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Topology of the Efficient Sets of Convex Sets in $\mathbf{R}^{2}$ *

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Abstract. In this paper we prove that if a convex subset of $\mathbb{R}^2$ has compact sections with respect to the closure $\overline{C}$ of a convex cone C (but the set itself may be noncompact) then its efficient set with respect to C is homeomorphic to a polyhedral convex set, provided $\overline{C}$ is a pointed cone with nonempty interior.

1. Problem Statement and the Main Result

Problems concerning topological structure of the efficient solution sets form an interesting research field in Vector Optimization. The most important classical results can be found in [3-5, 8], and the references given in [5]. Some significant recent results can be found in [1, 6, 7], and references therein.

Knowing the characteristic topological properties (such as compactness, contractibility, arcwise connectedness, and connectedness) of the efficient set still does not mean that one knows its topology. To our knowledge, up to now the unique case where one knows fully the topology of the efficient set is the one described in [5, p. 144]. Namely, the following theorem holds.

Theorem 1.1. (See [5, Theorem 3.8, p. 144]) *If $A \subset \mathbb{R}^2$ is a nonempty convex compact set then the efficient set $E(A|C)$ of A with respect to a nonempty convex cone $C \subset \mathbb{R}^2$ is homeomorphic to a 0-dimensional or an 1-dimensional simplex,*

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provided the closure $\overline{C}$ *of* C *is a pointed cone.*

Recently, a long and detailed proof for the above theorem has been given in [2].

For the convenience of the reader, we recall now some standard notions.

Definition 1.1. (See [9, p. 11–12]) *An* m*-dimensional simplex is the convex hull of a family of* $m+1$ *affinely independent points* $\{b^0, b^1, \ldots, b^m\}$ *in a Euclidean space* $\mathbb{R}^n$ $(n \geq m)$*. A set* $M \subset \mathbb{R}^n$ *which can be expressed as the intersection of finitely many closed half spaces of* $\mathbb{R}^n$ *is called a polyhedral convex set. By the dimension of a convex set* M *one means the dimension of the affine hull of* M*.*

Definition 1.2. (See [5, p. 39–40]) *An efficient point of a set* $A \subset \mathbb{R}^n$ *with respect to (w.r.t.) a convex cone* $C \subset \mathbb{R}^n$ *is a vector* $x \in A$ *such that there exists no* $y \in A \setminus \{x\}$ *satisfying* $y \leq_C x$*. The last inequality means* $x - y \in C$*. The set of the efficient points of* A *w.r.t.* C *is denoted by* $E(A|C)$*. If the interior* $\mathrm{int}C$ *of* C *is nonempty, then a vector* $x \in A$ *is called a weakly efficient point of* A *w.r.t.* C *if* $x \in E(A|C')$*, where* $C' := \mathrm{int}C \cup \{0\}$*. The set of all weakly efficient points of* A *w.r.t.* C *is denoted by* $E^w(A|C)$*. A cone* $D \subset \mathbb{R}^n$ *is said to be pointed if* $D \cap (-D) = \{0\}$*.*

One may ask: *What happens if the compactness assumption on* A *in Theorem 1.1 is omitted?*

Let us consider one example.

Example 1.1. For $A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 \geq x_1^2\}$ and $C = \mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0\}$, one has

$$E^w(A|C) = E(A|C) = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1^2,\ x_1 \leq 0\}.$$

Since $E(A|C)$ is noncompact, it cannot be homeomorphic to any simplex. *However, it is homeomorphic to an* 1*-dimensional polyhedral convex set.*

The aim of this paper is to obtain a generalized version of Theorem 1.1 which is applicable to the class of nonempty convex subsets $A \subset \mathbb{R}^2$ having compact sections w.r.t. the cone $\overline{C}$. This class contains all nonempty compact convex subsets of $\mathbb{R}^2$.

Definition 1.3. ([5, Definition 2.3, p. 43]) *For a subset* $A \subset \mathbb{R}^n$ *and a convex cone* $D \subset \mathbb{R}^n$*, the sets*

$$S(x) = (x - D) \cap A \qquad (x \in \mathbb{R}^n)$$

are called the sections of A *w.r.t.* D*. A subset* $A \subset \mathbb{R}^n$ *is said to have compact sections w.r.t.* D *if* $S(x)$ *is compact for every* $x \in \mathbb{R}^n$*.*

Note that $S(x)$ is nonempty if and only if $x \in A + D$. Hence A has compact sections w.r.t. D if and only if $S(x)$ is compact for every $x \in A + D$.

In the sequel, it is assumed that $C \subset \mathbb{R}^2$ *is a convex cone such that the interior* $\mathrm{int}C$ *of* C *is a nonempty set and the closure* $\overline{C}$ *of* C *is a pointed cone.*

The main result of this paper is stated as follows.

Theorem 1.2. *Suppose that* $A \subset \mathbb{R}^2$ *is a nonempty convex set having compact sections w.r.t.* $\overline{C}$*. Then the set* $E(A|C)$ *(resp.,* $E^w(A|C)$*) is homeomorphic to a* 0*-dimensional or an* 1*-dimensional polyhedral convex set. This amounts to saying that* $E(A|C)$ *(resp.,* $E^w(A|C)$*) is homeomorphic to one of the following four subsets of the real line:* $\{0\},\ [0,1],\ [0,+\infty),\ (-\infty,+\infty)$.

The organization of the paper is as follows. In Sec. 2 we shall establish two auxiliary results. The proof of Theorem 1.2 will be given in Sec. 3. Sec. 4 presents some examples illustrating the result formulated in Theorem 1.2.

2. Lemmas

Let $A \subset R^2$ be a nonempty convex subset having compact sections w.r.t. $\overline{C}$. The scalar product of $x,\ y \in \mathbb{R}^2$ is denoted by $\langle x, y\rangle$. The norm of $x \in \mathbb{R}^2$ is defined by setting $\|x\| = \sqrt{\langle x, x\rangle}$. Fix a vector $e \in \mathrm{int}C$ with $\|e\| = 1$.

The following two lemmas will be crucial for proving Theorem 1.2.

Lemma 2.1. *The set* $T := \{t \in \mathbb{R} : S(te) \neq \emptyset\}$ *is nonempty, bounded from below, and closed.*

Proof. Let $t^0 \in \mathbb{R}$ be given arbitrarily. We show that

$$\mathbb{R}^2 = \bigcup_{t \geq t^0} (te - \overline{C}). \tag{1}$$

It is obvious that $\bigcup_{t\geq t^0} (te - \overline{C}) \subset \mathbb{R}^2$. To prove the opposite inclusion, pick any $x \in \mathbb{R}^2$. Since $\lim_{t\to+\infty} \frac{te-x}{t} = e \in \mathrm{int}\overline{C}$, there exists $t^1 \in (t^0, +\infty) \cap (0, +\infty)$ satisfying $\frac{t^1e-x}{t^1} \in \mathrm{int}\overline{C}$. Then $t^1e - x \in \mathrm{int}\overline{C}$. Hence $x \in t^1e - \mathrm{int}\overline{C} \subset t^1e - \overline{C}$. We conclude that $\mathbb{R}^2 = \bigcup_{t\geq t^0} (te-\overline{C})$. As $S(te) = (te-\overline{C})\cap A$, from (1) it follows that T is nonempty.

Suppose, contrary to our claim, that T is not bounded below. Then we can find a sequence $\{t^k\} \subset T,\ t^{k+1} < t^k$ for every $k \in \{1, 2, ...\}$, such that

$$\lim_{k\to+\infty} t^k = -\infty. \tag{2}$$

Since

$$(t^{k+1}e - \overline{C}) \cap A = (t^ke - (t^k - t^{k+1})e - \overline{C}) \cap A \subset (t^ke - \overline{C}) \cap A,$$

we have $S(t^{k+1}e) \subset S(t^ke)$ for all k. Consequently, for every $k \in \{1, 2, ...\}$,

$$S(t^ke) \subset S(t^1e). \tag{3}$$

For each $k \in \{1, 2, ...\}$, since $t^k \in T$, there exists some $x^k \in S(t^k e)$. The sequence $\{\|x^k\|\}$ is unbounded. Otherwise there exists a subsequence $\{\|x^{k_j}\|\}$ of $\{\|x^k\|\}$ such that $\lim_{j\to+\infty} \|x^{k_j}\| < +\infty$. By (2), $t^{k_j} < 0$ for j large enough. So the inclusion $x^{k_j} \in t^{k_j} e - \overline{C}$, for j large enough, implies that

$$\frac{x^{k_j}}{(-t^{k_j})} + e \in -\frac{1}{(-t^{k_j})}\overline{C} \subset -\overline{C}.$$

By the closedness of $-\overline{C}$, one has

$$\lim_{j\to\infty}\left(\frac{x^{k_j}}{(-t^{k_j})} + e\right) = e \in -\overline{C},$$

which is impossible because $\overline{C} \cap (-\overline{C}) = \{0\}$ by our assumption on C. We have thus shown that the sequence $\{\|x^k\|\}$ is unbounded. From this and (3) we deduce that $S(t^1 e)$ is a noncompact set, a contradiction. Therefore T must be bounded below.

Let $t^0 = \inf_{t\in T} t$. Our next claim is that $t^0 \in T$. Suppose the claim were false. Then we would have

$$S(t^0 e) = (t^0 e - \overline{C}) \cap A = \emptyset. \tag{4}$$

For every $k \in \{1, 2, ...\}$, since $t^k := t^0 + \dfrac{1}{k}$ belongs to T, one has $S(t^k e) \neq \emptyset$. So one can find a vector $x^k = t^k e - u^k \in S(t^k e)$, where $u^k \in \overline{C}$. Since $S(t^1 e)$ is a compact set by our assumption and since the inclusion $S(t^k e) \subset S(t^1 e)$ holds for every $k \in \{1, 2, ...\}$, there is no loss of generality in assuming that

$$\lim_{k\to\infty} x^k = x^0 \in S(t^1 e) \subset A. \tag{5}$$

As $u^k = t^k e - x^k$, we have $\lim_{k\to\infty} u^k = t^0 e - x^0$. By the closedness of $\overline{C}$,

$$t^0 e - x^0 \in \overline{C}. \tag{6}$$

From (5) and (6) it follows that $x^0 \in (t^0 e - \overline{C}) \cap A$. This implies that $S(t^0 e) \neq \emptyset$, contrary to (4). We have thus proved that $t^0 \in T$. Since $S(t^0 e) \subset S(te)$ for every $t > t^0$, the inclusion $t^0 \in T$ implies that $T = [t^0, +\infty)$. In particular, T is a closed set. The proof is complete. ■

Lemma 2.2. *The set $E(A|C)$ is arcwise connected, i.e., for any a, $b \in E(A|C)$ there exists a continuous function $\gamma : [0,1] \to E(A|C)$ such that $\gamma(0) = a$ and $\gamma(1) = b$.*

Proof. Choose a sequence $\{t^k\} \subset T$ such that $t^k < t^{k+1}$ for every $k \in \{1, 2, , ..\}$ and $\lim_{k\to\infty} t^k = +\infty$. Since the inclusions

$$S(t^k e) \subset S(te) \subset S(t^{k+1} e)$$

hold for any $k \in \{1, 2, ...\}$ and $t \in (t^k, t^{k+1})$, from (1) it follows that

$$A = \bigcup_{k=1}^{\infty} S(t^k e). \tag{7}$$

For each k, from the inclusion $S(t^k e) \subset S(t^{k+1}e)$ it follows that

$$E(S(t^k e)|C) \subset E(S(t^{k+1}e)|C). \tag{8}$$

Indeed, if (8) were false then there would exist $x \in E(S(t^k e)|C) \setminus E(S(t^{k+1}e)|C)$. So there is $y \in S(t^{k+1}e) \setminus \{x\}$ such that $y \leq_C x$, i.e.,

$$x - y \in C \setminus \{0\}. \tag{9}$$

One has

$$y = x - (x - y) \subset x - C \setminus \{0\} \subset t^k e - \overline{C} - C \setminus \{0\} \subset t^k e - \overline{C}.$$

Hence

$$y \in S(t^k e). \tag{10}$$

From (9) and (10) we deduce that $x \notin E(S(t^k e)|C)$, which is impossible. So (8) is proved. We wish to show that

$$E(A|C) = \bigcup_{k=1}^{\infty} E(S(t^k e)|C). \tag{11}$$

Given any $x \in E(A|C) \subset A$, from (7) we conclude that there exists $k \in \{1, 2, ...\}$ such that $x \in S(t^k e)$. Then $x \in E(S(t^k e)|C)$, since otherwise, there exists $y \in S(t^k e) \subset A$ such that $x - y \in C \setminus \{0\}$; hence $x \notin E(A|C)$, a contradiction. We have thus shown that $E(A|C) \subset \bigcup_{k=1}^{\infty} E(S(t^k e)|C)$. To prove the opposite inclusion, pick any $x \in \bigcup_{k=1}^{\infty} E(S(t^k e)|C)$. Let $k \in \{1, 2, ...\}$ be such an index that $x \in E(S(t^k e)|C)$. We must have $x \in E(A|C)$. Indeed, assume to the contrary that there exists $y \in A$ such that $x - y \in C \setminus \{0\}$. Arguing similarly as in the proof of (8), we get $y \in S(t^k e)$. Then we have $x \notin E(S(t^k e)|C)$, which is impossible. Thus (11) is proved. For every $k \in \{1, 2, ...\}$, since A has compact sections w.r.t. $\overline{C}$ and $t^k \in T$ then $S(t^k e)$ is nonempty, convex, and compact. Hence we can deduce from Theorem 1.1 that for every $k \in \{1, 2, ...\}$ the set $E(S(t^k e)|C)$ is nonempty and arcwise connected. Combining this fact with (8) and (11) we can assert that $E(A|C)$ is arcwise connected. Indeed, for any $a, b \in E(A|C)$, according to (8) and (11) there exists $k \in \{1, 2, ...\}$ such that $a, b \in E(S(t^k e)|C)$. By the arcwise connectedness of $E(S(t^k e)|C)$, we can find a continuous function $\gamma : [0, 1] \to E(S(t^k e)|C)$ such that $\gamma(0) = a,\ \gamma(1) = b$. Since $E(S(t^k e)|C) \subset E(A|C)$ by (11), the function $\gamma : [0, 1] \to E(A|C)$ is well-defined, continuous, and it joins a with b. ∎

3. Proof of Theorem 1.2

By assumption, $A \subset \mathbb{R}^2$ is a nonempty convex subset having compact sections w.r.t. $\overline{C}$, $\text{int}C \neq \emptyset$, and $\overline{C}$ is pointed.

We shall prove that $E(A|C)$ is homeomorphic to a 0-dimensional or an 1-dimensional polyhedral convex set. The desired homeomorphism will be defined as the restriction of a parallel projection of $\mathbb{R}^2$ (onto an 1-dimensional affine manifold) on the efficient set $E(A|C)$.

Let $T = \{t \in \mathbb{R} : S(te) = (te - \overline{C}) \cap A \neq \emptyset\}$. According to Lemma 2.1, T is nonempty, bounded below, and closed. Let

$$t^0 = \min_{t \in T} t. \tag{12}$$

Fix a point $\bar{x} \in E(S(t^0 e)|C)$. From (11) it follows that $\bar{x} \in E(A|C)$. Then we must have $\bar{x} \in \partial(A+\overline{C}) := (A+\overline{C}) \setminus \mathrm{int}(A+\overline{C})$. Indeed, if $\bar{x} \in \mathrm{int}(A+\overline{C})$ then there exists $\rho > 0$ satisfying $\bar{x}+\rho v \in A+\overline{C}$ for every $v \in R^2$ with $\|v\| \leq 1$. Choose $\bar{v} \in \mathrm{int}C$ such that $\|\bar{v}\| = 1$. Since $\bar{x} - \rho\bar{v} \in A + \overline{C}$, there exist $x \in A,\ c \in \overline{C}$ such that $\bar{x} - \rho\bar{v} = x + c$. Then $\bar{x} - x = \rho\bar{v} + c \in \mathrm{int}C + \overline{C} \subset \mathrm{int}C \subset C$, contrary to the fact that $\bar{x} \in E(A|C)$.

By the closedness and the convexity of $A + \overline{C}$, there exists (see [9, Theorem 11.6]) a supporting hyperplane $H = \{x :\ \langle \xi, x\rangle = \alpha\}$ to $(A + \overline{C})$, where $\xi \in \mathbb{R}^2 \setminus \{0\}$ and $\alpha \in \mathbb{R}$, such that

$$(A + \overline{C}) \subset H_+ := \{x :\ \langle \xi, x\rangle \geq \alpha\}, \qquad \bar{x} \in H. \tag{13}$$

Then H is also a supporting hyperplane to A. We claim that

$$\langle \xi, e\rangle > 0. \tag{14}$$

Indeed, if $\langle \xi, e\rangle = 0$ then $\langle \xi, \bar{x}+e\rangle = \langle \xi, \bar{x}\rangle = \alpha$. Hence $\bar{x}+e \in H$. Besides, since

$$\bar{x} + e \in \bar{x} + \mathrm{int}C \subset \mathrm{int}(\bar{x} + \overline{C}) \subset \mathrm{int}(A + \overline{C}),$$

we obtain $\bar{x} + e \in H \cap \mathrm{int}(A + \overline{C})$. In particular, we get $H \cap \mathrm{int}(A + \overline{C}) \neq \emptyset$, contrary to (13). If $\langle \xi, e\rangle < 0$ then we have $\bar{x} + e \in A + \overline{C}$ and $\langle \xi, \bar{x} + e\rangle = \langle \xi, \bar{x}\rangle + \langle \xi, e\rangle < \langle \xi, \bar{x}\rangle = \alpha$, contrary to (13). Thus (14) must hold true. For each $x \in \mathbb{R}^2$, we set

$$L(x) = \{y \in R^2 : y = x - te \text{ for some } t \geq 0\}$$

and define

$$t(x) = \frac{\langle \xi, x\rangle - \alpha}{\langle \xi, e\rangle}. \tag{15}$$

By (13) and (14), $t(x) \geq 0$ for each $x \in A + \overline{C}$. It follows from (15) that $\langle \xi, x - t(x)e\rangle = \alpha$, so $x - t(x)e \in L(x) \cap H$. We construct a projection $\Pi_e : A + \overline{C} \to H$ from $A + \overline{C}$ into H parallel to the direction $-e$ by setting

$$\Pi_e(x) = x - t(x)e \in L(x) \cap H \quad (\forall x \in A + \overline{C}). \tag{16}$$

Using (14) one can see at once that $L(x) \cap H$ is a singleton; so vector $\Pi_e(x)$ defined in (16) is just the unique element of $L(x) \cap H$.

From (15) and (16) it follows that $\Pi_e(\cdot)$ is continuous on $A + \overline{C}$. Hence the restriction map

$$\Pi_e :\ E(A|C) \longrightarrow \Pi_e(E(A|C)) \tag{17}$$

is also continuous. We claim that the map in (17) is one-to-one. If the claim were false, then we could find $x^1,\ x^2 \in E(A|C)$ such that $x^1 \neq x^2$ and $\Pi_e(x^1) = \Pi_e(x^2)$. The last equality means that there exist $t(x^1),\ t(x^2) \in R$ such that

$$x^1 - t(x^1)e = x^2 - t(x^2)e. \tag{18}$$

If $t(x^1) > t(x^2)$ then (18) implies that $x^1 - x^2 = (t(x^1) - t(x^2))e \in \text{int}C$, hence $x^1 \notin E(A|C)$, a contradiction. Likewise, if $t(x^1) < t(x^2)$ then (18) implies that $x^2 \notin E(A|C)$, which is impossible. Thus $t(x^1) = t(x^2)$. Combining this with (18) we obtain $x^1 = x^2$, a contradiction. Our claim has been proved.

Consider the inverse map of the one in (17):

$$\Pi_e^{-1} : \ \Pi_e(E(A|C)) \longrightarrow E(A|C). \tag{19}$$

We wish to show that this map is continuous. Fix any $\bar{u} \in \Pi_e(E(A|C))$. Let $\{u^k\} \subset \Pi_e(E(A|C))$ be any sequence satisfying

$$\lim_{k\to\infty} u^k = \bar{u}. \tag{20}$$

Let $\bar{x} = \Pi_e^{-1}(\bar{u})$ and $x^k = \Pi_e^{-1}(u^k)$ for $k = 1, 2, ...$ Clearly, from (16) it follows that $\bar{u} = \bar{x} - t(\bar{x})e$ and $u^k = x^k - t(x^k)e$ for every $k = 1, 2, ...$ Therefore

$$\bar{x} = \bar{u} + t(\bar{x})e, \quad x^k = u^k + t(x^k)e \quad (\forall k \in \{1, 2, ...\}). \tag{21}$$

Our task is to show that $\lim_{k\to\infty} x^k = \bar{x}$. First, we observe that the sequence $\{t(x^k)\}$ is bounded. Indeed, since $t(x) \geq 0$ for every $x \in A + \overline{C}$, we have $t(x^k) \geq 0$ for every k. If $\{t(x^k)\}$ is not bounded above then, without loss of generality, we can assume that $\lim\limits_{k\to\infty} t(x^k) = +\infty$. So $t(x^k) > t(\bar{x})$ for k large enough, and (21) yields

$$\frac{x^k - \bar{x}}{t(x^k) - t(\bar{x})} = e + \frac{u^k - \bar{u}}{t(x^k) - t(\bar{x})}. \tag{22}$$

Taking account of (20), (22) and the fact that $e \in \text{int}C$, we obtain

$$\frac{x^k - \bar{x}}{t(x^k) - t(\bar{x})} \in \text{int}C$$

for k large enough. The last inclusion implies that $x^k - \bar{x} \in \text{int}C$. From this we obtain $x^k \notin E(A|C)$ for every k large enough, which is impossible. Thus the sequence $\{t(x^k)\}$ must be bounded. Combining this with (20) and (21) we see that the sequence $\{x^k\}$ is also bounded.

From (1) it follows that there exists $\hat{t} \geq t^0$ such that $x^k \in \hat{t}e - \overline{C}$ for all k. Indeed, for any $t, t' \geq t^0, \ t' > t$, we have

$$\begin{aligned} te - \overline{C} \subset t'e - (t' - t)e - \overline{C} &\subset t'e - \text{int}C - \overline{C} \\ &\subset t'e - \text{int}C \\ &\subset \text{int}(t'e - \overline{C}). \end{aligned}$$

Then from (1) it follows that

$$R^2 = \bigcup_{t \geq t^0} \text{int}(te - \overline{C}).$$

Thus the family $\left\{\text{int}(te - \overline{C})\right\}_{t \geq t^0}$ forms an open covering of the compact set $\Delta := \overline{\{x^k : k = 1, 2, ...\}}$. Consequently, there exists some $\hat{t} \geq t^0$ such that $x^k \in \text{int}(\hat{t}e - \overline{C})$ for all k.

From the inclusions $x^k \in \hat{t}e - \overline{C}$ and $x^k \in A$ we obtain $x^k \in S(\hat{t}e)$. Since $x^k \in E(A|C)$, we must have $x^k \in E(S(\hat{t}e)|C) \subset E(A|C)$ for every $k = 1, 2, ...$ By Theorem 1.1, $E(S(\hat{t}e)|C)$ is a compact set.

Let $\{x^{k_j}\}$ be an arbitrary subsequence of $\{x^k\}$ converging to some $\hat{x} \in R^2$. By the compactness of $E(S(\hat{t}e)|C)$,

$$\hat{x} = \lim_{j\to\infty} x^{k_j} \in E(S(\hat{t}e)|C) \subset E(A|C). \tag{23}$$

By (20), $\lim_{j\to\infty} u^{k_j} = \bar{u}$. Since $t(x)$ is a continuous function, taking account of (21) and (23), we have

$$\bar{u} = \bar{x} - t(\bar{x})e = \hat{x} - t(\hat{x})e. \tag{24}$$

If $t(\hat{x}) > t(\bar{x})$ then (24) implies that $\hat{x} - \bar{x} = (t(\hat{x}) - t(\bar{x}))e \in \text{int}C$. Hence $\hat{x} \notin E(A|C)$, contrary to (23). Similarly, if $t(\hat{x}) < t(\bar{x})$ then (24) implies that $\bar{x} \notin E(A|C)$, which is impossible. Thus $t(\hat{x}) = t(\bar{x})$. From (24) it follows that $\hat{x} = \bar{x}$. Since $\bar{x}$ is the limit of any converging subsequence of the bounded sequence $\{x^k\}$, we can deduce that the sequence $\{x^k\}$ converges to $\bar{x}$. We have thus proved that the map Π_e^{-1} in (19) is continuous at any point $\bar{u} \in \Pi_e(E(A|C))$.

From what has been said we conclude that the map $\Pi_e(\cdot)$ in (17) is a homeomorphism. From Lemma 2.2 it follows that $\Pi_e(E(A|C))$ is arcwise connected. Combining these with the obvious facts that $\Pi_e(E(A|C)) \subset H$ and $\dim H = 1$, we deduce that $E(A|C)$ is homeomorphic to $\Pi_e(E(A|C))$, which is a 0-dimensional or an 1-dimensional polyhedral convex set.

For proving the second assertion about the topology of $E^w(A|C)$, we set $C' = \text{int}C \cup \{0\}$. Since $\text{int}C' = \text{int}C$ and $\overline{C'} = \overline{C}$, we deduce that $\text{int}C' \neq \emptyset$ and A has compact sections w.r.t. the pointed convex cone $\overline{C'}$. Applying the first assertion of the theorem to the pair (A, C') instead of (A, C), we conclude that $E(A|C')$ is homeomorphic to a 0-dimensional or an 1-dimensional polyhedral convex set. The proof is complete because $E^w(A|C) = E(A|C')$ by definition. ∎

4. Some Examples and Remarks

In Example 1.1 we have dealt with a set $A \subset \mathbb{R}^2$ for which the set of the weakly efficient points coincides with the set of the efficient points and both of them are homeomorphic to the 1-dimensional polyhedral convex set $[0, +\infty)$ of the real line. It is easy to check that the set A has compact sections w.r.t. the cone R^2_+. Thus Theorem 1.2 can be applied to this example.

To see the variety of convex sets and convex cones to which Theorem 1.2 can be applied, we consider several additional examples. For simplicity, in the following examples we always take

$$C = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 > 0\} \cup \{(0, 0)\}.$$

Note that all the sets A under consideration have compact sections w.r.t. $\overline{C} = \mathbb{R}^2_+$.

Example 4.1. Let $A = \left\{x = (x_1, x_2) \in R^2 : x_2 \geq \frac{1}{x_1 - 1} + 1,\ x_1 > 1\right\}$. Then

$$E^w(A|C) = E(A|C) = \Big\{x = (x_1, x_2) : x_2 = \frac{1}{x_1 - 1} + 1,\ x_1 > 1\Big\},$$

and both the efficient sets are homeomorphic to the 1-dimensional polyhedral convex set $(-\infty, +\infty)$ of the real line.

Example 4.2. Let $A = \Big\{x \in R^2 : 2 > x_1 \geq 1,\ x_2 \geq \frac{2}{x_1}\Big\} \cup \{x \in R^2 : x_1 \geq 2,\ x_2 \geq x_1\}$. We have

$$E^w(A|C) =$$
$$\{x : x_1 = 1,\ x_2 \geq 2\} \cup \Big\{x : 2 > x_1 > 1, x_2 = \frac{2}{x_1}\} \cup \{x : x_1 \geq 2, x_2 = 1\Big\},$$
$$E(A|C) = \Big\{x : 2 > x_1 \geq 1, x_2 = \frac{2}{x_1}\Big\} \cup \{x : x_1 \geq 2, x_1 = 1\}.$$

So $E^w(A|C)$ (resp., $E(A|C)$ is homeomorphic to the interval $(-\infty, +\infty)$ (resp., the closed interval $[0, +\infty)$) of the real line.

Example 4.3. Let $A = \{x \in \mathbb{R}^2 : 2x_2 \geq x_1 \geq x_2\}$. We have $E^w(A|C) = E(A|C) = \{(0, 0)\}$, so both the efficient sets are homeomorphic to the 0-dimensional polyhedral convex set $\{0\}$ of the real line.

Example 4.4. Let $A = \{x \in R^2 : x_2 = -x_1 + 1\}$. Then the efficient sets $E^w(A|C)$ and $E(A|C)$ coincide with A, which is homeomorphic to the real line.

Example 4.5. Let $A = \{x \in R^2 : x_1 \geq 1,\ x_2 \geq 1\}$. We have

$$E^w(A|C) = \{x : x_1 = 1, x_2 \geq 1\} \cup \{x : x_1 \geq 1, x_2 = 1\},$$
$$E(A|C) = \{x : x_1 \geq 1, x_2 = 1\}.$$

Thus $E^w(A|C)$ (resp., $E(A|C)$ is homeomorphic to the interval $(-\infty, +\infty)$ (resp., the closed interval $[0, +\infty)$) of the real line.

In [3], a result on the topology of the efficient set $E(A|C)$ of a (possibly unbounded) set $A \subset \mathbb{R}^2$ w.r.t. a pointed, closed, convex cone $D \subset \mathbb{R}^2$ was given. Actually, a "vector maximization problem" is considered in [3], while a "vector minimization problem" is considered in our paper. The efficient set in [3] is composed of those $x \in A$ such that there exists no $y \in A \setminus \{x\}$ satisfying $y \geq_D x$ (i.e., $y - x \in D$). It is clear that the set coincides with the set $E(A|C)$, where $C := -D$, defined as in Definition 1.2. In what follows, all the material taken from [3] will be adapted to our notation.

Definition 4.1. (See [3, Definition 2.3]) *It is said that A is D-convex, D-closed if $A + D$ is convex, closed, respectively. If there is a bounded set Δ such that $A + D \subset \Delta + D$, then A is called D-bounded. If A is D-closed and D-bounded, then it is called D-compact.*

Consider the set A defined in Example 4.1 and choose $D = \mathbb{R}^2_+$. It is easy to check that A is D-convex and D-closed. In order to show that A is D-bounded, one can choose $\Delta = \{(1,1)\}$ and observe that $A + D \subset \Delta + D$. Therefore A is a D-convex, D-compact set.

The following statement will clarify the relation between the notions "D-compact sets" and "sets having compact sections w.r.t. D".

Proposition 4.1. *Suppose that $D \subset \mathbb{R}^n$ is a nonempty, pointed, closed, convex cone; $A \subset R^n$ a closed set. If A is D-compact then A has compact sections with respect to D.*

Proof. Let $\Delta \subset \mathbb{R}^n$ be a bounded set satisfying

$$A + D \subset \Delta + D. \tag{25}$$

Given any $x \in A + D$, we have to show that $S(x) := (x - D) \cap A$ is compact. Since A and D are assumed closed, we see that $S(x)$ is closed. It remains to prove that $S(x)$ is bounded. From (25) it follows that

$$S(x) = (x - D) \cap A \subset (x - D) \cap (\Delta + D). \tag{26}$$

If $S(x)$ is unbounded then there exists an unbounded sequence $\{y^k\} \subset S(x)$. By (26), there are some sequences $\{d^k\} \subset D$, $\{u^k\} \subset \Delta$, and $\{\tilde{d}^k\} \subset D$, such that

$$y^k = x - d^k = u^k + \tilde{d}^k \quad (\forall k = 1, 2, ...). \tag{27}$$

From the boundedness of Δ we deduce that the sequence $\{u^k\}$ is bounded. Since $\{y^k\}$ is unbounded, $\{d^k\}$ must be unbounded. Without loss of generality we can assume that $\|d^k\| \to +\infty$ and $d^k/\|d^k\| \to v$ for some $v \in R^n$, $\|v\| = 1$. Dividing both sides of the second equality in (27) by $\|d^k\|$ and taking limit as $k \to \infty$, we obtain

$$-v = \lim_{k \to \infty} \frac{\tilde{d}^k}{\|d^k\|}. \tag{28}$$

Since $\tilde{d}^k/\|d^k\| \in D$ for every k and since D is closed, from (28) it follows that $-v = \tilde{v}$ for some $\tilde{v} \in D$. The last inequality contradicts the assumption that D is a pointed cone. We have thus shown that $S(x)$ is bounded. The proof is complete. ∎

Note that the sets A defined in Examples 1.1 and 4.4 have compact sections w.r.t. $D = \mathbb{R}^2_+$, but they are not D-compact. Combining this fact with Proposition 4.1 we can conclude that, in general, the class of closed subsets of $\mathbb{R}^n$ having compact sections w.r.t. a pointed, closed, convex cone D is strictly larger than the class of closed subsets of $\mathbb{R}^n$ which are D-compact in the sense of Definition 4.1.

Note that if D is a closed, convex cone and there exists a compact convex set Δ such that $A + D = \Delta + D$ then A is a D-convex, D-compact set.

Theorem 4.1. ([3, Theorem 3.1]) *Suppose that $D \subset \mathbb{R}^2$ is a pointed, closed, convex cone; $A \subset \mathbb{R}^2$ a nonempty set such that there exists a compact convex* *set* $\Delta \subset \mathbb{R}^2$ *such that* $A + D = \Delta + D$. *Then* $E(A|D)$ *is homeomorphic to a* 0*-dimensional or an* 1*-dimensional simplex.*

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