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Short Communication

Some Remarks on a Global Optimality Criterion

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With reference to a note by N. V. Thoai recently published in *Journal of Mathematical Analysis and Applications* [5], I would like to point out that the main theorem established in this paper (Proposition 4) is wrong. It is unfortunate that this incorrect result has been republished in full in at least two subsequent papers [2, 3] and a book [4].

Consider the problem

$$\inf \{g(x) - h(x) : x \in X\}, \tag{DC}$$

where g and h are two finite convex functions on $\mathbb{R}^p$, and X is a closed convex subset of $\mathbb{R}^p$. The following wrong theorem has been established in [6].

Proposition 4. *Assume that problem* (DC) *is solvable. Then a point $x^* \in X$ is an optimal solution of* (DC) *if and only if there is $t^* \in R$ such that*

$$0 = \inf\{-h(x) + t : x \in X,\, t \in \mathbb{R},\, g(x) - t \le g(x^*) - t^*\}. \tag{1}$$

To see that this proposition is false it suffices to consider the problem (DC) with $g(x) = \|x\|$, $h(x) = 0$, $X = \{x \in \mathbb{R}^p : \|x\| \le 1\}$. Clearly $0 = \min\{\|x\| : \|x\| \le 1\}$, but if we take any x^* such that $0 < \|x^*\| \le 1$, then for $t^* = \|x^*\|$, we will have (1) namely: $0 = \min\{t : \|x\| \le 1, t \in \mathbb{R}, \|x\| - t \le \|x^*\| - t^*\}$. Therefore, according to Proposition 4, x^* is optimal, with $\|x^*\| > 0$, a contradiction.

In fact Proposition 4 should be corrected as follows:

(P) *A vector $x^* \in X$ is optimal solution of problem* (DC) *if and only if*

$$0 = \inf\{-h(x) + t : x \in X, t \in \mathbb{R}, g(x) - t \leq g(x^*) - h(x^*)\} \quad (2)$$

In contrast with the complicated proof in [6] based on a trivial proposition called the "reciprocity principle", the proof of (P) is straightforward:

$$\begin{aligned} x^* \in X \text{ optimal } &\Leftrightarrow 0 = \inf\{-h(x) + g(x) - [g(x^*) - h(x^*)] : x \in X\} \\ &\Leftrightarrow 0 = \inf\{-h(x) + t : x \in X, g(x) - t \leq g(x^*) - h(x^*)\}. \quad \blacksquare \end{aligned}$$

Similarly, no reciprocity principle is needed for establishing Proposition 3 [5] which states a well known optimality criterion for problem

$$\omega^* = \inf\{\omega(z) : z \in Z, \psi(z) \leq 0\}, \quad \text{(C1)}$$

where $\omega(z)$ is a convex function, $\psi(z)$ is a concave function, and Z is a closed convex set. In fact, assuming regularity of (C1) and the existence of $z^0 \in Z$ such that $\omega(z^0) < \omega^*$ (assumption (A)), we have that

$$\begin{aligned} &z^* \in Z \text{ optimal to (C1)} \\ &\Leftrightarrow 0 = \inf\{\omega(z) - \omega(z^*) : z \in Z, \psi(z) \leq 0\} \\ &\Leftrightarrow 0 = \inf\{\omega(z) - \omega(z^*) : z \in Z, \psi(z) < 0\} \text{ by regularity of (C1)} \\ &\Leftrightarrow 0 = \inf\{\psi(z) : z \in Z, \omega(z) < \omega(z^*)\} \text{ because } \psi(z^*) \leq 0 \\ &\Leftrightarrow 0 = \inf\{\psi(z^0 + \alpha(z^* - z^0)) : 0 < \alpha < 1\} \text{ by assumption (A)} \\ &\Leftrightarrow 0 = \inf\{\psi(z^0 + \alpha(z^* - z^0)) : 0 < \alpha \leq 1\} \\ &\Leftrightarrow 0 = \inf\{\psi(z) : z \in Z, \omega(z) \leq \omega(Z^*)\}. \quad (3) \end{aligned}$$

The above proof is in essence the one given originally in [7], where this optimality criterion was first formulated and proved (see also [5]). Also the correct optimality criterion (P) is nothing but an application of the just mentioned optimality criterion to problem (C1).

Proposition 5 [6] states a well known fact, first proved in [9] (see formula (2)) in [9], or the proof of Proposition 11 in [8]).

It should also be noted that the proof of Lemma 1 in [6] is awkward and confused, while the proofs of Propositions 4 and 5 are mathematically incorrect. Furthermore, the proof of Hiriart–Urruty's theorem presented in [6] is actually the same as that originally given in [9] (reference 9 in [6]) several years earlier and now widely known (see also [1]). To sum up, every result in [6] is either trivial (Proposition 2), or flattly false (Proposition 4) or already well known (Propositions 3, 5, 6 and the proof of Proposition 6).

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