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Weak Holomorphic Extension of Fréchet-Valued Functions from Compact Subsets of $\mathbb{C}^n$

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Abstract. The aim of this paper is to establish the equivalence between the notion of weakly holomorphic extension and that of holomorphic extension of Fréchet-valued functions on compact subsets in $\mathbb{C}^n$.

1. Introduction

Let E, F be locally convex spaces and $X \subset E$ be a subset of E. A function $f : X \longrightarrow F$ is called to be holomorphic on X if f can be extended to a holomorphic function $\hat{f} : U \longrightarrow F$, where U is a neighborhood of X in E.

By $H(X, F)$ we denote the space of F-valued holomorphic functions on X.

A function $f : X \longrightarrow F$ is called to be weakly holomorphic if for every $u \in F'$, the topological dual space of F, the function $uf : X \longrightarrow C$ is holomorphic on X.

By $H_w(X, F)$ we denote the space of F-valued weakly holomorphic functions on X.

For a long time, the problem of finding conditions for X and F such that

$$H_w(X, F) = H(X, F) \tag{1}$$

has been interested by many authors. In the case X is an open subset and F' is a Baire space (1) was first proved by Bogdanowicz [4] and, after that, by Ligocka–Siciak [13] and in the more general case by Van [21]. Next, when X is a compact set of uniqueness in a Banach space, by using the closed graph theorem Waelbroeck proved the equality (1) for the case where F is a Banach space (see [27]). The idea of using the closed graph theorem of Waelbroeck in establishing the equality (1) for Banach-valued functions is a starting point of our investigation when F is a Fréchet space. Next, under the assumption that F' is still a Baire space Khue and Tac [11] have shown that (1) holds in the case either E is a nuclear metric space or F is nuclear. Note that the Baireness of F' plays a very important role in the works of the above authors. The case where F' is not a Baire space, in particular, when F is a Fréchet space which is not Banach, (1) is established by Hai [9].

In [9] Hai proved that (1) holds for every $\widetilde{L}$-regular compact subset in any nuclear Fréchet space E if and only if F is a Fréchet space having property (DN).

Recently, some authors have investigated the equality (1) for Fréchet-valued real analytic functions on open subsets $X \subset \mathbb{R}^n$. The first result in this direction may be found in [7]. In the special case authors have proved the equivalence between weakly real analytic and real analytic extension for Fréchet-valued functions in relation with the linear topological invariant (DN). Recent year Bonet and Domanski [5] completely solved this problem for Fréchet-valued functions also in the relation with the linear topological invariant (DN).

This paper is devoted to establish (1) in the case where X is a compact subset of uniqueness in $\mathbb{C}^n$ in the terms of the property (LB_∞).

Namely, we prove the following

The main theorem. *Let F be a Fréchet space. Then F has property* (LB_∞) *if and only if*

$$H_w(X,F) = H(X,F)$$

for every compact set of uniqueness X in $\mathbb{C}^n$ and for every $n \geq 1$.

The proof of this theorem is relied on the auxiliary Theorem 1 which is of independent interest.

2. Preliminaries

2.1. Köthe Sequence Spaces. Now we introduce the Köthe sequence spaces $\Lambda(B)$.

Let $B = (b_{j,k})_{j,k=1,2,\ldots}$ be a matrix satisfying the following:

(1) $0 \leq b_{j,k} \leq b_{j,k+1}$ for all $j,k \geq 1$,

(2) For each $j \geq 1$ there exists $k \geq 1$ such that $b_{j,k} > 0$,

Then we define the Köthe sequence space $\Lambda(B)$ by:

$$\Lambda(B) = \big\{\xi = (\xi_j) \in \ C^N : \|\xi\|_k = \sum_{j=1}^{\infty} |\xi_j| b_{j,k} < +\infty \quad \text{for all} \quad k \geq 1\big\}.$$

Obviously $\Lambda(B)$ is a Fréchet space under the natural topology induced by the semi-norms $(\|.\|_k)_{k\geq 1}$.

Let $\alpha = (\alpha_n)_{n\geq 1}$ be an unbounded increasing sequence of positive real numbers $\big(\lim\limits_{n\to+\infty} \alpha_n = +\infty\big)$ and $0 < r_k \nearrow R$.

By putting $b_{j,k} = \ r_k^{\alpha_j}$ we define the power series space

$$\Lambda_R(\alpha) = \big\{\xi = (\xi_j) \subset C^N : \|\xi\|_k = \sum_{j=1}^{\infty} |\xi_j| r_k^{\alpha_j} < +\infty \quad \text{for all} \quad k \geq 1\big\}.$$

$\Lambda_R(\alpha)$ is called a power series space. In the case $R = 1$ (resp $R = \infty$), $\Lambda_1(\alpha)$ (resp. $\Lambda_\infty(\alpha)$) is called the power series space of finite (resp. infinite) type.

2.2. For a Fréchet space F we always assume that its topology is defined by an increasing system $(\|.\|_n)_{n\geq 1}$ of semi-norms .
Then by F_n we denote the canonical Banach space associated to the semi-norm $\|.\|_n$. Let $U_n = \{x \in F : \|x\|_n \leq 1\}$.

We consider that $\{U_n\}_{n\geq 1}$ is a decreasing neighborhood basis of $0 \in F$. For $u \in F'$, (the topological dual space of F), we define

$$\begin{aligned} \|u\|_k^* &= \sup\{\ |u(x)| : \|x\|_k \leq 1\} \\ &= \sup\{\ |u(x)| : x \in U_k\}. \end{aligned}$$

In [22] Vogt has introduced and investigated the following linear topological invariants on a Fréchet space F with the topology defined by an increasing sequence of semi-norms $(\|\,.\,\|_k)_{k\geq 1}$. We say that F has the property

$(\overline{\text{DN}})$ if $\exists p\ \forall q\ \exists k\ \forall d > 0\ \exists C > 0$:

$$\|x\|_q^{1+d} \leq C\|x\|_k\ \|x\|_p^d \quad \text{for all} \quad x \in F;$$

(LB_∞) if $\forall\{\rho_n\} \nearrow +\infty\ \exists p\ \forall q\ \exists k(q) \geq q, C > 0\ \forall x \in F\ \exists q \leq m \leq k(q)$:

$$\|x\|_q^{1+\rho_m} \leq C\|x\|_m\ \|x\|_p^{\rho_m}.$$

In [22] Vogt proved that F has the property (LB_∞) if and only if

$$L(\Lambda_\infty(\alpha), F) = \text{LB}(\Lambda_\infty(\alpha), F)$$

for any exponent sequence $\alpha = (\alpha_n)_{n\geq 1}$, where $L(\Lambda_\infty(\alpha), F)$ denotes the space of continuous linear maps from $\Lambda_\infty(\alpha)$ to F and $\text{LB}(\Lambda_\infty(\alpha), F)$ consists of continuous linear maps which are bounded on some neighborhood of $0 \in \Lambda_\infty(\alpha)$.

2.3. Let X be a subset of $\mathbb{C}^n$ and E a locally convex space. A function f defined on X with values in E is called holomorphic on X if it can be extended to a holomorphic function on a neighborhood of X in $\mathbb{C}^n$. If this holds for uf with all $u \in E'$, the topological dual space of E, then we say that f is weakly holomorphic on X.

Let V be an open subset of $\mathbb{C}^n$. We let

$$H^\infty(V) = \{f \in H(V) : \|f\|_V = \sup\{|f(x)| : x \in V\} < \infty\},$$

where $H(V)$ is the space of holomorphic functions on V. $H^\infty(V)$ is a Banach space with the norm $\|\,.\,\|_V$

Let X be a compact subset of $\mathbb{C}^n$. On $\bigcup_{\substack{V\supset X \\ V open}} H^\infty(V)$, we define the equivalence relation $\sim$ as follows: $f \sim g$ if there exists a neighborhood W of X such that $f|_W = g|_W$.

We denote by $H(X)$ the vector space of equivalent classes and the elements of $H(X)$ are called germs of holomorphic functions on X. $H(X)$ is equipped with the inductive limit topology

$$H(X) = \lim \mathrm{ind} H^\infty(V).$$

Now let X be a compact subset of $\mathbb{C}^n$. X is called a compact subset of uniqueness if for every $f \in H(X), f|_X = 0$, then $f = 0$ on some neighborhood of X.

2.4. Let Ω be an open subset of $\mathbb{C}^n$. A set $E \subset \Omega$ is said to be *pluripolar* if for each $a \in E$ there exist a connected neighborhood V of a in Ω and a plurisubharmonic function

$$\varphi : V \longrightarrow [-\infty, +\infty), \quad \varphi \not\equiv -\infty,$$

on V such that $E \cap V \subset \{z \in V : \varphi(z) = -\infty\}$. It has been proved in [12] that any locally pluripolar set $E \subset \Omega$ is globally pluripolar, i.e, there exists a plurisubharmonic function ψ on Ω, $\psi \not\equiv -\infty$ on every connected component of Ω and $E \subset \{\psi = -\infty\}$.

2.5. Let $\Omega \subset \mathbb{C}^n$ be an open subset and $E \subset \Omega$. The relative extremal function associated to the pair (E, Ω) is defined by (see [12])

$$u_{E,\Omega}(z) = \sup\{v(z) : v \in PSH(\Omega), \ v \leq 0 \ \text{ on } \ E, \ v \leq 1 \ \text{ on } \ \Omega\}, \quad z \in \Omega,$$

where $PSH(\Omega)$ denotes the set of plurisubharmonic functions on Ω. We denote by $\omega(z, E, \Omega) = u^*_{E,\Omega}(z)$ the upper regularization of $u_{E,\Omega}(z)$. Then the function $\omega(z, E, \Omega)$ is plurisubharmonic on Ω.

We say that E is pluriregular at a point $a \in \overline{E}$ if $\omega(a, E \cap V, V) = 0$ for every open neighborhood V of a.

Denote by E^* the set of all points $a \in \overline{E} \cap \Omega$ such that E is pluriregular at a.

By a result of Bedford–Taylor [2] if E is non-pluripolar then $E^* \neq \emptyset$ and E^* is non-pluripolar.

2.6. Let Ω be an open subset of $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_s}$ and F be a Fréchet space.

A function $f : \Omega \longrightarrow F$ is called to be *p-separately analytic* $(1 \leq p < s)$ if for every $x^0 = (x^0_1, ..., x^0_s) \in \Omega$ and for every sequence $1 \leq i_1 < ... < i_p \leq s$ the function

$$(x_{i_1}, ..., x_{i_p}) \longrightarrow f(x^0_1, ..., x_{i_1}, ...x_{i_p}, ..., x^0_s)$$

is analytic in a neighborhood of $(x^0_{i_1}, ..., x^0_{i_p})$.
For a p-separately analytic function $f : \Omega \longrightarrow F$, let

$$A(f) = \{x \in \Omega : f \ \text{ is analytic in a neighborhood of } \ x\}$$

denote its analytic set, and $S(f) = \Omega \setminus A(f)$ its singular set.

2.7. Let K, L be compact sets in $\mathbb{C}^n, \mathbb{C}^m$ respectively and F a Fréchet space. Let $f : K \times L \longrightarrow F$ be an arbitrary function. The function f is said to be *separately holomorphic* if the following conditions are satisfied:
(i) For every $x \in K$, the function $f_x : L \longrightarrow F$ is extended holomorphically to a neighborhood V_x of L in $\mathbb{C}^n$, where $f_x(y) = f(x, y)$ for $x \in K, \ y \in L$.

(ii) For every $y \in L$, the function $f_y : K \longrightarrow F$ is given by $f_y(x) = f(x, y)$ for $x \in K$, is extended holomorphically to a neighborhood U_y of K in $\mathbb{C}^n$.

2.8. Let E be a locally convex space and $A \subset E$. By Conv$A$ we denote the balanced convex envelope of A, that means

$$\mathrm{Conv}A = \Big\{ \sum_{i=1}^{n} \lambda_i x_i,\ x_i \in A,\ \sum_{i=1}^{n} |\lambda_i| \leq 1 \Big\}.$$

3. Weakly Holomophic Extension of Fréchet-Valued Functions on Compact Subsets of Uniqueness in $\mathbb{C}^n$

The aim of this section is to establish the equivalence between the notion of weakly holomorphic extension and that of holomorphic extension of Fréchet-valued functions on compact subsets of uniqueness in $\mathbb{C}^n$. In order to obtain this equivalence the linear topological invariant (LB_∞) plays an essential role. Hence, first we investigate the following

3.1. Stability of Property (LB_∞) for Second Duals

Now we prove the following

Theorem 1. *Let F be a Fréchet space having property* (LB_∞). *Then* $(F'_{bor})'_\beta$ *and, in particular,* F''_β *has also property* (LB_∞) *where* F'_{bor} *denotes the dual space* F' *equipped with the bornological topology associated to the strong topology* β.

In order to prove the above theorem we need the following

Lemma 1. *A Fréchet space F has propety (LB_∞) if and only if* $\forall \{\rho_n\} \nearrow +\infty\ \exists p\ \forall q\ \exists n_0 \geq q, D > 0$ *such that*

$$U_q^0 \subseteq D\ \mathcal{C}\ell_{\sigma(F',F)}\ \mathrm{Conv}\Big(\bigcup_{q \leq k \leq n_0} \bigcap_{r>0} \Big\{ r^{\rho_k} U_k^0 + \frac{1}{r}\, U_p^0 \Big\} \Big). \tag{2}$$

Proof

Sufficiency. Let A denote the set in the right part of (2). For $x \in F$ we have

$$\|x\|_q = \sup\{\ |u(x)| : u \ \in\ U_q^0\}.$$

By (2) it follows that

$$\|x\|_q \leq \sup\{|u(x)| : u \ \in\ A\}.$$

On the other hand, it is easy to see that A is $\sigma(F', F)$-compact. However, since x is $\sigma(F', F)$-continuous for all $x \ \in\ F$ then in order to obtain the estimation

$$\sup\{|u(x)| : u \ \in\ A\}$$

it suffices to estimate

$$\sup\Big\{\ |u(x)| : u \in D\ \mathrm{Conv}\Big(\bigcup_{q \leq k \leq n_0} \bigcap_{r>0} \Big\{ r^{\rho_k} U_k^0 + \frac{1}{r}\, U_p^0 \Big\} \Big) \Big\}.$$

Let $u \in D\ \mathrm{Conv}\Big(\bigcup_{q\le k\le n_0}\bigcap_{r>0}\Big\{r^{\rho_k}U_k^0+\frac{1}{r}\,U_p^0\Big\}\Big)$. Write $u = D\sum_{j\in I}\lambda_j u_j, \sum_{j\in I}|\lambda_j|\le 1$, where I is finite and $u_j \in \bigcup_{q\le k\le n_0}\bigcap_{r>0}\Big\{r^{\rho_k}U_k^0+\frac{1}{r}\,U_p^0\Big\}$.

By the balanced convexity of the sets $\bigcap_{r>0}\Big\{r^{\rho_k}U_k^0+\frac{1}{r}\,U_p^0\Big\}$ for all $q\le k\le n_0$ one can assume that

$$u = D\sum_{j=q}^{n_0}\lambda_j u_j,$$

where $u_j \in \bigcap_{r>0}\Big\{r^{\rho_j}U_j^0+\frac{1}{r}\,U_p^0\Big\}$, $j=q,...,n_0$.

Hence

$$|u(x)|\le D\max_{q\le j\le n_0}|u_j(x)|.$$

For each $r>0$ write $u_j = r^{\rho_j}v_r+\frac{1}{r}w_r, v_r\in U_j^0, w_r\in U_p^0$.
Then

$$|u_j(x)|\le\ r^{\rho_j}|v_r(x)|+\frac{1}{r}|w_r(x)|\le r^{\rho_j}\|x\|_j\ +\frac{1}{r}\|x\|_p\quad\text{for all}\quad r>0.$$

Hence

$$|u_j(x)|\le\ C_j\|x\|_j^{\frac{1}{1+\rho_j}}\|x\|_p^{\frac{\rho_j}{1+\rho_j}},\quad\text{where}\quad C_j=(\ \rho_j^{\frac{-\rho_j}{1+\rho_j}}\ +\ \rho_j^{\frac{1}{1+\rho_j}}),\quad q\le j\le n_0.$$

This implies that $|u(x)|\ \le\ D\max_{q\le j\le n_0}C_j\|x\|_j^{\frac{1}{1+\rho_j}}\|x\|_p^{\frac{\rho_j}{1+\rho_j}}$, for all $u\ \in\ D$ $\mathrm{Conv}\Big(\bigcup_{q\le k\le n_0}\bigcap_{r>0}\Big\{r^{\rho_k}U_k^0+\frac{1}{r}\,U_p^0\Big\}\Big)$. Hence, by the above argument it follows that

$$\|x\|_q\le D\max_{q\le j\le n_0}C_j\|x\|_j^{\frac{1}{1+\rho_j}}\|x\|_p^{\frac{\rho_j}{1+\rho_j}}.$$

This yields that there exists $q\le k\le n_0$ such that

$$\|x\|_q\le DC_k\|x\|_k^{\frac{1}{1+\rho_k}}\|x\|_p^{\frac{\rho_k}{1+\rho_k}},$$

or, equivalently,

$$\|x\|_q^{1+\rho_k}\le\ C\|x\|_k\|x\|_p^{\rho_k},$$

where $C=(DC_k)^{1+\rho_k}$ only depends on q. Hence F has the property (LB_∞).

Necessity. Now let F have property (LB_∞). Then

$\forall\{\rho_k\}\nearrow+\infty\ \exists p\ \forall q\ \exists n_0\ge q, D>0\ \forall x\in F\ \exists q\le k\le n_0$:

$$\|x\|_q^{1+\rho_k}\le D\|x\|_k\|x\|_p^{\rho_k}.$$

We prove that (2) holds for the above D.

Assume that (2) is false. This means that we can find

$$u \in U_q^0 \setminus D\, \mathcal{C}\ell_{\sigma(F',F)} \operatorname{Conv}\Big(\bigcup_{q \le k \le n_0} \bigcap_{r>0} \Big\{ r^{\rho_k} U_k^0 + \frac{1}{r}\, U_p^0 \Big\} \Big).$$

Since $F = (F', \sigma(F', F))'$, by Hahn–Banach theorem, it follows that there exist $x \in F$ with $|u(x)| > 1$ and $|v(x)| \le 1$ for all

$$v \in D \bigcup_{q \le k \le n_0} \bigcap_{r>0} \Big\{ r^{\rho_k} U_k^0 + \frac{1}{r}\, U_p^0 \Big\}.$$

Hence

$$\sup \Big\{ |v(x)| : v \in D \bigcap_{r>0} \Big\{ r^{\rho_k} U_k^0 + \frac{1}{r}\, U_p^0 \Big\} \Big\} \le 1$$

for all $q \le k \le n_0$. This shows that for every $q \le k \le n_0$ and every $r > 0$ we have

$$r^{\rho_k} \|x\|_k \ + \frac{1}{r} \|x\|_p \le \frac{1}{D}.$$

Hence,

$$\big(\rho_k{}^{\frac{-\rho_k}{1+\rho_k}} \ + \ \rho_k^{\frac{1}{1+\rho_k}} \big) \|x\|_k^{\frac{1}{1+\rho_k}} \|x\|_p^{\frac{\rho_k}{1+\rho_k}} \le \frac{1}{D}$$

for all $q \le k \le n_0$.

However, from

$$\rho_k{}^{\frac{-\rho_k}{1+\rho_k}} \ + \ \rho_k^{\frac{1}{1+\rho_k}} \ge 1$$

for all $q \le k \le n_0$ we derive that

$$\|x\|_k^{\frac{1}{1+\rho_k}} \|x\|_p^{\frac{\rho_k}{1+\rho_k}} \le \frac{1}{D}.$$

Hence

$$D \|x\|_k^{\frac{1}{1+\rho_k}} \|x\|_p^{\frac{\rho_k}{1+\rho_k}} \le 1$$

for all $q \le k \le n_0$.
By the hypothesis we claim that $\|x\|_q \le 1$. Hence $|u(x)| \le 1$ which is impossible.

Proof of Theorem 1. Assume that F has property (LB_∞). By Lemma 1 we have $\forall \{\rho_n\} \nearrow +\infty \ \exists p \ \forall q \ \exists n_0 \ge q, D > 0$ such that

$$\begin{aligned} U_q^0 &\subseteq D\, \mathcal{C}\ell_{\sigma(F',F)} \operatorname{Conv}\Big(\bigcup_{q \le k \le n_0} \bigcap_{r>0} \Big\{ r^{\rho_k} U_k^0 + \frac{1}{r}\, U_p^0 \Big\} \Big) \\ &\subseteq D \sum_{q \le k \le n_0} \bigcap_{r>0} \Big\{ r^{\rho_k} U_k^0 + \frac{1}{r}\, U_p^0 \Big\}. \end{aligned}$$

The last inclusion is a consequence of the $\sigma(F', F)$-compactness and the balanced convexity of $\bigcap_{r>0} \{ r^{\rho_k} U_k^0 + \frac{1}{r} U_p^0 \}$. Then for $u \in (F'_{bor})'_\beta$ we have

$$\begin{aligned} \|u\|_q^{**} &= \sup\{ |u(y)| : y \in U_q^0 \} \\ &\le D \sum_{q \le k \le n_0} \|u\|^{**}_{\bigcap_{r>0} \{ r^{\rho_k} U_k^0 + \frac{1}{r}\, U_p^0 \}} \\ &\le D \sum_{q \le k \le n_0} C_k \|u\|_k^{**\,\frac{1}{1+\rho_k}} \|u\|_p^{**\,\frac{\rho_k}{1+\rho_k}} \end{aligned}$$

where $C_k = (\rho_k^{\frac{-\rho_k}{1+\rho_k}} + \rho_k^{\frac{1}{1+\rho_k}})$.
Hence

$$\|u\|_q^{**} \leq Dn_0 \max_{q\leq k\leq n_0} \Big(C_k \|u\|_k^{**\frac{1}{1+\rho_k}} \|u\|_p^{**\frac{\rho_k}{1+\rho_k}} \Big).$$

This shows that $(F'_{bor})'_\beta$ has the property (LB_∞).

3.2. Holomorphic Functions on Compact Sets in $\mathbb{C}^n$

The main result of this section is to prove the main theorem. However in order to meet reader's convenience we recall the contents of this theorem.

The main theorem . *Let F be a Fréchet space. Then F has property* (LB_∞) *if and only if*

$$H_w(X, F) = H(X, F)$$

holds for every compact set of uniqueness X in $\mathbb{C}^n$ *and for every* $n \geq 1$.

Proof

Necessity. Assume that F has property (LB_∞).
It suffices to prove that $H_w(X, F) \subset H(X, F)$. Let $f \in H_w(X, F)$. By the uniqueness of X we may define a linear map

$$\hat{f} : F'_{bor} \longrightarrow H(X)$$

given by

$$\hat{f}(u) = \widehat{uf}$$

for $u \in F'_{bor}$, where $\widehat{uf}$ denotes the holomorphic extension of uf to some neighborhood of X in $\mathbb{C}^n$.

Again by the uniqueness of X it follows that $\hat{f}$ has a closed graph. Hence, from the closed graph theorem of Grothendieck [8] it follows that $\hat{f}$ is continuous. On the other hand, by [14] $[H(X)]'_\beta$ is isomorphic to a quotient space of $s \cong \Lambda_\infty(\log(n+1)_{n\geq 1})$, where s is the space of rapidly decreasing sequences.

Now, by a result of Vogt [22] we infer that

$$L(F'_{bor}, H(X)) = LB(F'_{bor}, H(X))$$

because

$$(\hat{f})' : (H(X))'_\beta \longrightarrow (F'_{bor})'_\beta$$

is continuous and by Theorem 1 $(F'_{bor})'_\beta$ has property (LB_∞). Hence, we can find a neighborhood W of $0 \in F'_{bor}$ such that $\hat{f}(W)$ is bounded in $H(X)$. Since $H(X) = \lim_p \text{ind} H^\infty(V_p)$ is regular [6], we derive that there exists p such that $\hat{f}(W)$ is contained and bounded in $H^\infty(V_p)$. Then $\hat{f}(F'_{bor}) \subset H^\infty(V_p)$.

Now we define a function $g: V_p \longrightarrow (F'_{bor})'_\beta$ by setting

$$g(z)(u) = \hat{f}(u)(z) \quad \text{for} \quad z \in V_p,\ u \in F'_{bor}.$$

Then $g : V_p \longrightarrow (F'_{bor})'_\beta$ is holomorphic. Moreover, it is easy to see that $g|_X = f$. Again by the uniqueness of X and by the identity theorem, we infer that there exists a neighborhood V of X such that $g: V \longrightarrow F$ is holomorphic and it is a desired holomorphic extension of f.

Sufficiency. First by [16] there exists a compact polar set of uniqueness $X \subset \mathbb{C}$. Then by a result of Zaharjuta [25] $H(X) \cong H(\{0\})$. Hence

$$[H(X)]'_\beta \cong H(\{0\})'_\beta \cong H(\mathbb{C}_\infty \setminus \{0\}) \cong H(\mathbb{C}).$$

Then second isomorphism is a result of a dual theorem of Grothendieck. By Vogt [22] in order to prove that F has property (LB_∞) it suffices to prove that

$$L([H(X)]'_\beta, F) = LB([H(X)]'_\beta, F).$$

Let $S \in L([H(X)]'_\beta, F)$ be given. Then we can define a map $f : X \longrightarrow F$ given by $f(z) = S(\delta_z)$ where $\delta_z(h) = h(z)$ for $z \in X$ and $h \in H(X)$ is the Dirac functional. It is easy to see that $f \in H_w(X, F)$. Using the hypothesis we can find a neighborhood U of X in $\mathbb{C}$ and a F-valued holomorphic function $\hat{f}$ on U such that $\hat{f}|_X = f$.

By shrinking U we may assume that $B = \hat{f}(U)$ is bounded in F. Then

$$|S'(u)(z)| = |\hat{f}(z)(u)| \leq 1 \text{ for } z \in U,\ u \in B^0,$$

where B^0 denotes the polar of B in F'_β. This shows that $S'(B^0)$ is contained and bounded in $H^\infty(U)$ and, hence, $S'(B^0)$ is bounded in $(H'(X)_\beta)' = H(X)$. Put $C = (S'(B^0))^0$. Then C is a neighborhood of $0 \in [H(X)]'_\beta$.

Now for $u \in B^0$ and $t \in C$ we have

$$|S(t)(u)| = |S'(u)(t)| \leq 1.$$

Thus $S(C) \subset B^{00} \cap F$ and, hence, is bounded in F. The desired conclusion follows.

4. Singular Sets of Fréchet-Valued Separately Holomorphic Functions on the Product of Compact Subsets

In this section we extend results of Saint Raymond - Siciak - Blocki on structure of singular sets of Fréchet-valued separately holomorphic functions to products of compact subsets and to open subsets.

For the product of compact subsets by using the results of Alehyane and Zeriahi [1] we establish a result on structure of singular sets of Fréchet-valued separately holomorphic functions on the product of two compact subsets. The analogous problem for a product of more two compact subsets is still an open problem.

For the case of open subsets it is easy to see that the results of Saint Raymond - Siciak and more general, of Blocki are still valid for functions with Banach values. However, it is difficult to study the problem for functions with Fréchet values because such functions are given by a sequence of functions with Banach values. Hence, in general, the singular set of the given function is not able be contained in the union of singular sets of component functions with Banach values. Here we only prove results of Saint Raymond - Siciak - Blocki for functions with values in Fréchet spaces having the linear topological invariant $(\overline{\mathrm{DN}})$ and the general case is still open.

Let K, L be non-pluripolar connected compact subsets in $\mathbb{C}^n$, $\mathbb{C}^m$ respectively and F a Fréchet space. Let $f : K \times L \longrightarrow F$ be a separately holomorphic function. We put $A(f) = \{(x, y) \in K \times L : f$ is extended holomorphically to an open neighborhood of (x, y) in $\mathbb{C}^n \times \mathbb{C}^m\}$, and $S(f) = K \times L \setminus A(f)$. Now we have

Theorem 2. *Let S_1 and S_2 be the projections of $S(f)$ to $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Then S_1 is pluripolar in $\mathbb{C}^n$ and S_2 is pluripolar in $\mathbb{C}^m$.*

The proof of Theorem 2 relies on the following

Lemma 2. *Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ be domains and $K \subset U$, $L \subset V$ non-pluripolar closed subsets and F a Fréchet space. Let*

$$Y = (K \times V) \cup (U \times L)$$

and $f : Y \longrightarrow F$ be a function satisfying

(i) *The function $f_z : V \longrightarrow F$ given by*

$$f_z(w) = f(z, w), \quad w \in V,$$

is holomorphic on V for all $z \in K$;

(ii) *The function $f^w : U \longrightarrow F$ given by*

$$f^w(z) = f(z, w), \quad z \in U,$$

is holomorphic on U for all $w \in L$

Then f is extended holomorphically to an open neighborhood Ω of $Y^* = (K \times L^*) \cup (K^* \times L)$ which is the union of connected components of $\widetilde{Y}$ meeting $K^* \times L^*$, where

$$\widetilde{Y} = \{(z, w) \in U \times V : \omega(z, K^*, U) + \omega(w, L^*, V) < 1\}.$$

Proof. Let $u \in F'$ be an arbitrary continuous linear form on F. Consider the separately holohorphic function

$$uf : Y \longrightarrow \mathbb{C}.$$

By a result of Alehyane and Zeriahi [1, Théorème 2.2.4] uf has a holomorphic extension $\widehat{uf} : \widetilde{Y} \longrightarrow \mathbb{C}$. Let Ω be the union of connected components of $\widetilde{Y}$ meeting $K^* \times L^*$. By the indentity principle we can define the map

$$S : F'_{bor} \longrightarrow H(\Omega),$$

given by

$$S(u)(z) = \widehat{uf}(z), \quad z \in \Omega, \ \ u \in F'_{bor},$$

where F'_{bor} is F' equipped with the bornological topology associated with the strong topology β.

By the definition of Ω and by using the identity principle, it follows that S is linear and has a closed graph. Hence, in view of the closed graph theorem of Grothendieck [8] we derive that S is continuous.

Now we can define the map $\widehat{f} : \Omega \longrightarrow [F'_{bor}]'_\beta$ by the formula

$$\widehat{f}(z)(u) = S(u)(z), \quad z \in \Omega, \ \ u \in F'_{bor}.$$

For each $u \in F'_{bor}$ we have

$$\widehat{f}(z)(u) \ = \ S(u)(z) \ = \ \widehat{uf}(z) \ , \ z \ \in \ \Omega$$

and hence, we deduce that $\widehat{f} : \Omega \longrightarrow [F'_{bor}]'_\beta$ is holomorphic. However, F is a closed subspace of $[F'_{bor}]'_\beta$ and $\widehat{f}(K^* \times L^*) = f(K^* \times L^*)$ and by the definition of Ω and the identity principle it follows that $\widehat{f} : \Omega \longrightarrow F$ is holomorphic. The lemma is proved.

Proof of Theorem 2. Since K and L are connected compact subsets in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, then there exists a connected decreasing neighborhood basis $\{U_k\}_{k\geq 1}$ of K in $\mathbb{C}^n$ (resp. $\{V_k\}_{k\geq 1}$ of L in $\mathbb{C}^n$). Suppose that S_2 is not pluripolar. For each $k \ \geq \ 1$ put

$$S_2^k = \{w \in S_2 : f^w \in H(U_k, F)\}.$$

Then $S_2 = \bigcup_{k\geq 1} S_2^k$. Thus there exists $k_0 \geq 1$ such that S_2^k is not pluripolar for every $k \geq k_0$. Moreover, without loss of generality, we may assume that for every $k \geq k_0$, S_2^k is closed. Similarly, for each $j \geq 1$ put

$$Z_j = \{z \in K : f_z \in H(V_j, F)\}.$$

Then $K = \bigcup_{j\geq 1} Z_j$ and, hence, there exists $j_0 \geq 1$ such that Z_{j_0} is not pluripolar. We also assume that Z_{j_0} is closed. For each $k \geq k_0$ consider the separately holomorphic function

$$f : (U_k \times S_2^k) \cup (Z_{j_0} \times V_{j_0}) \longrightarrow F.$$

By Lemma 2 f is extended holomorphically to a neighborhood of $U_k \times (S_2^k)^*$. Thus f extends holomorphically to a neighborhood of $\bigcup_{k \ \geq k_0} U_k \times (S_2^k)^*$.

This follows that

$$S \subset S_1 \times (S_2 \setminus \bigcup_{k\geq k_0} U_k \times (S_2^k)^*).$$

Hence

$$S_2 \subset S_2 \setminus \bigcup_{k\geq k_0} (S_2^k)^*, \quad \text{or} \quad \bigcup_{k\geq k_0} (S_2^k)^* = \emptyset.$$

This is impossible, because S_2^k is non-pluripolar for $k \geq k_0$ then by a result of Bedford–Taylor [2] $\bigcup_{k\geq k_0} (S_2^k)^* \neq \emptyset$.

In 1992, by omitting the assumption $p \geq s/2$, Blocki [3] gave a complete characterization of singular sets of separately analytic functions. At the end of this paper we want to extend the result of Blocki to Fréchet-valued separately analytic functions. However as the above we only establish this result in the case where F is a Fréchet space having the property $(\overline{\mathrm{DN}})$. Namely, we prove the following

Theorem 3. *Let $\Omega \subset \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_s}$ be an open subset and F a Fréchet space having the topological linear invariant $(\overline{DN})$. Assume that $f : \Omega \longrightarrow F$ is a p-separately analytic function $(1 \leq p < s)$. Then for every $1 \leq j_1 < ... <$*

$j_q \leq s,\ q = s - p$ the projection of $S(f)$ on $\mathbb{R}^{n_{j_1}} \times \ldots \times \mathbb{R}^{n_{j_q}}$ is pluripolar in $\mathbb{C}^{n_{j_1}} \times \ldots \times \mathbb{C}^{n_{j_q}}$.

First of all, by repeating arguments used in the paper of Blocki [3], we remark that Theorem 3 still holds in the case where F is a Banach space. By using this remark we give the proof of Theorem 3 as follows.

Proof of Theorem 3. Let F be a Fréchet space with the topology defined by an increasing sequence of semi-norms

$$\|\cdot\|_1 \ \leq\ \|\cdot\|_2 \ \leq \ldots \leq\ \|.\|_n \ \leq \ldots$$

By F_k we denote the Banach space associated to the *k-th* semi-norm. Since F has the property $(\overline{DN})$, we have

$(\overline{\mathrm{DN}})$ $\exists p\ \forall\ q\ \exists\ k\ \forall\ d > 0\ \exists\ C > 0:$

$$\|\cdot\|_q^{1+d} \leq C\|\cdot\|_k\|\cdot\|_p^d.$$

Write $F = \mathrm{limproj}F_k$. Let $\Pi_k : F \longrightarrow F_k$ be the canonical projection and put $f_k = \Pi_k \circ f$. If $f : \Omega \longrightarrow F$ is a p-separately analytic function then it is easy to see that for each $k \ \geq\ 1$ the function $f_k : \Omega \longrightarrow F_k$ is also p-separately analytic. Now we prove that if F has the property $(\overline{\mathrm{DN}})$ then

$$S(f) \subset \bigcup_{k=1}^{\infty} S(f_k). \tag{*}$$

Indeed, let $x_0 \notin \bigcup_{k=1}^{\infty} S(f_k)$. Hence, f_k is analytic at x_0 for all $k \geq 1$. Write the Taylor expansion of f_k at x_0 in the form

$$f_k(x_0 + h) = \sum_{n=0}^{\infty} P_{n,f_k}(x_0, h),$$

where the right series is convergent in the ball $B\Big(x_0, \dfrac{1}{R_{f_k}(x_0)}\Big)$ with

$$R_{f_k}(x_0) = \frac{1}{\limsup_n \|P_{n,f_k}(x_0)\|_k^{\frac{1}{n}}}.$$

Since F has the property $(\overline{\mathrm{DN}})$ then for $h \ \in\ \mathbb{C}^{n_1} \times \ldots \times \mathbb{C}^{n_s}$, $\|h\| \leq 1$ we have

$$\|P_{n,f_q}(x_0, h)\|_q^{1+d} \leq C\|P_{n,f_k}(x_0, h)\|_k\ \|P_{n,f_p}(x_0, h)\|_p^d.$$

Hence,

$$\|P_{n,f_q}(x_0)\|_q^{1+d} \leq C\|P_{n,f_k}(x_0)\|_k\|P_{n,f_p}(x_0)\|_p^d.$$

This implies that

$$\big(\|P_{n,f_q}(x_0)\|_q^{\frac{1}{n}}\big)^{1+d} \leq C^{\frac{1}{n}}\ \|P_{n,f_k}(x_0)\|_k^{\frac{1}{n}}\ \big(\|P_{n,f_p}(x_0)\|_p^{\frac{1}{n}}\big)^d.$$

and we derive that

$$R_{f_q} \geq (R_{f_k})^{\frac{1}{1+d}}\ \big(R_{f_p}\big)^{\frac{d}{1+d}}.$$

On the other hand, from the property $(\overline{\text{DN}})$ we can choose d sufficiently large such that

$$(R_{f_k})^{\frac{1}{1+d}} > \frac{1}{2} \quad \text{and} \quad (R_{f_p})^{\frac{d}{1+d}} > \frac{1}{2} R_{f_p}.$$

Thus, if d is chosen sufficiently large then for all $q \geq 1$ we have

$$R_{f_q} > \frac{1}{4} R_{f_p} = R,$$

and the function f_q is analytic on $B(x_0, R)$. This fact shows that f is analytic at x_0, and, hence $x_0 \notin S(f)$. The inclusion $(*)$ has been proved. Let $\Pi_{j_1,\ldots,j_q}$ denote the projection of $S(f)$ onto $\mathbb{R}^{n_{j_1}} \times \ldots \times \mathbb{R}^{n_{j_q}}$, $q = s - p$. From $(*)$ we have

$$\Pi_{j_1,\ldots,j_q}((S(f)) \subset \bigcup_{k=1}^{\infty} \Pi_{j_1,\ldots,j_q}(S(f_k)),$$

and the pluripolarity of $\Pi_{j_1,\ldots,j_q}((S(f))$ in $\mathbb{C}^{n_{j_1}} \times \ldots \times \mathbb{C}^{n_{j_q}}$ follows.

Theorem 3 is proved.

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