f = \sum_{k=1}^{n} P_k f &= \sum_{k=1}^{n}\sum_{j=1}^{n} m_{kj}(S+N_j)P_j\varphi \\ &= \sum_{k=1}^{n}\sum_{j=1}^{n} P_k m_j(S+N_j)P_j\varphi \\ &= \sum_{j=1}^{n}\Big(\sum_{k=1}^{n} P_k\Big) m_j(S+N_j)P_j\varphi \\ &= \sum_{j=1}^{n} m_j(S+N_j)P_j\varphi. \end{aligned}$$

Thus, $\varphi(t)$ is a solution of (11). The theorem is proved. ■

Remark. Due to the results of Theorem 1, it is enough to solve system (12) in X instead of solving (11) in X.

Theorem 2. *Let all assumptions of Theorem* 1 *be satisfied and let*

$$\det M(t) = \det\Big[m_{kj}(t)\Big]_{k,j=1,\dots,n} \neq 0.$$

Then, for every $f \in X$, the equation (11) *has the following unique solution*

$$\varphi(t) = \sum_{k=1}^{n}\sum_{j=1}^{n}(S+N_k)P_k b_{kj}(t)P_j f, \tag{13}$$

where $b_{kj}(t)$ $(k,j=1,2,\dots,n)$ are the elements of the inverse matrix $\Big[M(t)\Big]^{-1}$

Proof. We rewrite system (12) in the following form

$$MK\Phi = F,$$

where

$$K = \Big[\delta_{kj}(S+N_j)\Big]_{k,j=1,\dots,n},$$
$$\Phi = (\varphi_1, \varphi_2, \dots, \varphi_n)^{\mathrm{t}}$$
$$F = (P_1 f, P_2 f, \dots, P_n f)^{\mathrm{t}}.$$

On the other hand, from the Sokhotski-Plemelij formula, it follows (see [1]) that

$$N_j^2 = 0, \quad N_j P = 0, \quad Q N_j = 0.$$

We then have $N_j S = -N_j$, $SN_j = N_j$. It means that $(S+N_j)^2 = I$, $j = 1,2,\dots,n$. Hence, K is invertible and $K^{-1} = K$. Thus, we obtain $\Phi = KM^{-1}F$, i.e.

$$\varphi_k = \sum_{j=1}^{n} (S+N_k)(b_{kj} P_j f).$$

Using Theorem 1, we conclude that every solution of (11) is of the form (13). The proof is complete.

3.2. We now deal with equation (9) without the assumption $a_j(t) \equiv 0$ on Γ, $j = 1, \dots, n$.

$$\sum_{j=1}^{n} a_j^*(t)(P_j\varphi)(t) + \sum_{j=1}^{n} m_j(t)\big((S+N_j)P_j\varphi\big)(t) = f(t). \tag{14}$$

In a way similar to the proof of Theorem 1, we can prove easily the following theorem:

Theorem 3. *Let $n_j(\tau,t)$ $(j = 1,2,\dots,n)$ admit an analytic continuation onto D^+ in both variables. Suppose that $n_j(\epsilon_1\tau, t) = n_j(\tau, \epsilon_1 t) = n_j(\tau,t)$.*

a) *If $\varphi \in X$ is a solution of* (14) *then $(P_1\varphi, P_2\varphi, \dots, P_n\varphi)$ is a solution of the following system*

$$\sum_{j=1}^{n} \Big[a_{kj}^* + m_{kj}(S+N_j)\Big]\varphi_j = P_k f, \quad k = 1,2,\dots,n, \tag{15}$$

where

$$a_{kj}^*(t) = \frac{1}{n}\sum_{\nu=1}^{n} \epsilon_{\nu+1}^{j-k} a_j^*(\epsilon_{\nu+1} t),$$
$$m_{kj}(t) = \frac{1}{n}\sum_{\nu=1}^{n} \epsilon_{\nu+1}^{j-k} m_j(\epsilon_{\nu+1} t).$$

b) *If $(\varphi_1, \varphi_2, \dots, \varphi_n)$ is a solution of* (15) *then*

$$\varphi = \sum_{j=1}^{n} P_j \varphi_j$$

is a solution of (14).

Write

$$\begin{aligned} \Phi &= (\varphi_1, \varphi_2, \dots, \varphi_n)^{\mathrm{t}}, \\ F &= (P_1 f, P_2 f, \dots, P_n f)^{\mathrm{t}}, \\ A(t) &= \Big[a^*_{kj}(t)\Big]_{k,j=1,\dots,n}, \\ B(t) &= \Big[m_{kj}(t)\Big]_{k,j=1,\dots,n}, \\ L(\tau,t) &= \Big[m_{kj}(t) n_j(\tau,t)\Big]_{k,j=1,\dots,n}, \end{aligned}$$

and

$$\begin{aligned} L &= \Big[m_{kj} N_j\Big]_{k,j=1,\dots,n}, \\ S &= \Big[\delta_{kj} S\Big]_{k,j=1,\dots,n} \end{aligned}$$

as function-matrices and operator-matrices respectively. Then the system (15) can be written in the form

$$A(t)\Phi(t) + B(t)(S\Phi(t)) + L\Phi(t) = F(t). \tag{16}$$

Denote by $\mathcal{H}(D^+, D^+)$ the set of all two-variable functions $l(z,w)$ admitting an analytic continuation onto D^+ in each variable and satisfying the following condition

$$l(\epsilon_1 \tau, t) = l(\tau, \epsilon_1 t) = l(\tau, t) \quad \text{on} \quad \Gamma \times \Gamma.$$

Denote by $\mathcal{H}_{n\times n}(D^+, D^+)$ the set of all function-matrices of order $n \times n$ whose elements belong to $\mathcal{H}(D^+, D^+)$.

Theorem 4. *Suppose that matrices $D_\pm(t) = A(t) \pm B(t)$ are invertible and*

$$\Big[A(t) \pm B(t)\Big]^{-1} L(\tau, t) \in \mathcal{H}_{n\times n}(D^+, D^+).$$

Then the system (16) *can be represented in the following form*

$$\big(AI + BS\big)\big(I + M\big)\Phi = F, \tag{17}$$

where M is an integral operator with the kernel

$$\frac{1}{2\pi i} M(\tau, t) = \big[A(t) + B(t)\big]^{-1} L(\tau, t). \tag{18}$$

Proof. According to the assumption and Lemma 2, we get $SM = M$. Hence

$$\big(AI + BS\big)\big(I + M\big) = AI + BS + L.$$

The proof is complete.

Now we can present the solvability in a closed form of the equation (9) by the following theorem

Theorem 5. *Suppose that all assumptions of Theorem* 4 *are satisfied. Then the equation* (9) *has solutions if and only if the corresponding Riemann boundary value problem*

$$(AI + BS)\Phi = F \tag{19}$$

has a solution. If that is the case, the equation (9) *can be solved in a closed form by means of Riemann boundary value problems.*

Proof. From Lemma 2, it follows that $(I - M)(I + M) = I$. This means that the operator $(I + M)$ is invertible. Hence, the sufficient and necessary condition is proved. The solvability in a closed form follows immediately from Theorem 3, representation (16) and the theory of the Riemann boundary value problems (see [1, 9]).

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