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A Remark on Non-Markov Property of a Fractional Brownian Motion

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Abstract. In this short remark we would like to give a simple proof for the non-Markov property of a fractional Brownian motion. Such a proof cannot be found anywhere else so far.

Introduction

Fractional Brownian motion plays an intensive role in study of stochastic dynamical systems that exhibit a long range dependence between states of the system (see [2] for example).

It is known that, in general, a fractional Brownian motion is not a semimartingale and it is not a Markov process. The proof of the first assertion can be found in [4]. Up to now there does not exist any strict and direct proof for the second assertion. One proved only that a fractional Brownian motion B_t is a long memory process in the sense that B_t and B_{t+k} is still corrected for enough large k (see [1, 5 - 6]) and the non-Markov property is deduced from this as a consequence. And now we would like to give a simple proof for the second assertion based only on a property of covariance function of a Gaussian Markov process.

Recall that a fractional Brownian motion X_t is a centered Gaussian process whose covariance function $R(t,s) = EX_tX_s$ is given by

$$R(t,s) = \frac{1}{2}(|t|^\alpha + |s|^\alpha - |t-s|^\alpha), \tag{1}$$

where α is a real parameter, $0 < \alpha < 2$. In the case where $\alpha = 1, R(t,s) = \frac{1}{2}(|t| + |s| - |t-s|) = \min(t,s)$ we have an ordinary standard Brownian motion.

We will prove that X_t is not a Markov process for $\alpha \neq 1$, based on a known necessary condition for Markov property as follows [6-7]:

Lemma. *Suppose that $R(t,s)$ is the covariance function of a centered Gaussian process X_t. If X_t is a Markov process then for any $t > s > t_0$ we have*

$$R(t,t_0) = \frac{R(t,s)R(t_0,t_0)}{R(s,s)}, \quad t > s > t_0. \tag{2}$$

Proof. Consider two Gaussian variables

$$U = X_s \quad \text{and} \quad V = X_t - \frac{R(t,s)}{R(s,s)} X_s.$$

We have $EU = EV = 0$ and $EUV = 0$, so U and V are two uncorrelated Gaussian variables. Therefore they are independent, and by consequence

$$\begin{aligned} E\Big[X_t - \frac{R(t,s)}{R(s,s)} X_s \Big| X_s = \xi\Big] &= E\Big[X_t - \frac{R(t,s)}{R(s,s)} X_s\Big] \\ &= E(X_t) - \frac{R(t,s)}{R(s,s)} E(X_s) = 0, \end{aligned}$$

where ξ is any possible value of X_s. Then we get

$$E\Big[X_t \Big| X_s = \xi\Big] - \frac{R(t,s)}{R(s,s)} E\Big[X_s \Big| X_s = \xi\Big] = 0$$

or

$$E\Big(X_t \Big| X_s = \xi\Big) = \frac{R(t,s)}{R(s,s)} \xi. \tag{3}$$

Denote by $P(x,t|x_0,t_0)$ the transition probability of X_t. Then the Chapman-Kolmogorov equation can be written as

$$P(x,t|x_0,t_0) = \int_{-\infty}^{\infty} P(x,t|\xi,s) P(d\xi,s|x_0,t_0) \tag{4}$$

We have

$$\begin{aligned} E\Big[X_t \Big| X_{t_0} = x_0\Big] &= \int_{-\infty}^{\infty} x dP(x,t|x_0,t_0) \\ &= \int_{-\infty}^{\infty} x P(dx,t|x_0,t_0) \\ &= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} P(dx,t|\xi,s) P(d\xi,s|x_0,t_0) \\ &= \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} x P(dx,t|\xi,s)\Big] P(d\xi,s|x_0,t_0) \\ &= \int_{-\infty}^{\infty} \Big[E(X_t|X_s = \xi)\Big] P(d\xi,s|x_0,t_0) \\ &= \int_{-\infty}^{\infty} \frac{R(t,s)}{R(s,s)} \xi P(d\xi,s|x_0,t_0) \\ &= \frac{R(t,s)}{R(s,s)} \int_{-\infty}^{\infty} \xi P(d\xi,s|x_0,t_0) \end{aligned}$$

$$= \frac{R(t,s)}{R(s,s)} E\big[x_s \big| X_{t_0} = x_0\big]$$
$$= \frac{R(t,s)}{R(s,s)} x_0, \quad t_0 < s < t. \tag{5}$$

On the other hand we have from (5)

$$E\big[X_t \big| X_{t_0} = x_0\big] = \frac{R(t,t_0)}{R(t_0,t_0)} x_0. \tag{6}$$

In comparison (5) and (6) we get

$$\frac{R(t,s)}{R(s,s)} = \frac{R(t,t_0)}{R(t_0,t_0)}$$

or

$$R(t,t_0) = \frac{R(t,s)R(t_0,t_0)}{R(s,s)}, \quad t > s > t_0.$$

We are now in a position to prove the following

Theorem. *A fractional Brownian motion is not a Markov process for $\alpha \neq 1$.*

Proof. We will prove that for a fractional Brownian motion X_t the relation (2) is not true.

Fix t $(t > s)$ and consider the function

$$f(s) = \frac{R(t,s)}{s^\alpha} = \frac{1}{2}\Big[\big(\frac{t}{s}\big)^\alpha - 1 - \big(\frac{t}{s} - 1\big)^\alpha\Big], t > s. \tag{7}$$

We take the derivative $f'(s)$ of $f(s)$:

$$f'(s) = -\frac{1}{2}\frac{\alpha t}{s^2}\Big[\big(\frac{t}{s}\big)^{\alpha-1} - \big(\frac{t}{s} - 1\big)^{\alpha-1}\Big], t > s. \tag{8}$$

We see that, for $s < t$

$$\begin{aligned} f'(s) > 0 \quad &\text{if} \quad \alpha < 1 \\ f'(s) < 0 \quad &\text{if} \quad \alpha > 1 \\ f'(s) = 0 \quad &\text{if} \quad \alpha = 1. \end{aligned}$$

So $f(s)$ is either an increasing function or a decreasing function and therefore either $f(s) > f(t_0)$ or $f(s) < f(t_0)$, and

$$f(s) = f(t_0) \quad \text{only for} \quad \alpha = 1, \quad (s > t_0). \tag{9}$$

Now for $\alpha \neq 1$,

$$\frac{R(t,s)}{s^\alpha} \neq \frac{R(t,t_0)}{t_0^\alpha}$$

or

$$t_0^\alpha R(t,s) \neq s^\alpha R(t,t_0).$$

So we have

$$R(t_0,t_0)R(t,s) \neq R(s,s)R(t,t_0) \quad \text{for} \quad t > s > t_0. \tag{10}$$

or

$$R(t,t_0) \neq \frac{R(t,s)R(t_0,t_0)}{R(s,s)}, \quad t > s > t_0.$$

The equality (9) occurs only when $\alpha = 1$, then $f'(s) = 0$ and $f(s)$ is identically equal to a constant:

$$\frac{R(t,s)}{s^\alpha} = \frac{R(t,t_0)}{t_0^\alpha}$$

or

$$R(t,t_0) = \frac{R(t,s)R(t,t_0)}{R(s,s)}. \tag{11}$$

We know that this is the case of an ordinary standard Brownian motion. The proof is thus complete.

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