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Submanifolds with Constant Scalar Curvature

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Abstract. Let M^n be a compact orientable submanifold in a Riemannian manifold of constant sectional curvature with flat normal bundle. This paper gives some intrinsic conditions for M^n to be totally umbilical or a local product of several totally umbilical submanifolds.

1. Introduction

A theorem of Nomizu–Smyth [4] states that if M is a nonnegatively curved compact submanifold with constant mean curvature in the Euclidean space, or the Euclidean sphere, then it is the standard sphere or the Riemannian product of two spheres. Later Yano and Ishihara [6], Smyth [5] and Yau [7] extended this result to submanifolds, independently. On the other hand, Cheng and Yau [2] studied compact hypersurfaces with constant scalar curvature and obtained a similar result. Recently, Zheng [8] extended the Cheng-Yau's result to compact submanifolds and proved the following: Let M^n be an n-dimensional compact orientable hypersurface with constant scalar curvature and nonnegative sectional curvature in a real space form of constant sectional curvature c. Suppose that M^n has flat normal bundle, if the normalized scalar curvature of M^n is greater than c, then M^n is either totally umbilical, or locally the Riemannian product of several totally umbilical constantly curved submanifolds.

Compare Zheng's result with Cheng-Yau's, we conjecture that the condition the normalized scalar curvature of M^n is greater than c can be changed to that the normalized scalar curvature of M^n is greater than or equal to c. In this paper we confirm this conjecture, i.e. we prove the following

Theorem A. *Let M^n be an n-dimensional compact orientable submanifold with* *constant scalar curvature and with nonnegative sectional curvature immersed in a Riemannian manifold of constant sectional curvature c. Suppose that M^n has flat normal bundle, if the normalized scalar curvature of M^n is greater than or equal to c. Then M^n is either totally umbilical, or locally the Riemannian product of several totally umbilical constantly curved submanifolds.*

Also we will use Cheng-Yau's technique to prove the following theorem

Theorem B. *Let M^n be an n-dimensional compact orientable submanifold with nonnegative sectional curvature immersed in a Riemannian manifold of constant sectional curvature c. Suppose that M^n has flat normal bundle, if the normalized scalar curvature R of M^n is proportional to the mean curvature H of M^n, that is*

$$R = aH, \quad a^2 > \frac{4nc}{n-1},$$

where a is a constant. Then M^n is either totally umbilical, or locally the Riemannian product of several totally umbilical constantly curved submanifolds.

2. Preliminaries

Let M^n be an n-dimensional submanifold immersed in an $(n+p)$-dimensional Riemannian manifold $M^{n+p}(c)$ of constant curvature c. We choose a local field of orthonormal frames $e_1, \ldots, e_{n+p}$ in $M^{n+p}(c)$ such that at each point of M^n, $e_1, \ldots, e_n$ span the tangent space of M^n and form an orthonormal frame there. Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field. In this paper, we use the following convention on the range of indices:

$$1 \leq i, j, k, \ldots \leq n; \quad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

Then we have the equation of Gauss

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\alpha}(h^{\alpha}_{ik}h^{\alpha}_{jl} - h^{\alpha}_{il}h^{\alpha}_{jk}), \tag{1}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{\alpha} h_{\alpha}e_{\alpha} = \sum_{i,j,\alpha} h^{\alpha}_{ij}\omega_i \otimes \omega_j \otimes e_{\alpha} \tag{2}$$

is the second fundamental form of M^n and the square length of the second fundamental form is defined by

$$S = \sum_{\alpha,i,j}(h^{\alpha}_{ij})^2 = ||h||^2. \tag{3}$$

From (1) we have

$$n^2H^2 - \|h\|^2 = n(n-1)(R-c), \tag{4}$$

where R is the normalized scalar curvature of M^n.

The mean curvature vector N of M^n is defined by

$$N = \frac{1}{n} \sum_{\alpha} \text{ tr } (h_\alpha) e_\alpha = \frac{1}{n} \sum_{\alpha} (\sum_i h_{ii}^\alpha) e_\alpha, \tag{5}$$

and it is well known that N is independent of the choice of unit normal vectors $e_{n+1}, \dots, e_{n+p}$ to M^n. The length of the mean curvature vector is called the mean curvature of M^n, denoted by H. From now on we impose the following choice in our basis: if $N \neq 0$, we choose the first unit normal vector e_{n+1} to M^n in the direction N. Therefore we have

$$H = \frac{1}{n} \text{ tr } h_{n+1} = \frac{1}{n} \sum_i h_{ii}^{n+1} \geq 0, \tag{6}$$

$$\text{tr } h_\alpha = \sum_i h_{ii}^\alpha = 0, \quad \alpha = n+2, \dots, n+p. \tag{7}$$

If there exist p functions ρ_α such that $h_{ij}^\alpha = \rho_\alpha \delta_{ij}$ at each point of M^n, we call M^n a totally umbilical submanifold. For a totally umbilical submanifold, we have

$$\rho_\alpha = \frac{1}{n} \text{ tr } h_\alpha = \frac{1}{n} \sum_i h_{ii}^\alpha. \tag{8}$$

Let h_{ijk}^α and h_{ijkl}^α denote the covariant derivative and the second covariant derivative of h_{ij}^α, respectively. Then we have

$$h_{ijk}^\alpha = h_{ikj}^\alpha, \tag{9}$$

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{jm}^\alpha R_{mikl} + \sum_\beta h_{ij}^\beta R_{\alpha\beta kl}, \tag{10}$$

where $R_{\alpha\beta kl}$ are the components of the normal curvature tensor of M^n, that is

$$R_{\alpha\beta kl} = \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta). \tag{11}$$

If $R_{\alpha\beta kl} = 0$ at point x of M^n we say that the normal connection of M^n is flat at x and it is well known [1] that $R_{\alpha\beta kl} = 0$ at x if and only if h_α are simultaneously diagonalizable at x.

For later use, we denote $S_\alpha = \sum_{i,j} (h_{ij}^\alpha)^2$ and $S_I = \sum_{\beta > n+1} S_\beta$, so $S = S_{n+1} + S_I$.

In codimension one case, Cheng-Yau [2] gave a lower estimation for $|\nabla\sigma|^2$, the square of the length of the covariant derivative of σ. They proved that, for a hypersurface in a space form of constant scalar curvature c, if the normalized scalar curvature R is constant and $R \geq c$, then $|\nabla\sigma|^2 \geq n^2 |\nabla H|^2$.

In higher codimension cases, we have the following

Lemma 2.1. *Let M^n be a connected submanifold in $M^{n+p}(c)$ with nowhere zero mean curvature H. If R is constant and $R \geq c$, then*

$$|\nabla\sigma|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \geq n^2 |\nabla H|^2. \tag{12}$$

Moreover,
(i) *when $R-c>0$, if the equality in (12) holds on M^n, then H is constant.*
(ii) *when $R-c=0$, if the equality in (12) holds on M^n, then either H is constant or M^n lies in a totally geodesic subspace $M^{n+1}(c)$ of $M^{n+p}(c)$.*

Proof. From the assumption that $R \geq c$, we have $n^2H^2-S=n(n-1)(R-c) \geq 0$. Taking the covariant derivative on both sides of this equality, we get

$$n^2 H\, H_k = \sum_{i,j,\alpha} h_{ij}^\alpha\, h_{ijk}^\alpha, \quad k=1,\cdots,n.$$

For every k, it follows from Cauchy-Schwarz's inequality that

$$n^4H^2H_k^2 = \Big(\sum_{i,j,\alpha} h_{ij}^\alpha\, h_{ijk}^\alpha\Big)^2 \leq S\sum_{i,j,\alpha}(h_{ijk}^\alpha)^2, \tag{13}$$

where the equality holds if and only if there exists a real function c_k such that

$$h_{ijk}^\alpha = c_k\, h_{ij}^\alpha \tag{14}$$

for all i,j and α. Taking sum on both sides of (13) with respect to k, we have

$$n^4H^2|\nabla H|^2 = n^4H^2\sum_k H_k^2 \leq S\sum_{(i,j,k,\alpha)}(h_{ijk}^\alpha)^2 \leq n^2H^2\sum_{(i,j,k,\alpha)}(h_{ijk}^\alpha)^2. \tag{15}$$

Therefore (12) holds on M^n.

Suppose that $\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2 = n^2|\nabla H|^2$ holds on M^n. It follows from (15) that

$$0 \leq n^3(n-1)(R-c)|\nabla H|^2 \leq S\Big(\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2 - n^2|\nabla H|^2\Big). \tag{16}$$

Hence $(R-c)|\nabla H|^2=0$ on M^n. If $R>c$, then $|\nabla H|^2=0$ on M^n, that is H is a constant.

If $R-c=0$, then $S=n^2H^2$ on M^n. In this case, the equality in (13) holds for all k. From (14) we have

$$h_{ijk}^{n+1} = c_k h_{ij}^{n+1}. \tag{17}$$

Taking sum on both sides of (17) with respect to $i=j$, we have

$$H_k = c_k H. \tag{18}$$

Multiplying the both sides of (17) by H and using (18), we have

$$Hh_{ijk}^{n+1} = H_k h_{ij}^{n+1}. \tag{19}$$

Taking sum on both sides of (19) with respect to $j=k$, we have

$$(nH)H_i = H_j h_{ij}^{n+1}. \tag{20}$$

From (19) and the fact $\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2 = n^2|\nabla H|^2$, we have $|\nabla S_I|^2=0$ on M^n. Hence S_I is constant. From (20) and the fact $S=n^2H^2$, we have $S_I|\nabla H|^2=0$

on M^n. Thus if $S_I \neq 0$ on M^n, then $|\nabla H|^2 = 0$ on M^n, that is H is constant. If $S_I = 0$, then M^n lies in a totally geodesic subspace $M^{n+1}(c)$ of $M^{n+p}(c)$. This completes the proof of Lemma 2.1.

3. Proof of Theorem A

We know the Laplacian Δh_{ij}^α of the fundamental form h_{ij}^α is defined to be $\sum_k h_{ijkk}^\alpha$, and hence, using (9), (10) and the assumption that M^n has flat normal bundle, we have

$$\begin{aligned}\Delta h_{ij}^\alpha &= \sum_k (h_{ijkk}^\alpha - h_{ikjk}^\alpha) + \sum_k (h_{ikjk}^\alpha - h_{ikkj}^\alpha) \\ &+ \sum_k (h_{ikkj}^\alpha - h_{kkij}^\alpha) + (\text{tr } h_\alpha)_{ij} \\ &= \sum_{m,k} h_{im}^\alpha R_{mkjk} + \sum_{m,k} h_{mk}^\alpha R_{mijk} + (\text{tr } h_\alpha)_{ij}, \end{aligned} \tag{21}$$

where $(\text{tr } h_\alpha)_{ij}$ denotes the second covariant derivative of $(\text{tr } h_\alpha)$. Since the normal bundle of M^n is flat, we choose $e_1, \dots, e_n$ such that

$$h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}, \quad \alpha = n+1, \dots, n+p. \tag{22}$$

Then the Laplacian of $\|h\|^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2$ is given by

$$\begin{aligned}\frac{1}{2}\Delta\|h\|^2 = \frac{1}{2}n^2 \Delta H^2 &= \|\nabla h\|^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha \\ &= \|\nabla h\|^2 + n\sum_i \lambda_i^{n+1} H_{ii} + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \end{aligned} \tag{23}$$

We define an operator $\Box$ acting on f by

$$\Box f = \sum_{i,j} (nH\delta_{ij} - h_{ij}^{n+1}) f_{ij}. \tag{24}$$

Since $(nH\delta_{ij} - h_{ij}^{n+1})$ is trace-free it follows from [2] that the operator $\Box$ is self-adjoint relative to the L^2-inner product of M^n, i.e.,

$$\int_{M^n} f \cdot \Box g = \int_{M^n} g \cdot \Box f. \tag{25}$$

Thus we have

$$\begin{aligned}\Box H &= \sum_{i,j} (nH\delta_{ij} - h_{ij}^{n+1}) H_{ij} = nH\sum_i H_{ii} - \sum_i \lambda_i^{n+1} H_{ii} \\ &= \frac{1}{2}n(\Delta H^2 - 2\|\nabla H\|^2) - \sum_i \lambda_i^{n+1} H_{ii}. \end{aligned} \tag{26}$$

From (23) and (26), we have

$$n\Box H = \|\nabla h\|^2 - n^2\|\nabla H\|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{27}$$

Since $\Box$ is self-adjoint, we have

$$0 = \int_{M^n} \Big\{\|\nabla h\|^2 - n^2\|\nabla H\|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2\Big\}. \tag{28}$$

Thus, by hypothesis of the theorem and Lemma 2.1, we have either H is constant or $S_I = 0$ on M^n. If H is constant on M^n, then our theorem follows immediately from a result of Yano and Ishihara [6]. Otherwise, if $S_I = 0$ on M^n, then M^n lies in a totally geodesic subspace $M^{n+1}(c)$ of $M^{n+p}(c)$, hence then our theorem follows directly from the result of Cheng-Yau [2]. This completes the proof of Theorem A.

4. Proof of Theorem B

From (4) and the assumption $R = aH$, we have

$$|h|^2 = n^2H^2 + n(n-1)(c - aH). \tag{29}$$

Taking the covariant derivative of (29), we have for each k

$$(2n^2H - n(n-1)a)H_k = \sum_{i,j,\alpha} h_{ij}^\alpha h_{ijk}^\alpha$$

and hence, by Cauchy-Schwarz's inequality, we have

$$(2n^2H - n(n-1)a)^2|\nabla H|^2 \le 4\sum_{i,j,\alpha}(h_{ij}^\alpha)^2 \sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2,$$

that is

$$(2n^2H - n(n-1)a)^2|\nabla H|^2 \le 4|h|^2|\nabla h|^2. \tag{30}$$

From (23), (26), (29) and (30), we have

$$\begin{aligned} n\Box H &= |\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2 \\ &\ge |h|^{-2}\Big[\frac{(2n^2H - n(n-1)a)^2}{4} - n^2|h|^2\Big]|\nabla H|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2 \\ &= |h|^{-2}n^2(n-1)\Big(\frac{(n-1)a^2}{4} - nc\Big)|\nabla H|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2, \end{aligned} \tag{31}$$

Since $\Box$ is self-adjoint, we conclude that

$$0 \ge \int_{M^n} \Big\{|h|^{-2}n^2(n-1)\Big(\frac{(n-1)a^2}{4} - nc\Big)|\nabla H|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2\Big\}. \tag{32}$$

Thus, by hypothesis, $|\nabla H|^2 = 0$, so H is constant on M^n. Therefore our theorem follows immediately from a result of Yano and Ishihara [6, Lemma 2.8] and this completes the proof of the Theorem B.

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