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On Unique Range Sets for P-Adic Holomorphic Maps

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Abstract. The purpose of this paper is to give a uniqueness result for holomorphic maps in terms of partial multiplicities.

1. Introduction

In 1926, Nevanlinna proved that two non-constant meromorphic functions of one complex variable which attain same five distinct values at the same points, must be identical.

It is observed that p-adic entire functions of one complex variable behave in many ways more like polynomials than entire functions. In 1971, Adams and Straus [1] proved the following theorem.

Theorem A. *Let f, g be two non-constant p-adic entire functions such that for two distinct (finite) values a, b we have $f(x) = a \Leftrightarrow g(x) = a$ and $f(x) = b \Leftrightarrow g(x) = b$. Then $f \equiv g$.*

For p-adic meromorphic functions, Adams and Straus [1] and Hu-Yang [4] obtained the following result similar to Nevanlinna's.

Theorem B. *Let f, g be two non-constant p-adic meromorphic functions such that for four distinct values a_1, a_2, a_3, a_4 we have $f(x) = a_i \Leftrightarrow g(x) = a_i, i = 1, 2, 3, 4$. Then $f \equiv g$.*

Ru [7] extended Theorem B to p-adic holomorphic curves. In this note we prove a similar theorem for the case of holomorphic maps from $\mathbb{C}_p^m$ to $\mathbb{P}^n(\mathbb{C}_p)$.

2. Height of P-Adic Holomorphic Functions of Several Variables

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers and $\mathbb{C}_p$ the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{Q}_p$ is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on $\mathbb{C}_p$ which extends ord_p.

We use the notations

$$\begin{aligned}
b_{(m)} &= (b_1, ..., b_m), \quad b_i(b) = (b_1, ..., b_{i-1}, b, b_{i+1}, ..., b_m),\\
\widehat{(b_i)} &= (b_1, ..., b_{i-1}, b_{i+1}, ..., b_m),\\
D_r &= \big\{z \in \mathbb{C}_p : |z| \le r, r > 0\big\},\\
D_{r_{(m)}} &= D_{r_1} \times \cdots \times D_{r_m}, \quad \text{where} \quad r_{(m)} = (r_1, \ldots, r_m) \text{ for } r_i \in \mathbb{R}_+,\\
D_{<r_{(m)}>} &= D_{<r_1>} \times \cdots \times D_{<r_m>}, \gamma_i \in \mathbb{N}, \gamma = (\gamma_1, ..., \gamma_m), \ |\gamma| = \gamma_1 + \cdots + \gamma_m,\\
z^\gamma &= z_1^{\gamma_1} ... z_m^{\gamma_m}, \quad r^\gamma = r_1^{\gamma_1} ... r_m^{\gamma_m}, \ t_i = v(r_i), \quad i = 1, ..., m.
\end{aligned}$$

Notice that the set of $(r_1, ..., r_m) \in \mathbb{R}_+^m$ such that there exist $x_1, ..., x_m \in \mathbb{C}_p$ with $|x_i| = r_i, i = 1, ..., m$, is dense in $\mathbb{R}_+^m$. Therefore, without loss of generality one can assume that $D_{<r_{(m)}>} \ne \emptyset$.

Let f be a non-zero holomorphic function in $D_{r_{(m)}}$ represented by a convergent series

$$f = \sum_{|\gamma| \ge 0} a_\gamma z^\gamma, \quad |z_i| \le r_i \ \ for \ \ i = 1, \ldots, m.$$

Set $\gamma t = \gamma_1 t_1 + \cdots + \gamma_m t_m$. Then we have

$$\lim_{|\gamma| \to \infty} (v(a_\gamma) + \gamma t) = + \infty.$$

Hence, there exists a $(\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m$ such that $v(a_\gamma) + \gamma t$ is minimal.

Definition 2.1. *The height of the function $f(z_{(m)})$ is defined by*

$$H_f(t_{(m)}) = \min_{0 \le |\gamma| < \infty} (v(a_\gamma) + \gamma t).$$

We also use the notation

$$H_f^+(t_{(m)}) = -H_f(t_{(m)}).$$

Set

$$\begin{aligned}
I_f(t_{(m)}) &= \Big\{(\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m : v(a_\gamma) + \gamma t = H_f(t_{(m)})\Big\},\\
n_{i,f}(0, r_{(m)}) &= \max\Big\{\gamma_i : \exists \ (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(t_{(m)})\Big\}.
\end{aligned}$$

For each $u = (u_1, ..., u_m) \in D_{r_{(m)}}$, set

$$f_{i,u}(z) = f(u_1, ..., u_{i-1}, z_i, u_{i+1}, ... u_m), z = z_i \in D_{r_i}.$$

Notice that, there always exists $u \in D_{r_{(m)}}$ such that $n_{i,f}(0, r_{(m)})$ is equal to the number of zeros with absolute value $\leq r_i$ of the one - variable function $f_{i,u}(z)$ (see [2, Theorem 3.1]).

For a an element of $\mathbb{C}_p$ and f a holomorphic function on $D_{r_{(m)}}$, which is not identically equal to a, define

$$n_{i,f}(a, r_{(m)}) = n_{i,f-a}(0, r_{(m)}), \quad i = 1, \dots, m.$$

Fix real numbers $\rho_1, \dots, \rho_m$ with $0 < \rho_i \leq r_i,\ i = 1, \dots, m$. For each $x \in \mathbb{R}$, set

$$A_i(x) = (\rho_1, \dots, \rho_{i-1}, x, r_{i+1}, \dots, r_m),\ i = 1, \dots, m.$$

Define *the counting function* $N_f(a, t_{(m)})$ by

$$N_f(a, t_{(m)}) = \frac{1}{\ln p} \sum_{k=1}^{m} \int_{\rho_k}^{r_k} \frac{n_{k,f}(a, A_k(x))}{x} dx.$$

If $a = 0$, then set $N_f(t_{(m)}) = N_f(0, t_{(m)})$.

Theorem 2.2.

1) *Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Then*

$$H_f^+(t_{(m)}) - H_f^+(c_{(m)}) = N_f(t_{(m)}),$$

and for each $i = 1, 2, ..., m$, there exists a set $\mathcal{U}^i_{f,A_i(r_i)}$ such that if $u^i \in \mathcal{U}^i_{f,A_i(r_i)}$, then

$$n_{f_{i,u^i}}(0, x) = n_{i,f}(0, A_i(x)),\ \rho_i \leq x \leq r_i.$$

2) *Let $f_s(z_{(m)}), s = 1, 2, \dots, q$, be q non-zero holomorphic functions on $D_{r_{(m)}}$. Then for each $i = 1, 2, ..., m$, there exists $u^i \in \mathcal{U}^i_{f_s,A_i(r_i)}$ for all $s = 1, 2, \dots, q$.*

The proof of Theorem 2.2 follows immediately from [2, Theorem 3.1 and Theorem 3.2].
Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Then there exists a set $\mathcal{U}^i_{f,A_i(r_i)}, i = 1, ..., m$, as in the statement of Theorem 2.2.
Set

$$v = (u^1, \dots, u^m), u^i \in \mathcal{U}^i_{f,A_i(r_i)},$$
$$N_{f_v}(t_{(m)}) = N_{f_{1,u^1}}(t_1) + \cdots + N_{f_{m,u^m}}(t_m),$$
$$V = \{v : N_{f_v}(t_{(m)}) = N_f(t_{(m)})\}.$$

By Theorem 2.2, V is a non-empty set,

$$\begin{aligned} N_{f_v}(t_{(m)}) = &\sum\nolimits_{c_1 > v(a) \geq t_1}^{1} (v(a) - t_1) + n_{f_{1,u^1}}(0, \rho_1)(c_1 - t_1) \\ &+ \cdots + \sum\nolimits_{c_m > v(a) \geq t_m}^{m} (v(a) - t_m) + n_{f_m,u^m}(0, \rho_m)(c_m - t_m), \end{aligned} \tag{2.1}$$

where $c_i = v(\rho_i)$ and

$$\sum\nolimits_{c_i > v(a) \geq t_i}^{i} (v(a) - t_i)$$

is taken on all of zeros a of f_{i,u^i} (counting multiplicity) with $c_i > v(a) \geq t_i, i = 1, 2, ..., m$.

Definition 2.3. *For every positive integer k denote by $N_{k,f_v}(t_{(m)})$ the sum (2.1), where every zero a of functions f_{i,u^i} for all $i = 1, \ldots, m$, is counted with multiplicity if its multiplicity is less than or equal to k, and k times otherwise. Notice that, there always exists $v \in V$ such that $N_{k,f_v}(t_{(m)})$ is maximum. Set*

$$N_{k,f}(t_{(m)}) = \max_{v \in V} N_{k,f_v}(t_{(m)}).$$

We call $N_{k,f}(t_{(m)})$ the k-truncated counting function of f.

Let f be a non-zero holomorphic function on $D_{r_{(m)}}$ and take $k, \ell \in \mathbb{N}^*$. Write

$$\begin{aligned} N_{k,f}(t_{(m)}) = & \sum\nolimits_{c_1 > v(a) \geq t_1}^{1} (v(a) - t_1) + n_{f_{1,u^1}}(0, \rho_1)(c_1 - t_1) \\ & + \cdots + \sum\nolimits_{c_m > v(a) \geq t_m}^{m} (v(a) - t_m) + n_{f_{m,u^m}}(c_m - t_m), \end{aligned} \tag{2.2}$$

where every zero a of functions f_{i,u^i} for $i = 1, \ldots, m$, is counted with multiplicity as in the statement of Definition 2.3.

By $N_{k,f}^{\leq \ell}(t_{(m)})$ (resp. $N_{k,f}^{>\ell}(t_{(m)})$) we denote the sum taken over all of zeros a with multiplicity less than or equal to ℓ (resp. at least $\ell + 1$). Then

$$N_{k,f}(t_{(m)}) = N_{k,f}^{\leq \ell}(t_{(m)}) + N_{k,f}^{>\ell}(t_{(m)}).$$

Let f be a non-zero holomorphic function on $D_{r_{(m)}}, a = (a_1, \ldots, a_m) \in D_{r_{(m)}}$, and let f represented by a convergent series

$$f(z_{(m)}) = \sum_{|\gamma|=0}^{\infty} a_\gamma (z_1 - a_1)^{\gamma_1} \ldots (z_m - a_m)^{\gamma_m}, \quad z_{(m)} \in D_{r_{(m)}}.$$

Set

$$v_f(a) = \min \ \{|\gamma| : a_\gamma \neq 0\}.$$

For each $i = 1, 2, \ldots, m$, write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}(\widehat{z_i - a_i})(z_i - a_i)^k.$$

Set

$g_{i,k}(z_1, ..., z_{i-1}, z_{i+1}, ..., z_m) = f_{i,k}(\widehat{z_i - a_i})$, $b_{i,k} = g_{i,k}(a_1, ..., a_{i-1}, a_{i+1}, ..., a_m)$.

Then $f_{i,a}(z) = \sum\limits_{k=0}^{\infty} b_{i,k}(z_i - a_i)^k$. Set

$$v_{i,f}(a) = \begin{cases} \min \ \{k : b_{i,k} \neq 0\} & \text{if } f_{i,a}(z) \not\equiv 0 \\ +\infty & \text{if } f_{i,a}(z) \equiv 0, \end{cases}$$

$$\text{ord}_{i,f}(a) = \begin{cases} \min \ \{k : g_{i,k}\widehat{(z_i)} \not\equiv 0, k \neq 0\} \\ +\infty & \text{if } g_{i,k}\widehat{(z_i)} \equiv 0 \text{ for all } k \neq 0. \end{cases}$$

If $f(a) = 0$, then a (resp.,a_i) is a zero of $f(z_{(m)})$ (resp.,$f_{i,a}(z)$). Then the numbers $v_f(a), v_{i,f}(a), \text{ord}_{i,f}(a)$ are called *multiplicity, i-th partial multiplicity, i-th partial order,* respectively, of a.

3. Uniqueness Problems Without Counting Multiplicity in Several Variables

By a *holomorphic map*

$$f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n(\mathbb{C}_p) = \mathbb{P}^n,$$

we mean an equivalence class of $(n+1)$-tuples of entire functions $(f_1, \dots, f_{n+1})$ such that $f_1, \dots, f_{n+1}$ do not have any common factor in the ring of entire functions on $\mathbb{C}_p^m$, where two $(n+1)$-tuples $(f_1, \dots, f_{n+1})$ and $(g_1, \dots, g_{n+1})$ are equivalent if there exists a constant c such that $f_i = cg_i$ for all i. We identify f with its representation by a collection of entire functions on $\mathbb{C}_p^m$

$$f = (f_1, \dots, f_{n+1}).$$

Definition 3.1. *The height of a holomorphic map f is defined by*

$$H_f(t_{(m)}) = \min_{1 \leq i \leq n+1} H_{f_i}(t_{(m)}).$$

We also use the notation

$$H_f^+(t_{(m)}) = -H_f(t_{(m)}).$$

Notice that $H_f(t_{(m)})$ is well defined up to an additive constant.

Hyperplanes $H_1, \dots, H_q$ in $\mathbb{P}^n, q \geq n+1$ are said to be in *general position* if any $n+1$ of them are linearly independent. Let H be a hyperplane of $\mathbb{P}^n$ such that the image of f is not contained in H, and H is defined by the equation $F = 0$.
We set

$$N_f(H, t_{(m)}) = N_{F\circ f}(t_{(m)}), H_f(H, t_{(m)}) = H_{F\circ f}(t_{(m)}, H_f^+(H, t_{(m)}) = -H_f(H, t_{(m)}),$$
$$\overline{E}_f(H) = \{z_{(m)} \in \mathbb{C}_p^m : F \circ f(z_{(m)}) = 0 \ \text{ignoring multiplicities}\}.$$

For a positive integer k, define the set

$$\overline{E}_f(H, k) = \{z_{(m)} \in \mathbb{C}_p^m : F \circ f(z_{(m)}) = 0 \text{ ignoring multiplicities}, v_{i,F\circ f}(z_{(m)}) \leq k \text{ for } i \in \{1, \dots, m\} \text{ such that } v_{i,F\circ f}(z_{(m)}) \neq \infty\}.$$

A holomorphic map $f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ is called *linearly non-degenerate* if the image of f is not contained in any hyperplane of $\mathbb{P}^n$.

Theorem 3.2. *Let H_i be hyperplanes in general position in $\mathbb{P}^n$, defined by the equations $F_i = 0$ and let $k_i \in \mathbb{N}^*, i = 1, \dots, q$. Let $f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be a linearly non-degenerate holomorphic map . Suppose that every zero of $F_i \circ f$ $(i = 1, \dots, q)$ satisfies the conditions that either all its partial multiplicities $(\neq \infty)$ are less than or equal to n, or all its partial orders are at least $n+1$. Then*

$$\Big(\sum_{i=1}^{q} \frac{k_i}{k_i+1} - n - 1\Big) H_f^+(t_{(m)}) \leq \sum_{i=1}^{q} \frac{k_i}{k_i+1} N_{n,f}^{\leq k_i}(H_i, t_{(m)}) + BT + 0(1),$$

where $0(1)$ *is bounded when* $T = \max_{1 \leq i \leq m} t_i \to -\infty$ *and* $n \leq B \leq \dfrac{n(n+1)}{2}$.

The following theorem generalizes Theorem B.

Theorem 3.3. *Let $f = (f_1, \dots, f_{n+1}), g = (g_1, \dots, g_{n+1}) : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be two linearly non-degenerate holomorphic maps. Let H_i be hyperplanes in general position in $\mathbb{P}^n$, defined by the equations $F_i = 0$. Let $k_i \in \mathbb{N}^*, i = 1, \dots, q$ with $k_1 \geq k_2 \geq \dots \geq k_q$,*

$$\sum_{i=2n^2+1}^{q} \frac{k_i}{k_i+1} \geq n+1. \tag{3.9}$$

Assume that every zero of $F_i \circ f$, $F_i \circ g$ $(i = 1, \dots, q)$ satisfies the conditions that either all its partial multiplicities $(\neq \infty)$ are less than or equal to n, or all its partial orders are at least $n+1$. Moreover let $\overline{E}_f(H_i, k_i) = \overline{E}_g(H_i, k_i), i = 1, 2, \dots, q$, and $f(z_{(m)}) = g(z_{(m)})$ for every point $z_{(m)} \in \bigcup_{i=1}^{q} \overline{E}_f(H_i, k_i)$. Then $f \equiv g$.

Theorem 3.3 gives the following

Corollary 3.4. *Let $f = (f_1, \dots, f_{n+1}), g = (g_1, \dots, g_{n+1}) : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be two linearly non-degenerate holomorphic maps. Let H_i be hyperplanes in general position in $\mathbb{P}^n$, defined by the equations $F_i = 0$, and let $q \geq 2n^2 + n + 1$. Assume that every zero of $F_i \circ f$, $F_i \circ g$ $(i = 1, \dots, q)$ satisfies the conditions that either all its partial multiplicities $(\neq \infty)$ are less than or equal to n, or all its partial orders are at least $n+1$ and $\overline{E}_f(H_i) = \overline{E}_g(H_i)$, $i = 1, 2, \dots, q$, and $f(z_{(m)}) = g(z_{(m)})$ for every point $z_{(m)} \in \bigcup_{i=1}^{q} \overline{E}_f(H_i)$. Then $f \equiv g$.*

Corollary 3.5. *Let $f = (f_1, \dots, f_{n+1}), g = (g_1, \dots, g_{n+1}) : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be two linearly non-degenerate holomorphic maps. Let H_i be hyperplanes in general position in $\mathbb{P}^n$, defined by the equations $F_i = 0$, and let $k_i \in \mathbb{N}^*, i = 1, \dots, q$, with $k_1 \geq k_2 \geq \dots \geq k_q$,*

$$\sum_{i=2n+1}^{q} \frac{k_i}{k_i+1} \geq n+1.$$

Assume that every zero of $F_i \circ f$, $F_i \circ g$ $(i = 1, \dots, q)$ satisfies the conditions that either all its partial multiplicities ($\neq \infty$) are less than or equal to n, or all its partial orders are at least $n+1$ and $\overline{E}_f(H_i, k_i) = \overline{E}_g(H_i, k_i), i = 1, 2, \dots, q$ and for each $i \neq j, \overline{E}_f(H_i, k_i) \cap \overline{E}_f(H_j, k_j) = \emptyset$, and $f(z_{(m)}) = g(z_{(m)})$ for every point $z_{(m)} \in \bigcup_{i=1}^{q} \overline{E}_f(H_i, k_i)$. Then $f \equiv g$.

Notice that, take $k_1 = k_2 = \cdots = k_q = k$ and for $k \longrightarrow \infty$, $m = 1$, from Corollary 3.5, we obtain the uniqueness theorem for p-adic holomorphic curves of Min Ru [7].

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