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Equisingular Families of Plane Curves with Many Connected Components

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Abstract. We present examples of equisingular families of complex projective plane curves with plural connected components that are not distinguished by the fundamental group of the complement.

1. Introduction

The topological fundamental group of the complement to a complex singular plane curve plays an important role in the study of families of singular plane curves. It has been used as a tool for distinguishing connected components of an equisingular family of plane curves.

According to [1], we define the equisingularity of plane curves as follows. Let C_1 and C_2 be two complex projective plane curves. Suppose that both of them are reduced and have a same degree. We say that C_1 and C_2 are *equisingular* if there exists an open neighborhood $T_i \subset \mathbb{P}^2$ of $C_i \subset \mathbb{P}^2$ for $i = 1$ and 2 such that (T_1, C_1) and (T_2, C_2) are diffeomorphic.

For an equivalence class of equisingularity of degree d, we consider the family $\mathcal{F} \subset \mathbb{P}_* H^0(\mathbb{P}^2, \mathcal{O}(d))$ of all projective plane curves that belong to the given class of equisingularity, and call it an equisingular family. Here $\mathbb{P}_* H^0(\mathbb{P}^2, \mathcal{O}(d))$ stands for the projective space of one-dimensional subspaces of the vector space $H^0(\mathbb{P}^2, \mathcal{O}(d))$, which parameterizes all plane curves of degree d.

The fact that an equisingular family may fail to be connected was first observed by Zariski in [20, 21] (see also [10]). He constructed a pair (C_1, C_2) of reduced curves of degree 6 with the following properties.

- For $i = 1$ and 2, the singular locus of C_i consists of 6 ordinary cusps. In particular, C_1 and C_2 are equisingular.

- The fundamental group $\pi_1(\mathbb{P}^2 \setminus C_1)$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$, while $\pi_1(\mathbb{P}^2 \setminus C_2)$ is isomorphic to the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

If C_1 and C_2 belonged to a same connected component of the equisingular family $\mathcal{F}_{6A_2} \subset \mathbb{P}_* H^0(\mathbb{P}^2, \mathcal{O}(6))$ of curves of degree 6 with 6 ordinary cusps, then $\mathbb{P}^2 \setminus C_1$ and $\mathbb{P}^2 \setminus C_2$ would be homeomorphic. Hence Zariski's example shows that $\mathcal{F}_{6A_2}$ is not connected.

After this classical example, many equisingular families with plural connected components have been constructed [1-3, 9-13, 15-19]. In these examples, the tool that was employed for distinguishing two distinct connected components is the fundamental groups $\pi_1(\mathbb{P}^2 \setminus C)$, or its derivatives like Alexander polynomials or the coverings of $\mathbb{P}^2$ branched along C.

In this paper, we give an example of equisingular families with plural connected components that can *not* be distinguished by $\pi_1(\mathbb{P}^2 \setminus C)$.

Let b and m be positive integers such that $b \geq 3$ and $b \equiv 0 \pmod m$. We put $n := b/m$.

Definition 1.1. *A projective plane curve $R \subset \mathbb{P}^2$ is said to be of type (b, m) if it satisfies the following;*

(1) *R consists of two irreducible components B and E of degree b and 3, respectively,*
(2) *both of B and E are non-singular,*
(3) *the set-theoretical intersection of B and E consists of $3n$ points, and*
(4) *at each intersection point, B and E intersect with multiplicity m.*

Let $\mathcal{F}_{b,m} \subset \mathbb{P}_* H^0(\mathbb{P}^2, \mathcal{O}(b+3))$ be the family of all curves of type (b, m). Any two curves R and R' of type (b, m) are equisingular, because the singular locus of a curve of type (b, m) always consists of $3n$ points of type A_{2m-1}. On the other hand, suppose that a curve $C \subset \mathbb{P}^2$ of degree $b + 3$ is equisingular to a curve of type (b, m). Then C consists of two nonsingular irreducible components, which intersect at $3n$ points. Since the intersection multiplicity of two germs of nonsingular curves is invariant under local diffeomorphisms, the intersection multiplicities at these $3n$ points are all m. Therefore the degrees of the irreducible components of C are b and 3, and hence C is of type (b, m). Thus $\mathcal{F}_{b,m}$ is an equisingular family.

The main result of this paper is as follows.

Theorem 1.2. *Suppose that $b \geq 4$, and let m be a divisor of b.*
(1) *The number of the connected components of $\mathcal{F}_{b,m}$ is equal to the number of divisors of m.*
(2) *Let R be a member of $\mathcal{F}_{b,m}$. Then the fundamental group $\pi_1(\mathbb{P}^2 \setminus R)$ is isomorphic to $\mathbb{Z}$ if b is not divisible by 3, while it is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ if b is divisible by 3.*

This result provides us in every degree ≥ 7 with the existence of equisingular families with plural connected components that cannot be distinguished by the fundamental group of the complement. It also gives us examples of equisingular families consisting of arbitrarily many connected components.

When $(b, m) = (3, 3)$, we also obtain an equisingular family with two connected components. In this case, the members belonging to distinct connected components of $\mathcal{F}_{3,3}$ have the complements with non-isomorphic fundamental groups.

The plan of this paper is as follows. In Sec. 2 we prove the assertion (1) of Theorem 1.2. In Sec. 3, we present a proposition that is useful in the calculation of the fundamental group of the complement to a projective plane curve. Using this proposition, we prove the assertion (2) of Theorem 1.2 in Sec. 4. In Sec. 5, we transplant the method of the construction of $\mathcal{F}_{b,m}$ to the case $b = 3$, and calculate the fundamental groups of the complements to the members of $\mathcal{F}_{3,3}$.

It is an interesting problem to determine whether $\mathbb{P}^2 \setminus R_1$ and $\mathbb{P}^2 \setminus R_2$ are homeomorphic or not for two curves R_1 and R_2 of a same type (b, m) that belong to distinct connected components of $\mathcal{F}_{b,m}$.

2. The Connected Components of $\mathcal{F}_{b,m}$

For a line bundle $L \to C$ of degree k on a curve C, we denote by $[L] \in \mathrm{Pic}^k(C)$ the isomorphism class of the line bundles containing L.

Let $R = B + E$ be a curve of type (b, m). We denote by D_R the reduced effective divisor $(B|_E)_{red}$ of degree $3n$ on E. Let H be a divisor of degree 3 on E that is obtained as the intersection of E and a line in $\mathbb{P}^2$. Then $\mathcal{O}_E(D_R - nH)$ is an invertible sheaf of degree 0 on E;

$$[\mathcal{O}_E(D_R - nH)] \in \mathrm{Pic}^0(E).$$

Note that $[\mathcal{O}_E(D_R - nH)]$ is an element of the kernel of the homomorphism $\mathrm{Pic}^0(E) \to \mathrm{Pic}^0(E)$ given by

$$[L] \mapsto [L^{\otimes m}],$$

because we have

$$\mathcal{O}_E(mD_R) \cong \mathcal{O}_{\mathbb{P}^2}(B)|_E \cong \mathcal{O}_E(bH) \cong \mathcal{O}_E(mnH).$$

Let $\lambda(R)$ denote the order of the isomorphism class $[\mathcal{O}_E(D_R - nH)]$ in $\mathrm{Pic}^0(E)$, which is a divisor of m. It is obvious that $\lambda(R)$ does not change under any continuous deformation of R in $\mathcal{F}_{b,m}$.

For a divisor μ of m, we write by $\mathcal{F}_{b,m}(\mu)$ the union of all connected components of $\mathcal{F}_{b,m}$ on which the function λ is constantly equal to μ;

$$\mathcal{F}_{b,m} = \coprod_{\mu | m} \mathcal{F}_{b,m}(\mu).$$

Then the first part (1) of Theorem1.2 follows from the following proposition.

Proposition 2.1. *For any divisor μ of m, the variety $\mathcal{F}_{b,m}(\mu)$ is irreducible and of dimension* $(b-1)(b-2)/2 + 3n + 8$.

Proof. Let $\mathcal{U}$ be the Zariski open dense subset of $\mathbb{P}_* H^0(\mathbb{P}^2, \mathcal{O}(3))$ parameterizing all non-singular cubic curves. We use the same letter to denote a non-singular cubic curve $E \subset \mathbb{P}^2$ and the corresponding point $E \in \mathcal{U}$. Let $\mathcal{E} \to \mathcal{U}$ be the universal family of the non-singular cubic curves, and let

$$\mathrm{Pic}^k(\mathcal{E}/\mathcal{U}) \longrightarrow \mathcal{U}$$

be the relative Picard variety of the isomorphism classes of line bundles of degree k on $\mathcal{E}$ over $\mathcal{U}$. A $\mathbb{C}$-valued point of $\mathrm{Pic}^k(\mathcal{E}/\mathcal{U})$ is a pair $(E, [L])$, where E is a non-singular cubic curve and $[L]$ is a point of $\mathrm{Pic}^k(E)$.

There is a section of $\mathrm{Pic}^3(\mathcal{E}/\mathcal{U}) \to \mathcal{U}$ given by

$$E \mapsto (E, [\mathcal{O}_E(H)]).$$

Using this section, we obtain an isomorphism

$$\phi : \mathrm{Pic}^{3n}(\mathcal{E}/\mathcal{U}) \xrightarrow{\sim} \mathrm{Pic}^0(\mathcal{E}/\mathcal{U})$$

over $\mathcal{U}$ given by

$$(E, [L]) \mapsto (E, [L \otimes \mathcal{O}_E(-nH)])$$

for $E \in \mathcal{U}$ and $[L] \in \mathrm{Pic}^{3n}(E)$.

Suppose that a divisor μ of m is given. Let $T_\mu \subset \mathrm{Pic}^0(\mathcal{E}/\mathcal{U})$ be the variety of all $(E, [L])$ such that $[L]$ is of order exactly μ in $\mathrm{Pic}^0(E)$. Then the projection

$$T_\mu \longrightarrow \mathcal{U}$$

is an étale covering of degree

$$\mu^2 \prod (1 - 1/p^2),$$

where the product is taken over all prime divisors of μ. By the definition of $\mathcal{F}_{b,m}(\mu)$, the map

$$R = B + E \mapsto (E, [\mathcal{O}_E(D_R)]) \in \mathrm{Pic}^{3n}(\mathcal{E}/\mathcal{U}), \quad \text{where} \quad D_R = (B|_E)_{red},$$

gives a morphism

$$\psi : \mathcal{F}_{b,m}(\mu) \to \phi^{-1}(T_\mu).$$

Since

$$\dim \phi^{-1}(T_\mu) = \dim T_\mu = \dim \mathcal{U} = 9,$$

Proposition 2.1 follows from the following two claims:

Claim 2.2. The variety $\phi^{-1}(T_\mu)$ is irreducible.

Claim 2.3. The fiber $\psi^{-1}(E, [L])$ of ψ over an arbitrary point $(E, [L])$ of $\phi^{-1}(T_\mu)$ is irreducible and of dimension $(b-1)(b-2)/2 + 3n - 1$.

Let us prove Claim 2.2. Since T_μ and $\phi^{-1}(T_\mu)$ are isomorphic, it is enough to show the irreducibility of T_μ. We fix a base point $E_b \in \mathcal{U}$, and consider the natural monodromy action of $\pi_1(\mathcal{U}, E_b)$ on the set of elements of order μ in $\mathrm{Pic}^0(E_b)$. In order to prove Claim 2.2, it suffices to show that this action is transitive. Consider the monodromy action of $\pi_1(\mathcal{U}, E_b)$ on the free $\mathbb{Z}$-module $H_1(E_b, \mathbb{Z})$ equipped with the intersection form;

$$\pi_1(\mathcal{U}, E_b) \longrightarrow \mathrm{Aut}(H_1(E_b, \mathbb{Z})) \cong SL_2(\mathbb{Z}).$$

It is known that this homomorphism is surjective (see [5, Sec. 4]). Since the natural action of $SL_2(\mathbb{Z})$ on $(\mathbb{Z}/\mu\mathbb{Z})^2$ is transitive on the set of elements of order exactly μ, Claim 2.2 is proved.

Next we prove Claim 2.3. Let E be a non-singular cubic curve. Let $S^k(E)$ denote the symmetric product of k copies of E, which parameterizes the effective divisors of degree k on E, and let $S^k_{red}(E) \subset S^k(E)$ be the Zariski open dense subset consisting of reduced divisors. Then, for each positive integer k, the natural homomorphism

$$\tau_k : S^k(E) \rightarrow \mathrm{Pic}^k(E)$$

is surjective, and, for any $[M] \in \mathrm{Pic}^k(E)$, the fiber $\tau_k^{-1}([M])$ is canonically isomorphic to the projective space $\mathbb{P}_* H^0(E, M)$ of dimension $k-1$. Moreover,

$$S^k_{red}(E) \cap \tau_k^{-1}([M])$$

is a Zariski open dense subset of $\tau_k^{-1}([M]) \cong \mathbb{P}_* H^0(E, M)$.

Let $(E, [L])$ be a point of $\phi^{-1}(T_\mu)$; that is, the isomorphism class $[L \otimes \mathcal{O}_E(-nH)]$ is of order μ in $\mathrm{Pic}^0(E)$. If $R \in \psi^{-1}(E, [L])$, then $\mathcal{O}_E(D_R)$ is isomorphic to L. Therefore the map $R \mapsto D_R$ induces a morphism

$$\rho_{(E,[L])} : \psi^{-1}(E, [L]) \rightarrow S^{3n}_{red}(E) \cap \tau_{3n}^{-1}([L]) \subset \mathbb{P}_* H^0(E, L).$$

Since $S^{3n}_{red}(E) \cap \tau_{3n}^{-1}([L])$ is irreducible and of dimension $3n-1$, it is enough to prove that, for any reduced divisor D of E with $\mathcal{O}_E(D) \cong L$, the fiber

$$\rho^{-1}_{(E,[L])}(D) \cong \big\{ R = E + B \in \mathcal{F}_{b,m}(\mu) \,|\, D_R = (B|_E)_{red} = D \big\}$$

of $\rho_{(E,[L])}$ over the point $D \in S^{3n}_{red}(E) \cap \tau_{3n}^{-1}([L])$ is irreducible and of dimension $(b-1)(b-2)/2$.

Let D be a point of $S^{3n}_{red}(E) \cap \tau_{3n}^{-1}([L])$. Since $[L \otimes \mathcal{O}_E(-nH)]$ is of order μ in $\mathrm{Pic}^0(E)$, we have

$$\mathcal{O}_E(\mu D) \cong L^{\otimes \mu} \cong \mathcal{O}_E(n\mu H).$$

Note that the restriction map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n\mu)) \rightarrow H^0(E, \mathcal{O}_E(n\mu H))$$

is surjective. By the isomorphism $\tau^{-1}_{3\mu n}([L^{\otimes \mu}]) \cong \mathbb{P}_* H^0(E, L^{\otimes \mu})$, there exists a non-zero element $h \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n\mu))$ such that the plane curve defined by $h = 0$ cuts out the divisor μD on E. We choose and fix such a homogeneous polynomial h. We also choose and fix a homogeneous polynomial f of degree 3 defining the plane curve E. We put

$$\nu := m/\mu.$$

Claim 2.4
(1) If $R = B + E$ is a member of $\rho^{-1}_{(E,[L])}(D)$, then B is defined by an equation of the form

$$f \cdot g_R + h^\nu = 0,$$

where g_R is a homogeneous polynomial of degree $b-3$ that is uniquely determined by R.
(2) If g is a general homogeneous polynomial of degree $b-3$, and B is the curve defined by $f \cdot g + h^\nu = 0$, then $R = B + E$ is a member of $\rho^{-1}_{(E,[L])}(D)$.

Claim 2.4 implies that $\rho^{-1}_{(E,[L])}(D)$ is isomorphic to a Zariski open dense subset of the vector space $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b-3))$ via the map $R \mapsto g_R$. Hence the fiber $\rho^{-1}_{(E,[L])}(D)$ is irreducible and of dimension $(b-1)(b-2)/2$.

Now all we have to do is to prove Claim 2.4.

Let $R = B + E$ be a curve belonging to $\rho^{-1}_{(E,[L])}(D)$, and let $\gamma = 0$ be a defining equation of the curve B. Then $\gamma = 0$ and $h^\nu = 0$ cut out a same divisor $mD = \mu\nu D$ on E. Hence, after multiplying γ by a suitable non-zero constant, $\gamma - h^\nu$ vanishes on E. Therefore γ is of the form $f \cdot g_R + h^\nu$. It is easy to see that $g_R \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b-3))$ is uniquely determined by the curve B, when f and h are fixed.

Conversely, let g be an element of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b-3))$. It is obvious that, if the curve B defined by $f \cdot g + h^\nu = 0$ is non-singular, then $R = B + E$ is of type (b, m), and belongs to the subset $\rho^{-1}_{(E,[L])}(D)$ of the family $\mathcal{F}_{b,m}$. We shall show that B is non-singular if g is general. The family of curves

$$\{f \cdot g + h^\nu = 0\}_{g \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b-3))}$$

is a linear system whose base locus Γ is given by $f = h = 0$. There is an element $g_0 \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b-3))$ such that $g_0(P) \neq 0$ for any $P \in \Gamma$. Then the curve defined by $f \cdot g_0 + h^\nu = 0$ is non-singular at each point of Γ, because $E = \{f = 0\}$ is non-singular. Hence, by Bertini's theorem, a general member of this linear system is non-singular. ∎

3. Fundamental Group of the Complement to a Plane Curve

The second part (2) of Theorem 1.2 is proved by using the following proposition, which can be applied to more general situations.

Let $L \subset \mathbb{P}_* H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ be a projective linear subspace of dimension ≥ 1, and let $\{C_u\}_{u \in L}$ be the corresponding linear system of plane curves of degree d. We denote by Ξ the locus of all $u \in L$ such that C_u is not reduced.

Proposition 3.1. *Suppose that there exists a hyperplane $H_L \subset L$ with the following properties;*
(i) *$\Xi \setminus H_L$ is of codimension ≥ 2 in $L \setminus H_L$, and*
(ii) *$\pi_1(\mathbb{P}^2 \setminus C_v)$ is abelian for a general point v of H_L.*
Then $\pi_1(\mathbb{P}^2 \setminus C_u)$ is also abelian for a general $u \in L$.

Remark 3.2. Proposition 3.1 is trivial if $H_L \not\subset \Xi$. Indeed, the assumption $H_L \not\subset \Xi$ implies that C_v is reduced when $v \in H_L$ is general, and hence there is a surjective homomorphism

$$\pi_1(\mathbb{P}^2 \setminus C_v) \;\to\; \pi_1(\mathbb{P}^2 \setminus C_u)$$

for a general $u \in L$.

Proof. We choose a general pencil $P \subset L$, and take an affine coordinate t on P such that P intersects H_L at $t = \infty$. Let C_∞ be the curve in this pencil corresponding to the point $t = \infty$. By the assumption (ii), $\pi_1(\mathbb{P}^2 \setminus C_\infty)$ is abelian. Consider the affine part

$$A := P \setminus \{t = \infty\}$$

of P. By the assumption (i), $A \cap \Xi$ is empty. Let $\mathcal{C} \subset \mathbb{P}^2 \times A$ be the universal family of curves of degree d over the affine part A of the pencil. We put

$$\mathcal{V} := (\mathbb{P}^2 \times A) \setminus \mathcal{C}.$$

Since $A \cap \Xi = \emptyset$, we can apply the following theorem [14, Theorem 1] to the second projection $\mathcal{V} \to A$, and conclude that $\pi_1(\mathcal{V})$ is isomorphic to $\pi_1(\mathbb{P}^2 \setminus C_u)$ for a general $u \in A$.

Theorem 3.3. *Let Y be a non-singular connected projective variety, and let $\mathcal{Z}$ be a reduced effective divisor of $Y \times \mathbb{A}^n$. For a point $a \in \mathbb{A}^n$, we denote by Z_a the scheme-theoretic intersection of $\mathcal{Z}$ with $Y \times \{a\}$, and regard it as a subscheme of Y. Suppose that the locus*

$$\{a \in \mathbb{A}^N \mid Z_a \quad \textit{is not a reduced divisor of} \quad Y\}$$

is contained in a Zariski closed subset of $\mathbb{A}^n$ with codimension ≥ 2. Then, for a general $a \in \mathbb{A}^n$, the inclusion

$$Y \setminus Z_a \hookrightarrow (Y \times \mathbb{A}^n) \setminus \mathcal{Z}$$

induces an isomorphism on the fundamental groups.

Thus it is enough to show that $\pi_1(\mathcal{V})$ is abelian. Let

$$\mathrm{pr} : \mathcal{V} \to \mathbb{P}^2.$$

be the first projection. If p is a point of $\mathbb{P}^2 \setminus C_\infty$, then there exists a unique point $u(p)$ of A such that the curve $C_{u(p)}$ contains p, and hence the fiber of pr over p is isomorphic to A minus a point. Therefore, the projection

$$\mathcal{V} \setminus \mathrm{pr}^{-1}(C_\infty) \to \mathbb{P}^2 \setminus C_\infty$$

is a locally trivial fiber bundle with fibers isomorphic to a complex plane with one point punctured. Moreover, this projection has a section

$$s : \mathbb{P}^2 \setminus C_\infty \to \mathcal{V} \setminus \mathrm{pr}^{-1}(C_\infty)$$

given by

$$p \mapsto (p, \sigma(p)), \quad \text{where} \quad t(\sigma(p)) = t(u(p)) + 1.$$

Hence we have the splitting homotopy exact sequence

$$1 \to \pi_1(A \setminus \{\text{a point}\}) \to \pi_1(\mathcal{V} \setminus \mathrm{pr}^{-1}(C_\infty)) \to \pi_1(\mathbb{P}^2 \setminus C_\infty) \to 1.$$

The monodromy action of $\pi_1(\mathbb{P}^2 \setminus C_\infty)$ on $\pi_1(A \setminus \{\text{a point }\})$ associated with the section s is trivial, because $\pi_1(A \setminus \{\text{a point }\}) \cong \mathbb{Z}$ has a canonical positive generator preserved by the monodromy. Therefore the fundamental group $\pi_1(\mathcal{V} \setminus \mathrm{pr}^{-1}(C_\infty))$ is isomorphic to the product of $\pi_1(\mathbb{P}^2 \setminus C_\infty)$ and $\mathbb{Z}$, and hence is abelian. The inclusion of $\mathcal{V} \setminus \mathrm{pr}^{-1}(C_\infty)$ into $\mathcal{V}$ induces a surjective homomorphism from $\pi_1(\mathcal{V} \setminus \mathrm{pr}^{-1}(C_\infty))$ to $\pi_1(\mathcal{V})$. Thus $\pi_1(\mathcal{V})$ is also abelian. ■

4. The Complement to a Curve of Type (b, m)

We prove the assertion (2) of Theorem 1.2.

First note that it is enough to show the commutativity of $\pi_1(\mathbb{P}^2 \setminus R)$, because if $\pi_1(\mathbb{P}^2 \setminus R)$ is abelian, then it is isomorphic to the first homology group $H_1(\mathbb{P}^2 \setminus R, \mathbb{Z})$, and hence is easily calculated from the degrees of the irreducible components B and E of the curve R.

Let E be a non-singular cubic curve defined by $f = 0$. Then, as was shown in Claim 2.4, there exists a homogeneous polynomial h of degree $n\mu$ such that, for a general homogeneous polynomial $g \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b-3))$ of degree $b-3$, the curve $R = B + E$ defined by

$$f(f \cdot g + h^\nu) = 0$$

is a member of $\mathcal{F}_{b,m}(\mu)$. Let $\{e_1, \dots, e_N\}$ be a basis of the vector space $H^0(\mathbb{P}^2, \mathcal{O}(b-3))$, where $N = (b-1)(b-2)/2$. We consider the N-dimensional linear system L in $\mathbb{P}_* H^0(\mathbb{P}^2, \mathcal{O}(b+3))$ given by

$$\{X_1 f^2 e_1 + \cdots + X_N f^2 e_N + Y f h^\nu \mid (X_1 : \cdots : X_N : Y) \in \mathbb{P}^N\},$$

whose general member is a member of $\mathcal{F}_{b,m}(\mu)$. Let R_u be the curve corresponding to a point $u \in L$.

Let $H_L \subset L$ be the hyperplane defined by $Y = 0$. We will apply Proposition 3.1 to this situation, and prove the commutativity of $\pi_1(\mathbb{P}^2 \setminus R_u)$ for a general $u \in L$. For this purpose, it is enough to check the following two points;

- if v is a general point of H_L, then $\pi_1(\mathbb{P}^2 \setminus R_v)$ is abelian, and
- the locus $\Xi \setminus H_L$ of all $u \in L \setminus H_L$ such that R_u is non-reduced is contained in a Zariski closed subset of codimension ≥ 2 in $L \setminus H_L$.

If v is a general point of H_L, then $R_v = 2E + D$ where $D = \{g = 0\}$ is a curve defined by a general homogeneous polynomial g of degree $b-3$. Since the curve $E \cup D$ has only nodes as its only singularities, $\mathbb{P}^2 \setminus R_v = \mathbb{P}^2 \setminus (E \cup D)$ has an abelian fundamental group by Fulton-Deligne's theorem on Zariski conjecture [4, 6-8]. Thus the first item is checked.

Let w be a general point of an irreducible component of $\Xi \setminus H_L$. We calculate the dimension of the tangent space to $\Xi \setminus H_L$ at w. Since $w \notin H_L$, the curve R_w is given by an equation

$$f(f \cdot g_w + h^\nu) = 0$$

by some $g_w \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b-3))$.

Let ε be a dual number; $\varepsilon^2 = 0$. The tangent space of the projective space L at w can be naturally identified with the vector space $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b -$

3)). The first order deformation of R_w to the direction corresponding to $\delta_g \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b-3))$ is defined by

$$f(f(g_w + \varepsilon\delta_g) + h^\nu) = 0.$$

Since R_w is non-reduced, there exist an integer $l \geq 2$ and homogeneous polynomials a and b such that $\deg a \geq 1$ and

$$f(f \cdot g_w + h^\nu) = a^l b.$$

Suppose that δ_g is contained in the tangent space to Ξ. Since w is a general point of an irreducible component of $\Xi \setminus H_L$, there exist $\delta_a \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(\deg a))$ and $\delta_b \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(\deg b))$ such that

$$f(f(g_w + \varepsilon\delta_g) + h^\nu) = (a + \varepsilon\delta_a)^l (b + \varepsilon\delta_b).$$

This implies

$$a^{l-1} | f^2 \delta_g. \tag{4.1}$$

Suppose that the curve defined by $a = 0$ were equal to the irreducible curve $E = \{f = 0\}$. Then $f(fg_w + h^\nu) = a^l b$ would coincide with $f^l b$ modulo multiplicative constant, and hence $h^\nu|_E$ would be zero because of $l \geq 2$. This contradicts the required property of h. Therefore the curve $\{a = 0\}$ does not coincide with E, and hence there is an irreducible factor a' of a such that

$$\{a' = 0\} \not\subset E.$$

Therefore (4.1) implies that δ_g must be divided by the homogeneous polynomial a' of degree ≥ 1. Since

$$\dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b - 3 - \deg a')) \leq \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b-3)) - 2$$

for all $b \geq 4$, every irreducible component of $\Xi \setminus H_L$ is of codimension ≥ 2 in $L \setminus H_L$. Thus the second item is checked, and the assertion (2) of Theorem 1.2 is proved. ∎

5. The Case of $\mathcal{F}_{3,3}$

Let $R \subset \mathbb{P}^2$ be a curve of type $(3,3)$; that is,

(1) R consists of two irreducible components E and E',
(2) both of E and E' are non-singular cubic curves,
(3) E and E' intersect at three points $\{p_1, p_2, p_3\}$, and
(4) at each point p_i, E and E' intersect with multiplicity 3.

The singular locus of a curve of type $(3,3)$ consists of three points of type A_5. We put

$$M := \mathcal{O}_E(p_1 + p_2 + p_3 - H) \quad \text{and} \quad M' := \mathcal{O}_{E'}(p_1 + p_2 + p_3 - H'),$$

where H and H' are hyperplane sections of E and E', respectively. Then we have $M^{\otimes 3} \cong \mathcal{O}_E$ and $M'^{\otimes 3} \cong \mathcal{O}_{E'}$. Therefore $[M] \in \mathrm{Pic}^0(E)$ and $[M'] \in \mathrm{Pic}^0(E')$ are either of order 1 or 3. We can easily check that the following conditions are equivalent

- $[M]$ is of order 3 in $\mathrm{Pic}^0(E)$,
- $[M']$ is of order 3 in $\mathrm{Pic}^0(E')$, and
- p_1, p_2 and p_3 are not on a line.

Having observed this fact, we can prove the following proposition in the same way as Proposition 2.1.

Proposition 5.1. *The family $\mathcal{F}_{3,3}$ consists of two connected components $\mathcal{F}_{3,3}(1)$ and $\mathcal{F}_{3,3}(3)$. A member $R \in \mathcal{F}_{3,3}$ is a member of $\mathcal{F}_{3,3}(1)$ if and only if p_1, p_2 and p_3 are on a line.*

In this case, we can distinguish the two connected components by the fundamental group of the complement.

Proposition 5.2. *If R is a member of $\mathcal{F}_{3,3}(3)$, then $\pi_1(\mathbb{P}^2 \setminus R)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, while if R is a member of $\mathcal{F}_{3,3}(1)$, then $\pi_1(\mathbb{P}^2 \setminus R)$ is isomorphic to the free product $\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.*

Proof. Suppose that $R = E + E'$ is a member of $\mathcal{F}_{3,3}(3)$. Consider the pencil E_t of cubic curves spanned by $E_0 := E$ and $E_\infty := E'$. We put

$$R_t := E_\infty + E_t.$$

It is obvious that R_t is a member of $\mathcal{F}_{3,3}(3)$ for a general t. Since the base points p_1, p_2 and p_3 of this pencil are not on a line, R_t is reduced for all $t \neq \infty$. On the other hand, the complement $\mathbb{P}^2 \setminus R_\infty = \mathbb{P}^2 \setminus E'$ has an abelian fundamental group. Applying Proposition 3.1 to this pencil with $H_L = \{t = \infty\}$, we see that $\pi_1(\mathbb{P}^2 \setminus R_t)$ is abelian for a general t.

Suppose that $R = E + E'$ is a member of $\mathcal{F}_{3,3}(1)$. Then the pencil E_t defined above contains a multiple line $3D$ of multiplicity 3, where D is the line passing through the points p_1, p_2 and p_3. Therefore R is defined by an equation of the form

$$f(f + l^3) = 0$$

where $E = \{f = 0\}$ and $D = \{l = 0\}$. On the other hand, if $f \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ and $l \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ are chosen generally, then the equation $f(f + l^3) = 0$ defines a member of $\mathcal{F}_{3,3}(1)$. Thus we can calculate $\pi_1(\mathbb{P}^2 \setminus R)$ by the following theorem [16, Theorem 1].

Let $\mathbb{A}^n$ be an affine space with affine coordinates $(\xi_1, \ldots, \xi_n)$ of weights

$$\deg \xi_i := a_i \quad (i = 1, \ldots, n),$$

and let Σ be a hypersurface defined by a quasi-homogeneous equation

$$\Phi(\xi_1, \ldots, \xi_n) = 0$$

of total degree d. We let $\mathbb{C}^\times$ act on $\mathbb{A}^n \setminus \Sigma$ by

$$(\xi_1, \ldots, \xi_n) \mapsto (\lambda^{a_1}\xi_1, \ldots, \lambda^{a_n}\xi_n) \quad (\lambda \in \mathbb{C}^\times).$$

This action induces a homomorphism

$$\pi_1(\mathbb{C}^\times) \longrightarrow \pi_1(\mathbb{A}^n \setminus \Sigma). \tag{5.1}$$

If we choose a general element

$$F := (F_1, \dots, F_n) \in H^0(\mathbb{P}^2, \mathcal{O}(a_1)) \times \cdots \times H^0(\mathbb{P}^2, \mathcal{O}(a_n)),$$

then the equation

$$\Phi(F_1, \dots, F_n) = 0$$

defines a projective plane curve $C_F \subset \mathbb{P}^2$ of degree d.

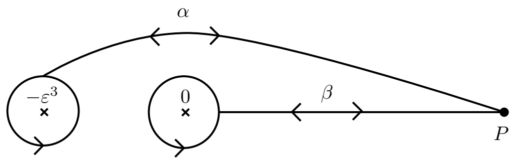

Figure 5.1. Generators of $\pi_1(p^{-1}(\varepsilon), P)$

Theorem 5.3. *Suppose that Σ is reduced, and that F is general. Then $\pi_1(\mathbb{P}^2 \setminus C_F)$ is isomorphic to the cokernel of the homomorphism* (5.1).

Consider the affine plane $\mathbb{A}^2$ with the affine coordinates (x, y) of weights

$$\deg x = 3 \quad \text{and} \quad \deg y = 1$$

Let W be the affine curve defined by the quasi-homogeneous equation

$$x(x + y^3) = 0$$

in $\mathbb{A}^2$. We consider the action of $\mathbb{C}^\times$ on the complement $\mathbb{A}^2 \setminus W$ given by

$$(x, y) \mapsto (\lambda^3 x, \lambda y),$$

where $\lambda \in \mathbb{C}^\times$. We choose a base point of $\mathbb{A}^2 \setminus W$ to be

$$P := (1, \varepsilon),$$

where ε is a sufficiently small positive real number. Let

$$p : \mathbb{A}^2 \setminus W \to \mathbb{A}^1$$

be the projection given by $(x, y) \mapsto y$. Then p is locally trivial over $\mathbb{A}^1 \setminus \{0\}$. We choose generators α and β of $\pi_1(p^{-1}(\varepsilon), P)$, which is a free group of rank 2, as in Figure 5.1. Then monodromy action on $\pi_1(p^{-1}(\varepsilon), P)$ along a small circle on $\mathbb{A}^1 \setminus \{0\}$ with the center 0 is as follows;

$$\alpha \mapsto (\alpha\beta)^3 \alpha (\alpha\beta)^{-3}, \quad \beta \mapsto (\alpha\beta)^3 \beta (\alpha\beta)^{-3},$$

because, when

$$a := \varepsilon \exp(2\pi i t)$$

moves along the small circle on $\mathbb{A}^1 \setminus \{0\}$ with the center 0, the point $x = -\varepsilon^3 \exp(6\pi i t)$, which is one of the two deleted points

$$x = 0 \quad \text{and} \quad x = -\varepsilon^3 \exp(6\pi i t)$$

of $p^{-1}(a)$ draws a small circle of radius ε^3 around the other fixed deleted point $x = 0$ three times. Hence, by the classical theorem of Zariski-van Kampen, we have

$$\pi_1(\mathbb{A}^2 \setminus W) \cong \langle \alpha, \beta \mid (\alpha\beta)^3 = (\beta\alpha)^3 \rangle,$$

On the other hand, the image of the homomorphism $\pi_1(\mathbb{C}^\times) \to \pi_1(\mathbb{A}^2 \setminus W)$ induced from the action of $\mathbb{C}^\times$ on $\mathbb{A}^2 \setminus W$ is generated by $(\alpha\beta)^3$, because the loop

$$(\lambda^3, \lambda\varepsilon) \in \mathbb{A}^2 \setminus W, \quad (\lambda = \exp(2\pi i t),\ t \in [0,1])$$

which is the orbit of P under the action of $\{\lambda \in \mathbb{C}^\times \mid |\lambda| = 1\}$, represents $(\alpha\beta)^3$ in $\pi_1(\mathbb{A}^2 \setminus W)$. Hence, by Theorem 5.3, we obtain

$$\pi_1(\mathbb{P}^2 \setminus R) \cong \langle \alpha, \beta \mid (\alpha\beta)^3 = (\beta\alpha)^3 = 1 \rangle \cong \mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

for all $R \in \mathcal{F}_{3,3}(1)$. ■

Let $R = E + E'$ be a member of $\mathcal{F}_{3,3}$. We choose a flex o of E, and consider the elliptic curve (E, o). We let the divisor $p_1 + p_2 + p_3$ on E approach to a non-reduced divisor $3q$ for some $q \in E$ in such a way that the limit curve $\tilde{R}$ consists of two non-singular cubic curves E and $\tilde{E}'$ touching to each other at only one point with multiplicity 9. Then $\tilde{R}$ is a sextic curve with only one singular point of type A_{17}. If $R \in \mathcal{F}_{3,3}(1)$, then q must be a point of order 1 or 3 of (E, o), while if $R \in \mathcal{F}_{3,3}(3)$, then q must be a point of order 9 on (E, o). These two limit curves form a Zariski pair, which is just one of the examples given by Artal Bartolo in [1].

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