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On Circumradii of Sets and Roughly Contractive Mappings

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Abstract. In this paper, circumradii and diameter of sets are used to estimate the distance between some subset S of a finite-dimensional normed space and any point from $\operatorname{conv} S \setminus S$, which is applied to prove a fixed-point theorem for roughly contractive mappings.

Let X be a normed linear space. For a bounded set $S \subset X$,

$$r_A(S) = \inf_{x\in A} \sup_{y\in S} \|x-y\| \quad \text{and} \quad \mathcal{C}_A(S) = \big\{x \in A : \sup_{y\in S} \|x-y\| = r_A(S)\big\}$$

are the *relative radius* and the *relative center set* of S with respect to $A \subset X$. In particular, $r_X(S)$ is the *absolute radius* of S, and $r_{\operatorname{conv} S}(S)$ is called its *self-radius*, because $r_{\operatorname{conv} S}(S) = r_S(S)$ if S is convex.

In general, $r_X(S)$ and $r_{\operatorname{conv} S}(S)$ may be different, and $\mathcal{C}_X(S) \cap \overline{\operatorname{conv}}\, S$ may be empty. For instance, let

$$S = \{e_1, e_2, e_3\} \subset X = \ell_3^3 \quad \text{with} \quad e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1) \tag{1}$$

Then, for $\mu = \sqrt{2} - 1 \approx 0.4142136$, we have

$$\begin{aligned} r_X(S) &= \big((1-\mu)^3 + 2\,\mu^3\big)^{1/3} \approx 0.7000991, \\ \mathcal{C}_X(S) &= \{y_\mu\}, \quad \text{where} \quad y_\mu = \mu\,(1,1,1), \end{aligned} \tag{2}$$

and

$$\begin{aligned} r_{\operatorname{conv} S}(S) &= \frac{1}{3}\,10^{1/3} \approx 0.7181449, \\ \mathcal{C}_{\operatorname{conv} S}(S) &= \{y_{1/3}\}, \quad \text{where} \quad y_{1/3} = \frac{1}{3}\,(1,1,1), \end{aligned} \tag{3}$$

i.e., $r_X(S) < r_{\operatorname{conv} S}(S)$ and $\mathcal{C}_X(S) \cap \overline{\operatorname{conv}}\, S = \mathcal{C}_X(S) \cap \operatorname{conv} S = \emptyset$.

In fact, due to Klee [6], $r_{\operatorname{conv} S}(S) = r_X(S)$ for all bounded $S \subset X$ is equivalent to X being two-dimensional or an inner-product space, and $C_X(S)$ intersects $\operatorname{conv} S$ for all bounded $S \subset X$ is equivalent to X being two-dimensional or a complete inner-product space. This classical result of Klee shows the role of absolute radius and center set for characterizing normed linear spaces.

Let us now use the above circumradii to estimate the distance between a set S and any point $z \in \operatorname{conv} S \setminus S$.

Proposition 1. *Suppose that X is some two-dimensional normed space or some Euclidean space, $S \subset X$, and $z \in \operatorname{conv} S \setminus S$. Then there exists $s \in S$ such that*

$$\|z - s\| \le r_X(S) = r_{\operatorname{conv} S}(S).$$

Proof. Due to Theorem 1 in [6], we have $r_X(S) = r_{\operatorname{conv} S}(S)$. Since $z \in \operatorname{conv} S \setminus S$ there exists a set $S_k = \{x_1, x_2, \ldots, x_k\} \subset S$ of $k \ge 2$ linearly independent points such that $z \in \operatorname{ri}(\operatorname{conv} S_k)$, where $\operatorname{ri} A$ denotes the relative interior of A. It remains to verify

$$\min_{1\le i\le k} \|z - x_i\| \le r_X(S). \tag{4}$$

Obviously, it holds for $k = 2$ that

$$\min_{1\le i\le 2} \|z - x_i\| \le \frac{1}{2}\|x_1 - x_2\| = r_{\operatorname{conv}\{x_1,x_2\}}(\{x_1, x_2\}) = r_X(\{x_1, x_2\}) \le r_X(S).$$

Assume (4) for $2 \le k \le l$, we are going to show that it also holds true for $k = l + 1$. Take a center c from the set $\mathcal{C}_X(S_k) \cap \operatorname{conv} S_k$, which is non-empty by Corollary 2 in [6]. Then

$$\max_{1\le i\le k} \|c - x_i\| \le r_X(S_k) \le r_X(S). \tag{5}$$

If $z = c$ then (4) follows from (5). Otherwise, since $c \in \operatorname{conv} S_k$ and $z \in \operatorname{ri}(\operatorname{conv} S_k)$, the ray from c through z cuts the boundary $\operatorname{conv} S_k \setminus \operatorname{ri}(\operatorname{conv} S_k)$ at some point

$$z' \in \operatorname{conv} S_{k'}, \quad \text{where} \quad S_{k'} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_{k'}}\} \subset S_k, \quad k' \le l.$$

If $z' \in S_k$ then $z \in [c, z']$ (where $[c, z']$ denotes the segment connecting c and z') implies

$$\|z - z'\| \le \|c - z'\| \le r_X(S).$$

If $z' \notin S_k$ then we can choose $S_{k'}$ so that $z' \in \operatorname{ri}(\operatorname{conv} S_{k'})$. By induction assumption, there is some $y \in S_{k'} \subset S$ such that $\|z' - y\| \le r_X(S)$. Therefore, it follows from the convexity of the norm and (5) that

$$\|z - y\| \le \max\{\|c - y\|, \|z' - y\|\} \le r_X(S),$$

which completes our proof. ∎

If $\dim X = n \geq 3$ and X is not a Euclidean space then it is not sure that, for any bounded set $S \subset X$ and any $z \in \operatorname{conv} S \setminus S$, there exists $s \in S$ satisfying

$$\|z - s\| \leq r_X(S) \quad \text{or} \quad \|z - s\| \leq r_{\operatorname{conv} S}(S).$$

To see this fact, let us consider example (1)–(2) again. For

$$\lambda = 3^{-1/3}\, r_X(S), \;\; y_{\lambda+\mu} = (\lambda + \mu)\,(1,1,1), \;\; \text{and} \;\; S' = \{e_1, e_2, e_3, y_{\lambda+\mu}\} \tag{6}$$

we have $y_{1/3} \in \operatorname{conv} S' \setminus S'$ but

$$\|y_{\lambda+\mu} - y_{1/3}\| = 3^{1/3}\,(\lambda + \mu - \frac{1}{3}) > r_X(S) \tag{7}$$

and

$$\|e_i - y_{1/3}\| = r_{\operatorname{conv} S}(S) > r_X(S), \;\; i = 1, 2, 3, \tag{8}$$

while

$$r_X(S) = r_X(S') = r_{\operatorname{conv} S'} S'. \tag{9}$$

Thus, the assertion of Proposition 3.9 in [7] is only true for two-dimensional normed spaces or for Euclidean spaces, and not for any normed spaces. But this fact does not influence the main results stated there, which were only formulated for Euclidean spaces.

Proposition 2. *Suppose that X is some two-dimensional strictly convex normed space or some Euclidean space, $S = \{x_1, x_2, \ldots, x_k\} \subset X$, and $z \in \operatorname{conv} S \setminus S$. Then it holds either*

$$\min_{1 \leq i \leq k} \|z - x_i\| < r_X(S) = r_{\operatorname{conv} S}(S) \tag{10}$$

or

$$\|z - x_i\| = r_X(S) = r_{\operatorname{conv} S}(S), \; i = 1, 2, \ldots, k. \tag{11}$$

Proof. We have to show that

$$\min_{1 \leq i \leq k} \|z - x_i\| \geq r_X(S) \tag{12}$$

implies (11). Let us prove by induction.

If $\dim S = 1$, then all points of S lie in some segment, say for instance, in the segment $[x_1, x_k]$ connecting x_1 and x_k. Then $r_X(S) = (1/2) \operatorname{diam} S = (1/2)\|x_1 - x_k\|$ and

$$\min_{1 \leq i \leq k} \|z - x_i\| < r_X(S) \;\text{ if }\; z \neq \frac{1}{2}\,(x_1 + x_k).$$

Therefore, (12) implies $z = (1/2)(x_1 + x_k)$, $\max_{1 \leq i \leq k} \|z - x_i\| \leq r_X(S)$, and finally (11).

Assume now that the assertion is true for $\dim S \leq l$, and (12) holds for some set $S = \{x_1, x_2, \ldots, x_k\}$ with $\dim S = l + 1 \geq 2$. We have to show (11) now. Due to Corollary 3 in [6], $\mathcal{C}_X(S)$ is a non-empty subset of $\operatorname{conv} S$. If $z \in \mathcal{C}_X(S)$ then

$$\max_{1\le i\le k}\|z-x_i\|\le r_X(S)$$

and (12) yield immediately (11). Next, we show that $z\notin\mathcal{C}_X(S)$ is impossible by considering two cases: $z\in\mathrm{ri}\,(\mathrm{conv}\,S)$ and $z\notin\mathrm{ri}\,(\mathrm{conv}\,S)$.

If $z\in\mathrm{ri}\,(\mathrm{conv}\,S)$, then the ray L from $c\in\mathcal{C}_X(S)$ through $z\notin\mathcal{C}_X(S)$ cuts the boundary $\mathrm{conv}\,S\setminus\mathrm{ri}\,(\mathrm{conv}\,S)$ at $z'\in\mathrm{conv}\,S_l$ for some $S_l\subset S$ with $\dim S_l\le l$. Therefore,

$$\|c-y\|\le r_X(S)\ \text{ and }\ \|z-y\|\ge r_X(S) \tag{13}$$

imply by the convexity of norm that

$$\|z'-y\|\ge r_X(S)\ge r_X(S_l)\ \text{ for all }\ y\in S_l.$$

This means at least $L\cap S_l=\emptyset$, and therefore, the function $g(x)=\|x-y\|$ is strictly convex on L, for all $y\in S_l$. Consequently, (13) and $z'\in L\setminus[c,z]$ yield

$$\|z'-y\|> r_X(S)\ge r_X(S_l)\ \text{ for all }\ y\in S_l,$$

i.e., (12) is satisfied for z' and S_l instead of z and S while (11) fails, which conflicts with induction assumption.

If $z\notin\mathrm{ri}\,(\mathrm{conv}\,S)$, then $z\in\mathrm{conv}\,S\setminus S$ implies $z\in\mathrm{ri}\,(\mathrm{conv}\,S_l)$ for some $S_l\subset S$ with $\dim S_l\le l$. The inequality

$$\min_{y\in S_l}\|z-y\|\ge r_X(S)\ge r_X(S_l)$$

and $\dim S_l\le l$ yield by induction assumption that

$$\|z-y\|=r_X(S)=r_X(S_l),\ \text{ for all }\ y\in S_l.$$

Therefore, for $c\in\mathcal{C}_X(S)$ and $z\notin\mathcal{C}_X(S)$, it follows from the strict convexity of the normed space X and

$$\|c-y\|\le r_X(S)=r_X(S_l)\ \text{ for all }\ y\in S_l$$

that

$$\left\|\frac{1}{2}(c+z)-y\right\|< r_X(S_l)\ \text{ for all }\ y\in S_l,$$

which conflicts with the definition of $r_X(S_l)$. Hence, $z\notin\mathcal{C}_X(S)$ is also impossible in case $z\notin\mathrm{ri}\,(\mathrm{conv}\,S)$. ■

The proof of the above proposition uses the fact that for such a space X, the absolute center c of any finite set S belongs to $\mathrm{conv}\,S$. It is no more true for an arbitrary strictly convex normed space, as stated in Proposition 3.10 of [7]. An example for this is given by (1)–(3) and (6)–(9), where it holds for $S'=\{e_1,e_2,e_3,y_{\lambda+\mu}\}$ and $y_{1/3}\in\mathrm{conv}\,S'\setminus S'$

$$\min\{\|y_{\lambda+\mu}-y_{1/3}\|,\|e_1-y_{1/3}\|,\|e_2-y_{1/3}\|,\|e_3-y_{1/3}\|\}> r_X(S')=r_{\mathrm{conv}\,S'}S'.$$

We have seen that such distance estimates by using circumradii (as in Propositions 1–2) cannot be obtained for other finite-dimensional spaces. But we can use the *diameter*

$$\operatorname{diam} S = \sup_{x,y \in S} \|x - y\|$$

to derive the following similar result.

Proposition 3. *Let S be a bounded subset of some n-dimensional normed space X and $z \in \operatorname{conv} S \setminus S$. Then there exists $s \in S$ such that*

$$\|z - s\| \le \frac{n}{n+1} \operatorname{diam} S. \tag{14}$$

If this normed space is strictly convex and $n \ge 2$ then

$$\inf_{s \in S} \|z - s\| < \frac{n}{n+1} \operatorname{diam} S. \tag{15}$$

Proof. By Carathéodory's theorem [2], there is a set $S_k = \{x_1, x_2, \dots, x_k\} \subset S$ of $k \le n + 1$ linearly independent points such that $z \in \operatorname{ri}(\operatorname{conv} S_k)$. Obviously, the function

$$f(x) = \sum_{i=1}^{k} f_i(x), \quad \text{where} \;\; f_i(x) = \|x - x_i\|,$$

is convex. Moreover, if the normed space X is strictly convex and $k \ge 3$ then f is strictly convex, because for every line L in X there exists an $x_i \in S_k$ lying outside of L which implies the strict convexity of f_i on L. Therefore, it follows from

$$f(x_j) = \sum_{i=1}^{k} \|x_j - x_i\| \le (k-1) \max_{1 \le i \le k} \|x_j - x_i\| \le (k-1) \operatorname{diam} S \quad \text{for } j = 1, 2, \dots, k$$

and $z \in \operatorname{ri}(\operatorname{conv} S_k)$ that

$$\min_{1 \le i \le k} \|z - x_i\| \le \frac{1}{k} f(z) \le \frac{1}{k} \max_{1 \le j \le k} f(x_j) \le \frac{k-1}{k} \operatorname{diam} S \le \frac{n}{n+1} \operatorname{diam} S, \tag{16}$$

i.e., (14) holds for $s \in S_k$ satisfying $\|z - s\| = \min_{1 \le i \le k} \|z - x_i\|$.

Assume now that the normed space X is strictly convex and $n \ge 2$. If $k = 2$ then $(k-1)/k = 1/2 < n/(n+1)$ and (16) yield

$$\min_{1 \le i \le k} \|z - x_i\| < \frac{n}{n+1} \operatorname{diam} S.$$

This strict inequality remains true if $k \ge 3$ because f is strictly convex, and therefore, $z \in \operatorname{ri}(\operatorname{conv} S_k)$ implies $f(z) < \max_{1 \le j \le k} f(x_j)$. Hence, in both cases, (15) follows from $\inf_{s \in S} \|z - s\| \le \min_{1 \le i \le k} \|z - x_i\|$. ∎

Propositions 1–3 can be applied for proving the following fixed-point theorem for so-called *r-roughly k-contractive mappings* $T : M \to M$ defined by

$$\|Tx - Ty\| \le k \|x - y\| + r \quad \text{for all} \;\; x, y \in M, \tag{17}$$

where $k \in (0, 1)$ and $r > 0$ are given.

Theorem 4. *Let $T : M \to M$ be an r-roughly k-contractive mapping on a closed and convex subset M of some n-dimensional normed space X. If $\dim X = 1$ then*

$$\exists x^* \in M : \ \|x^* - Tx^*\| \le \frac{1}{2}\, r. \tag{18}$$

If $\dim X \ge 2$ then

$$\forall \varepsilon > 0 \ \exists x^* \in M : \ \|x^* - Tx^*\| < \frac{n}{n+1}\, r + \varepsilon. \tag{19}$$

If, in addition, the normed space X is strictly convex then

$$\exists x^* \in M : \ \|x^* - Tx^*\| < \frac{n}{n+1}\, r, \tag{20}$$

or if X is the n-dimensional Euclidean space then

$$\exists x^* \in M : \ \|x^* - Tx^*\| \le \sqrt{\frac{n}{2(n+1)}}\, r. \tag{21}$$

Proof. (a) Fix an arbitrary point $x_0 \in M$ and define

$$\hat{B} = \{x \in X : \|x - x_0\| \le \hat{r}\}, \quad \text{where } \ \hat{r} = \frac{r + \|x_0 - Tx_0\|}{1 - k}\,.$$

Since (17) implies

$$\|Tx - x_0\| - \|x_0 - Tx_0\| \le \|Tx - Tx_0\| \le k\|x - x_0\| + r$$

it holds for $x \in \hat{B}$ that

$$\|Tx - x_0\| \le k\|x - x_0\| + r + \|x_0 - Tx_0\| \le k\hat{r} + (1-k)\hat{r} = \hat{r},$$

i.e., $Tx \in \hat{B}$. Hence, T maps the nonempty compact and convex subset $\hat{M} = M \cap \hat{B}$ into itself.
(b) Consider the set-valued map $\overline{T} : \hat{M} \to 2^{\hat{M}}$ defined by $\overline{T}(x) = \operatorname{conv} \overline{M}(x)$, where

$$\overline{M}(x) = \{y \in \hat{M} : \text{ there is a sequence } (x_i) \subset \hat{M} \text{ such that } x_i \to x,\ Tx_i \to y\}. \tag{22}$$

For all $x \in \hat{M}$, $\overline{M}(x) \ne \emptyset$ because $\hat{M}$ is compact. By definition, it is also closed. Moreover, (17) and (22) yield $\operatorname{diam} \overline{M}(x) \le r$, i.e., it is bounded. Since X is finite-dimensional, $\overline{M}(x)$ and $\operatorname{conv} \overline{M}(x)$ are compact (see [8]). Finally, $\overline{T}$ is upper semi-continuous (see [3]). Therefore, it follows from Kakutani's theorem (see [5] and [9]) that there exists a point $\bar{x} \in \hat{M}$ with

$$\bar{x} \in \overline{T}(\bar{x}) = \operatorname{conv} \overline{M}(\bar{x}). \tag{23}$$

(c) Assume $\dim X = 1$. If $\|\bar{x} - T\bar{x}\| \le (1/2)r$ then (18) holds for $x^* = \bar{x}$. Otherwise, if $\|\bar{x} - T\bar{x}\| > (1/2)r$, then $\operatorname{diam}(\overline{M}(\bar{x}) \cup T\bar{x}) \le r$, which follows from (17) and (22), $\overline{M}(\bar{x}) \cup T\bar{x} \subset \mathbb{R}$ and $\bar{x} \in \operatorname{conv} \overline{M}(\bar{x})$ imply that there exists $\bar{y} \in \overline{M}(\bar{x})$ satisfying $\|\bar{x} - \bar{y}\| < (1/2)r$. By choosing $\varepsilon = (1/2)r - \|\bar{x} - \bar{y}\| > 0$ and x^* satisfying (24), we have

$$\|x^* - Tx^*\| \le \|x^* - \bar{x}\| + \|\bar{x} - \bar{y}\| + \|\bar{y} - Tx^*\| < \|\bar{x} - \bar{y}\| + \varepsilon = \frac{1}{2}\, r,$$

i.e., (18) holds true.
(d) By $\operatorname{diam} \overline{M}(\bar{x}) \leq r$ and Proposition 3, there is a $\bar{y} \in \overline{M}(\bar{x})$ such that

$$\|\bar{x} - \bar{y}\| \leq \frac{n}{n+1} \operatorname{diam} \overline{M}(\bar{x}) \leq \frac{n}{n+1}\, r.$$

Due to (22), for any $\varepsilon > 0$, there exists a point $x^* \in \hat{M} \subset M$ such that

$$\|x^* - \bar{x}\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|Tx^* - \bar{y}\| < \frac{\varepsilon}{2}. \tag{24}$$

Consequently,

$$\|x^* - Tx^*\| \leq \|x^* - \bar{x}\| + \|\bar{x} - \bar{y}\| + \|\bar{y} - Tx^*\| < \frac{n}{n+1}\, r + \varepsilon,$$

i.e., (19) holds true.
(e) If X is an n-dimensional strictly normed space with $n \geq 2$ then, by Proposition 3 and (23), there is a $\bar{y} \in \overline{M}(\bar{x})$ such that

$$\|\bar{x} - \bar{y}\| < \frac{n}{n+1} \operatorname{diam} \overline{M}(\bar{x}) \leq \frac{n}{n+1}\, r.$$

By choosing $\varepsilon = n/(n+1)r - \|\bar{x} - \bar{y}\| > 0$ and x^* satisfying (24), we obtain

$$\|x^* - Tx^*\| \leq \|x^* - \bar{x}\| + \|\bar{x} - \bar{y}\| + \|\bar{y} - Tx^*\| < \|\bar{x} - \bar{y}\| + \varepsilon = \frac{n}{n+1}\, r,$$

i.e., (20) holds true.
(f) Assume now that X is the n-dimensional Euclidean space. By (23), there exists a finite set $\tilde{M} \subset \overline{M}(\bar{x})$ such that $\bar{x} \in \operatorname{conv} \tilde{M}$. Consider $S = \tilde{M} \cup T\bar{x}$. Obviously, $\bar{x} \in \operatorname{conv} S$. Moreover, it follows from (17) and (22) that $\operatorname{diam} S \leq r$. If $\|\bar{x} - T\bar{x}\| \leq r_X(S)$ then, by Jung's inequality [4]

$$r_X(S) \leq \sqrt{\frac{n}{2(n+1)}} \operatorname{diam} S, \tag{25}$$

we have

$$\|\bar{x} - T\bar{x}\| \leq r_X(S) \leq \sqrt{\frac{n}{2(n+1)}}\, r,$$

i.e., (21) is fulfilled for $x^* = \bar{x}$. Otherwise, if $\|\bar{x} - T\bar{x}\| > r_X(S)$ then Proposition 2 yields

$$\min_{s \in \tilde{M}} \|\bar{x} - s\| = \min_{s \in S} \|\bar{x} - s\| < r_X(S).$$

Consequently, there is a $\bar{y} \in \tilde{M} \subset \overline{M}(\bar{x})$ such that $\|\bar{x} - \bar{y}\| < r_{\operatorname{conv} S}(S)$. For

$$\varepsilon = r_{\operatorname{conv} S}(S) - \|\bar{x} - \bar{y}\| > 0,$$

(22) implies the existence of $x^* \in \hat{M} \subset M$ satisfying (24). Therefore, we have

$$\|x^* - Tx^*\| \leq \|x^* - \bar{x}\| + \|\bar{x} - \bar{y}\| + \|\bar{y} - Tx^*\| \leq \|\bar{x} - \bar{y}\| + \varepsilon = r_{\operatorname{conv} S}(S).$$

Hence, (21) follows from (25) and $\operatorname{diam} S \leq r$. ■

Note that r-roughly k-contractive mappings were investigated in [7]. We showed there that the usual iteration $x_{i+1} = Tx_i$ used in Banach's fixed-point theorem [1] is only suitable for approximating γ-invariant points satisfying $\|x^* - Tx^*\| \leq \gamma$ for $\gamma \geq r/(1-k) > r$. The existence of γ-invariant points with $\gamma \leq r$ is guaranteed if the domain M is convex, as stated by Theorem 3.11 in [7]. Since the proof of $\|x^* - Tx^*\| \leq n/(n+1)r$ for strictly convex spaces based on an incorrect proposition, we present a modified proof in this paper, even for the strict inequality $\|x^* - Tx^*\| < (n/n+1)r$.

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