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Some Comparison Results for Sequential Martingales in the Limits*

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Abstract. The class of sequential martingales in the limit is one of the best generalizations of martingales for which Riesz–Talagrand's decomposition and Doob-Chatterji's martingale limit theorem still hold. The main aim of this note is to give some constructive comparision theorems for the class.

1. Notations and Definitions

In the sequel, let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{A}$. By $L^1(\mathcal{F})$ we mean the Banach space of all (equivalence classes of) $\mathcal{F}$-measurable random variables (r.v'.s) X with

$$E(|X|) = \int_\Omega |X(\omega)| dP(\omega) < \infty.$$

For an increasing sequence of complete sub-σ-algebras $(\mathcal{A}_n)$ with $\mathcal{A}_n \uparrow \mathcal{A}$, we shall denote by T the set of all bounded stopping times with respect to $(\mathcal{A}_n)$. Then clearly, T becomes a directed set with the usual order "$\leq$", given by $\sigma \leq \tau$ if and only if $\sigma(\omega) \leq \tau(\omega)$, a.s. and the set N of all positive integers can be thus regarded as a cofinal subset of T.

From now on we shall consider only sequences (X_n) of r.v'.s such that each $X_n \in L^1(\mathcal{A}_n)$. This implies that with τ running over T, every such a sequence (X_n) induces an directed process (X_τ), where $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$, $\omega \in \Omega$ adapted to the increasing family of complete sub-σ-algebras of $\mathcal{A}$, given by

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$$\mathcal{A}_\tau = \{A \in \mathcal{A},\ A \cap \{\tau = n\} \in \mathcal{A}_n,\ n \in N\}.$$

For other related notions we refer to standard monographs of Neveu [7] and Edgar and Sucheston [2]. Here we are interested only in the following:

Definition 1.1. *A sequence (X_n) is said to be*
a) *a game which becomes fairer with time, if for every $\varepsilon > 0$, there exists $p \in N$ such that for every $n \geq p$ we have*

$$\sup_{p \leq q \leq n} P(|X_q(n) - X_q| > \varepsilon) < \varepsilon,$$

where given $\tau, \sigma \in T$, the notation $X_\sigma(\tau)$ denotes the $\mathcal{A}_\sigma$-conditional expectation of X_τ.
b) *a mil, if for every $\varepsilon > 0$, there exists $p \in N$ such that for every $n \geq p$ we have*

$$P\Big(\sup_{p \leq q \leq n} |X_q(n) - X_q| > \varepsilon\Big) < \varepsilon.$$

Although the class of games fairer with time introduced earlier by Blake (1970) is strictly larger than that of mils, later considered by Talagrand (1985), the Doob–Chatterji's martingale limit theorem still holds true however only for mils. Especially, it is worth noting that all the elegant structure results of Talagrand [8] can be extended further to the following important generalizations of mils.

Definition 1.2. *A sequence (X_n) is said to be*
a) *a quasi-martingale in the limit (briefly, a quasi-mil), if for every $\varepsilon > 0$, there exists $p \in N$ such that for every $m \geq p$, there exists $p_m \geq m$ such that for all $n \geq p_m$ we have*

$$P\Big(\sup_{p \leq q \leq m} |X_q(n) - X_q| > \varepsilon\Big) < \varepsilon. \tag{1.1}$$

b) *a Γ-martingale in the limit (briefly, a Γ-mil), where Γ is a cofinal subset of T, if for every $\varepsilon > 0$ there exists $p \in N$ such that for all $\tau \in T$ with $\tau \geq p$ we have*

$$P\Big(\sup_{p \leq q \leq \tau} |X_q(\tau) - X_q| > \varepsilon\Big) < \varepsilon. \tag{1.2}$$

In general, for any cofinal subset Π of T, (X_n) is said to be a Π-*sequential mil*, if there exists a cofinal sequence (σ_n) of Π such that (X_n) is a $\{\sigma_n\}$-mil, where $\{\sigma_n\}$ denotes the set of all elements of (σ_n). In particular, if $\Pi = T$ then such a sequence will be called simply *a sequential mil*.

Clearly, by the same definitions, every mil is both a quasi-mil and a sequential mil. The main aim of this note is to establish further comparison results for these two latter classes of martingale-like sequences.

2. Main Results

To complete the recent considerations given in [4, 5] by the first author of this note, we start with the following example which shows that unlike mils the last two notions are independent of games fairer with time.

Example 2.1. In general, neither the class of quasi-mils nor that of N-sequential mils is contained each in the other. Furthermore there exists an L^1-bounded sequence of real-valued r.v'.s which is both an N-sequential mil and a quasi-mil but it fails to be a game which becomes fairer with time.

Construction. Let $([0,1], \mathcal{B}_{[0,1)}, P)$ be the Lebesgue probability space on $[0,1)$, where $\mathcal{B}_{[0,1)}$ is the completion of the Borel σ-field with respect to the Lebesgue measure in P. For $n \geq 0$, let $a_n = \prod_{j=0}^{n} 2^j$, Q_n the partition of $[0,1)$ by a_n equal intervals $I_j^{(m)}$, $1 \leq j \leq a_n$ and $\mathcal{A}_n$ the σ-field generated by Q_n.

First define the sequence (X_n) of nonnegative real-valued functions by: For $n = 2p-1$, $p \in N$ set $X_n = 0$. For $n = 2p$, $p \in N$ define X_n as follows: Given an interval I of Q_{n-1}, set $X_n = 2^n$ on the first interval of Q_n which is contained in I and $X_n = 0$ elsewhere. It is easily seen that defined in such a way, $E(X_n) \leq 1$ and $P(X_n \neq 0) \leq 2^{-n}$. Then (X_n) converges to zero, a.s. But for every $n \in N_1 = \{2p-1,\ p \in N\}$ we have $X_q(n) = 0$ with $q < n$. It follows that (X_n) is an N_1-mil, hence an N-sequential mil. On the other hand, for every $n \in N \setminus N_1$ we have $X_q(n) = 1$ with $q < n$. Then by the a.s. convergence to zero of (X_n), it follows that (X_n) cannot be a quasi-mil. This proves that the class of all N-sequential mils is not contained in that of all quasi-mils.

Now let define the sequence (Y_n) of real-valued r.v'.s by: For $n < 2$, set $Y_n = 0$. For $n = 2p$ with $p \geq 1$ let define

$$Y_n = 2^n \quad \text{or} \quad Y_n = -2^n, \text{ resp.}$$

on the first interval of Q_n which is contained in I_{2k-1}^{n-1} or on I_{2k}^{n-1}, resp., with $1 \leq k \leq \frac{a_{n-1}}{2}$ and $Y_n = 0$ elsewhere. Then $E(|Y_n|) \leq 1$ and $P(Y_n \neq 0) \leq 2^{-n}$. Thus (Y_n) converges also to zero, a.s. But by the definition of (Y_n) we have

$$Y_{n-1}(n) = 1 \quad \text{on} \quad I_{2k-1}^{n-1} \quad \text{and} \quad Y_{n-1}(n) = -1 \quad \text{on} \quad I_{2k}^{n-1} \tag{2.1}$$

with $1 \leq k \leq \frac{a_{n-1}}{2}$. Hence (Y_n) is not an N-sequential mil. However, for all $n, q \in N$ with $n \geq 4$ and $q \leq n-2$ we have $Y_q(n) = 0$. Therefore by taking $p_m = m+2$, $m \in N$ we infer that for any $p, q, m, n \in N$ with $4 \leq p \leq q \leq m$ and $p_m \leq n$, the following holds

$$|Y_q(n) - Y_q| = |Y_q|.$$

This with the almost sure convergence to zero of (Y_n) shows that (Y_n) must be a quasi-mil. It proves the first statement given in the example.

Finally, set $Z_n = X_n$ for $n = 2p-1$, $p \in N$ and $Z_n = Y_n$ for $n = 2p$, $p \in N$. Then naturally, $E(|Z_n|) \leq 1$, (Z_n) converges to zero, a.s. and (Z_n) is even both an N-sequential mil and a quasi-mil. But by (2.1) and the almost sure convergence to zero of (Z_n) implies that (Z_n) cannot be a game fairer with time. It completes the construction of the example.

Now let G denote the set of all nondecreasing functions from N to N. Then endowed with the partial order $(\leq')$, defined by $f =' g$ if and only if $\operatorname{card}(\{f \neq g\}) < \infty$ and $f <' g$ if and only if $\operatorname{card}(\{f > g\}) < \infty$ and $\operatorname{card}(\{f < g\}) = \infty$, G becomes a directed set. Moreover, we have shown in [6] that *a sequence* (X_n) *is a quasi-mil if and only if there exists some* $g \in G$ *such that* (X_n) *is a mil of size* g, i.e., for every $\varepsilon > 0$ there exists $p \in N$ such that for all $m, n \in N$ with $m \geq p$ and $n \geq m + g(m)$ the inequality (1.1) is satisfied. To better imagine how large is the class of quasi-mils, the interested reader is refered to [4].

Having the previous example in hand, the remaining problem is to establish a best possible inclusion relation between the class of quasi-mils and that of sequential mils. To solve the problem we shall consider carefully in turn all possible situations of a probability space $(\Omega, \mathcal{A}, P)$ with its stochastic basic $(\mathcal{A}_n)$. For the first "almost general" situation as we shall see later, we get the following.

Theorem 2.2. *Suppose that there exists a decreasing sequence* (A_n) *adapted to* $(\mathcal{A}_n)$ *such that each* $P(A_n) > 0$ *and* $\lim_n P(A_n) = 0$. *Then every quasi-mil is a sequential-mil.*

Proof. To prove the theorem, it is useful to note that for a given quasi-mil (X_n), one can construct a strictly increasing sequence $(p(n))$ of N such that for every $k \in N$, there exists an increasing sequence $(p_n(k))$ such that for every $m \geq p(k)$ and $n \geq p_m(k)$ we have

$$P\Big(\max_{p(k) \leq q \leq m} |X_q(n) - X_q| > \frac{1}{k}\Big) < \frac{1}{k}.$$

Now let (A_n) be a sequence of $\mathcal{A}$ with all the properties given in the theorem. We define the increasing cofinal sequence (τ_n) of bounded stopping times by: For $m < p(1)$, set $\tau_m = m$. For any other m, say $p(j) \leq m < p(j+1)$, $j \in N$ set

$$k_m = \max\{p_m(i),\ i \leq j\},$$

and

$$\tau_m = \begin{cases} p(j) & \text{on} \quad A_{p(j)} \\ k_m & \text{elsewhere.} \end{cases}$$

We shall show that (X_n) becomes a $\{\tau_n\}$-mil, hence a sequential mil. For this purpose, let $0 < \varepsilon < 1$ be given. By the property of the sequence (A_n), there exists $k \in N$ such that $\frac{\varepsilon}{2} > \frac{1}{k}$, $P(A_k) < \frac{\varepsilon}{2}$ and for every $m, n \in N$ with $p(k) \leq m$ and $p_m(k) \leq n$ we have

$$P\Big(\max_{p(k) \leq q \leq n} |X_q(n) - X_q| > \frac{1}{k}\Big) < \frac{1}{k}. \tag{2.2}$$

But since $p(k) \leq m$, there exists a unique nonnegative integer k' such that $p(k+k') \leq m < p(k+k'+1)$. Then by the definition of τ_m, it follows from (2.2) that

$$\begin{aligned} &P\Big(\max_{p(k) \leq q \leq \tau_m} |X_q(\tau_m) - X_q| > \varepsilon\Big) \\ &= P\Big(\max_{p(k) \leq q \leq p(k+k')} |X_q(\tau_m) - X_q| > \varepsilon\Big) \end{aligned}$$

$$\le P\Big(\max_{p(k)\le q\le p(k+k')}|X_q(\tau_m)-X_q|>\frac{1}{k}\Big)$$
$$\le P(A_{p(k)})+P\Big(\max_{p(k)\le q\le p(k+k')}|X_q(I_{A^c_{p(k)}}X_{\tau_m})-I_{A^c_{p(k)}}X_q|>\frac{1}{k}\Big)$$

where given $A\in\mathcal{A}$, I_A denotes the characteristic function of A and $A^c=\Omega\setminus A$. But on $A^c_{p(k)}\subset A^c_{p(k+k')}$ we have $\tau_m=k_m$. Then by (2.2) one obtains

$$P\Big(\max_{p(k)\le q\le \tau_m}|X_q(\tau_m)-X_q|>\varepsilon\Big)$$
$$\le P(A_{p(k)})+P\Big(\max_{p(k)\le q\le p(k+k')}|X_q(I_{A^c_{p(k)}}X_{\tau_m})-I_{A^c_{p(k)}}X_q|>\frac{1}{k}\Big)$$
$$=P(A_{p(k)})+P\Big(I_{A^c_{p(k)}}\cdot\max_{p(k)\le q\le p(k+k')}|X_q(\tau_m)-X_q|>\frac{1}{k}\Big)$$
$$\le P(A_{p(k)})+P\Big(\max_{p(k)\le q\le p(k+k')}|X_q(k_m)-X_q|>\frac{1}{k}\Big)$$
$$<\frac{\varepsilon}{2}+\frac{1}{k}<\varepsilon.$$

This with (1.2) proves that (X_n) is a $\{\tau_n\}$-mil, hence a sequential mil, which completes the proof.

Having the theorem, a natural question is: how general is the assumption of the theorem? Before going to give an answer to this question, we need some additional notations. Indeed, let $\mathcal{A}$ and $(\mathcal{A}_n)$ be given as assumed in the first section. Let denote by $\mathcal{B}$ the collection of all atoms of $\mathcal{A}$ which do not belong to $\bigcup_n\mathcal{A}_n$. By $\mathcal{B}_1$ we mean the subset of all the elements $B\in\mathcal{B}$ such that there exists some $A\in\bigcup_n\mathcal{A}_n$ with

$$P(B\cap A)=P(B)\text{ and }P(A\cap C)=0,\ C\in\mathcal{B}\setminus\{B\}.$$

Since all $\mathcal{A}$ and $\mathcal{A}_n$ have been assumed to be complete, so instead of the above equalities, we write sometimes $B\subset A$ or $A\cap C=\emptyset$, resp. After a carefull analysis of $(\mathcal{A})$ and $(\mathcal{A}_n)$, we get the following some corollaries which show the previous theorem to be one of the best solution of the posed problem.

Corollary 2.3. *Suppose that either $\mathcal{A}$ is nonpurely atomic or there exists some $k\in N$ such that $\mathcal{A}_k$ is infinitely generated. Then the assumption of the Theorem* 2.2 *is satisfied. Hence every quasi-mil is a sequential mil.*

Proof. Suppose first that $\mathcal{A}$ is nonpurely atomic. Then $\mathcal{A}$ contains a nonatomic part B_0 with $P(B_0)>0$. So we can find $k_0\in N$ and $C_0\in\mathcal{A}_{k_0}$ with

$$P(C_0\Delta B_0)\le\frac{P(B_0)}{2},\text{ hence }0<P(C_0\cap B_0)\le P(B_0).$$

Since $C_0\cap B_0$ is nonatomic, there exists some $B_1\subset C_0\cap B_0$ with $0<P(B_1)\le\frac{P(B_0)}{2}$. Then we can find $k_1>k_0$ and $C_1\in\mathcal{A}_{k_1}$ with

$$P(C_1\Delta B_1)\le\frac{P(B_1)}{2},\text{ hence }0<P(C_1\cap B_1)\le 2^{-1}P(B_0).$$

Thus by induction, one can construct a sequence (B_n) of $\mathcal{A}$-measurable subsets of B_0 and a sequence (C_k) adapted to (A_{n_k}) for some strictly increasing sequence (k_n) of N such that for every $n \in N$ we have

$$B_n \subset B_{n-1} \cap C_{n-1}, \quad 0 < P(B_n) \le 2^{-n} P(B_0),$$
$$P(C_n \Delta B_n) \le \frac{P(B_{n-1})}{2}, \text{ and } 0 < P(C_n \cap B_n) \le 2^{-n} P(B_0).$$

Consequently, the decreasing sequence (D_n) defined by

$$D_n = \cap\{C_j,\ j \le n\},\ n \in N$$

is adapted to (A_{k_n}) and such that

$$0 \le \lim_n P(D_n) \le \lim_n P(C_n) \le \lim_n \frac{3P(B_n)}{2} = 0.$$

But for every $n \in N$, $P(D_n) \ge P(B_{n+1}) > 0$. Therefore, if we define the sequence (A_n) by

$$A_n = \begin{cases} \Omega & ,\ n < k \\ D_s & ,\ k_s \le n < k_{s+1},\ n \in N \end{cases}$$

then clearly (A_n) satisfies the assumption of the Theorem 2.2. Hence, the corollary follows. Further, if $\mathcal{A}$ is purely atomic and the second situation appears, then $\mathcal{A}_k$ is generated by a sequence (C_n) of all its different atoms. We shall prove the corollary by defining the sequence (A_n) as follows:

$$A_n = \begin{cases} \Omega & ,\ n < k \\ \cup(C_j,\ j \ge n) & ,\ n \ge k. \end{cases}$$

Then it is easily seen that the sequence (A_n) satisfies also the assumption of Theorem 2.2. This proves the corollary. ∎

Corollary 2.4. *Let $\mathcal{A}$ be purely atomic and infinitely generated. Suppose that either $\mathcal{B} = \emptyset$ or $\mathcal{B} = \mathcal{B}_1$ with* card $(\mathcal{B}) = \infty$*. Then the assumption of Theorem* 2.2 *is satisfied. Hence, every quasi-mil is a sequential mil.*

Proof. Let $\mathcal{A}$ be given in the corollary. By the previous corollary we would assume in addition that each card $(\mathcal{A}_n) < \infty$. Thus by the finite generatedness of each $\mathcal{A}_n$, it follows that every set A_n, given by

$$A_n = \Omega \setminus \cup\{A \in a(\mathcal{A}),\ A \in \mathcal{A}_n\}$$

belongs to $\mathcal{A}_n$ and $P(A_n) > 0$, where $a(\mathcal{A})$ denotes the set of all atoms of $\mathcal{A}$. Therefore, for the first case when $\mathcal{B} = \emptyset$, the sequence (A_n) automatically satisfies the assumption of the Theorem 2.2. Thus the corollary follows.

Suppose now that $\mathcal{B} = \mathcal{B}_1$ with card $(\mathcal{B}) = \infty$. Then $\mathcal{B}_1$ consists of a sequence (B_n) of all its elements. We claim first that card $(\mathcal{A} \setminus \mathcal{B}) = \infty$. Indeed, if card $(a(\mathcal{A}) \setminus \mathcal{B}) = \emptyset$ then $\Omega = \underset{n}{\cup} B_n$. By the definition of $\mathcal{B}_1$, there exists for example some $A \in \underset{n}{\cup} \mathcal{A}_n$ such that A contains no more elements of $\mathcal{B}$ except B_1. But since either $P(A \cap B_n) = 0$ or $P(A \cap B_n) = P(B_n)$ and $A = \underset{n}{\cup} A \cap B_n$, we have

$$P(A \cap B_n) = 0, \; n \neq 1$$

and

$$P(A \cap B_1) = P(A).$$

This also means that $B_1 = A \in \underset{n}{\cup} \mathcal{A}_n$ because of the completeness of each $\mathcal{A}_n$. It contradicts the definition of $B_1 \in \mathcal{B}$. In the next possibility when $a(\mathcal{A}) \setminus \mathcal{B} = (A_1, ..., A_p), \; p \geq 1$ there exists $A \in \underset{n}{\cup} \mathcal{A}_n$ such that

$$P(A \Delta B_1) < \min\left(P(B_1), P(A_j), \; j \leq p\right),$$

since $\mathcal{A}_n \uparrow \mathcal{A}$. Then

$$P(A \cap A_j) = 0, \; j \leq p \text{ and } P(A \cap B_1) \neq \emptyset.$$

Hence $P(B_1 \cap A) = P(B_1)$. But

$$P(A) = \sum_{j=1}^{p} P(A \cap A_j) + \sum_{j=1}^{\infty} P(A \cap B_j).$$

It follows that we have also $P(A \cap B_j) = 0, \; j \neq 1$. It means that

$$P(B_1) = P(A \cap B_1) = P(A).$$

In other words, $B_1 = A \in \underset{n}{\cup} \mathcal{A}_n$. This is impossible because of the definition of $B_1 \in \mathcal{B}$. The claim is therefore justified.

Having the claim in hand, we shall finish the proof for this case by the following construction. Let (C_n) denote a sequence of all elements of $a(\mathcal{A}) \setminus \mathcal{B}$. Clearly, by the definition of $\mathcal{B}$, each $C_p \in \underset{n}{\cup} \mathcal{A}_n$ and

$$\sum_n P(B_n) + \sum_p P(C_p) = 1. \tag{2.3}$$

On the other hand, by the definition of $\mathcal{B}_1$, there exists a strictly increasing sequence (k_n) of N and a sequence (D_n) such that for every $n \in N$, $C_n, D_n \in \mathcal{A}_{k_n}$ and each D_n contains no more elements of $\mathcal{B}$ except B_n. Consequently, by setting each

$$E_n = \Omega \setminus \Big[\big(\bigcup_{j=1}^{n} D_j\big) \cup \big(\bigcup_{j=1}^{n} C_j\big)\Big],$$

we see that (E_n) is decreasing, each $P(E_n) > 0$ since $B_j \subset E_n, \; j > n$ and

$$P(E_n) \leq \sum_{j>n} P(B_j) + \sum_{j>n} P(C_j).$$

Hence, by (2.3) we have $\lim\limits_n P(E_n) = 0$. Thus if we set

$$A_p = \begin{cases} \Omega, & p < k_1 \\ E_n, & k_n \leq p < k_{n+1}, \; n \in N \end{cases}$$

then by all the aforementioned properties of (E_n), we can conclude that the sequence (A_n) just constructed satisfies the assumption of the Theorem 2.2.

Consequently, the corollary also follows.

Now for simplicity in formulating the next important corollary, let denote by $\mathcal{C}_1$ the collection of all the elements $C \in (\mathcal{B} \setminus \mathcal{B}_1)$, for which there exists some $A \in \underset{n}{\cup} \mathcal{A}_n$ such that A contains no more elements of $(\mathcal{B} \setminus \mathcal{B}_1)$ but C. With the additional notation, we get the following.

Corollary 2.5. *Let $\mathcal{A}$ be purely atomic. Suppose that either $\mathcal{B} \setminus \mathcal{B}_1 = \mathcal{C}_1$ with* $\text{card}\,(\mathcal{C}_1) = \infty$ *or* $\text{card}\,(\mathcal{B}) = \infty$ *with* $\text{card}\,(\mathcal{B}_1) < \infty$*. Then the assumption of Theorem* 2.2 *is satisfied. Hence every quasi-mil is a sequential mil.*

Proof. Let $\mathcal{A}$ be as given in the corollary. Suppose first that $\mathcal{B} \setminus \mathcal{B}_1 = \mathcal{C}_1$ with $\text{card}\,(\mathcal{C}_1) = \infty$. Then $\mathcal{C}_1$ consists of a sequence (C_n) of all its elements. We claim in addition that if $\mathcal{C}_1 \neq \emptyset$ then $\text{card}\,(\mathcal{B}_1) = \infty$. Indeed, let $C \in \mathcal{C}_1$. Then by definition, there exists some $A \in \underset{n}{\cup} \mathcal{A}_n$ such that A contains no more elements of $(\mathcal{B} \setminus \mathcal{B}_1)$ except C. Suppose that A contains only a finite number of elements of $\mathcal{B}_1$. Then clearly A contains also only a positive finite number of elements of $\mathcal{B}$, say $B_1, B_2, ..., B_p$ with $1 \leq p < \infty$ and $B_1 = C$. But if $p = 1$, then by definition $C \in \mathcal{B}_1$. It contradicts the definition of $C \in \mathcal{C}_1$. On the other hand, if $p > 1$ then because $\mathcal{A}_n \uparrow \mathcal{A}$ there exists $A_1 \in \underset{n}{\cup} \mathcal{A}_n$ such that $A_1 \subset A$ and

$$P(A_1 \Delta C) < \min\,(P(B_j),\ j \leq p).$$

Consequently, A_1 does not contain any more elements of $\mathcal{B}$ except C. It means also that $C \in \mathcal{B}_1$ which contradicts the fact that $C \in \mathcal{C}_1$. Therefore A should contain an infinite number of elements of $\mathcal{B}_1$. Hence, the claim follows.

Let (B_n) be a sequence of all elements of $\mathcal{B}_1$. We conclude next that $\text{card}\,(a(\mathcal{A}) \setminus \mathcal{B}) = \infty$. Indeed, if $\text{card}\,(a(\mathcal{A}) \setminus \mathcal{B}) = 0$ then $\Omega = (\underset{n}{\cup} B_n) \cup (\underset{n}{\cup} C_n)$. On the other hand, by the definition of $B_1 \in \mathcal{B}_1$, there exists some $A \in \underset{n}{\cup} \mathcal{A}_n$ such that A contains no more elements of $\mathcal{B}$ except B_1. But since either

$$P(A \cap B_n) = 0 \text{ or } P(A \cap B_n) = P(B_n)$$

and either

$$P(A \cap C_n) = 0 \text{ or } P(A \cap C_n) = P(C_n)$$

it follows that

$$P(A \cap B_n) = 0,\ n \neq 1 \text{ and } P(A \cap C_n) = 0,\ n \in N.$$

Hence $P(A \cap B_1) = P(B_1) = P(A)$.

It also means that $B_1 = A \in \underset{n}{\cup} \mathcal{A}_n$. This contradicts the fact that $B_1 \in \mathcal{B}$. Further, if $a(\mathcal{A}) \setminus \mathcal{B} = \{A_1, A_2, ..., A_p\},\ p \geq 1$ then there exists some $A \in \underset{n}{\cup} \mathcal{A}_n$ such that

$$P(A \Delta B_1) < \min\,(P(B_1), P(A_j),\ j \leq p).$$

It means that $P(A \cap B_1) = P(B_1)$ and $P(A \cap A_j) = 0,\ j \leq p$. On the other hand we have also

$$P(A \cap B_n) = 0,\ n \neq 1 \text{ and } P(A \cap C_n) = 0,\ n \in N.$$

Then $P(A \cap B_1) = P(B_1) = P(A)$. It implies $B_1 \in \underset{n}{\cup} \mathcal{A}_n$, contradicting the fact that $B \in \mathcal{B}_1$. Therefore, the conclusion is verified. Let (D_n) denote thus a sequence of all elements of $(a(\mathcal{A}) \setminus \mathcal{B})$. It is clear that the three sequences (B_n), (C_n) and (D_n) perform a partition of Ω. Furthermore by the definition of $\mathcal{B}_1$ and $\mathcal{C}_1$, it follows that there exists a strictly increasing sequence (k_n) of N and two sequences (A_n^1) and (A_n^2) of $\underset{n}{\cup} \mathcal{A}_n$ such that all three sequences (D_n), (A_n^1) and (A_n^2) are adapted to $(\mathcal{A}_{k_n})$, each element A_n^1 contains no more elements of $\mathcal{B}$ except B_n and each A_n^2 contains no more elements of $\mathcal{B} \setminus \mathcal{B}_1$ except C_n. Consequently, the sequence (E_n), given by

$$E_n = \Omega \setminus \Big[\big(\bigcup_{j \le n} D_j \big) \cup \big(\bigcup_{j \le n} A_j^1 \big) \cup \big(\bigcup_{j \le n} A_j^2 \big) \Big]$$

is decreasing and adapted to $(\mathcal{A}_{k_n})$, with each $P(E_n) > 0$ since E_n contains all C_j with $j > n$. Furthermore, $\lim_n P(E_n) = 0$. Thus if we choose the sequence (A_n), defined by

$$A_n = \begin{cases} \Omega, & n < k_1 \\ E_p, & k_p \le n < k_{p+1},\ p \in N \end{cases}$$

then (A_n) satisfies the assumption of Theorem 2.2. Hence the corollary follows.

Suppose now that $\operatorname{card}(\mathcal{B}) = \infty$ and $\operatorname{card}(\mathcal{B}_1) < \infty$. We claim that if $B \in \mathcal{B} \setminus \mathcal{B}_1$ and $A \in \underset{n}{\cup} \mathcal{A}_n$ with $B \subset A$ then A contains even an infinite number of elements of $\mathcal{B} \setminus \mathcal{B}_1$. Indeed, suppose on the contrary that A contains only a positive finite number of elements of $\mathcal{B} \setminus \mathcal{B}_1$. Then by the assumption that $\operatorname{card}(\mathcal{B}_1) < \infty$ it follows that A also contains only a positive finite number of elements, say $\{B_1, ..., B_p\}$ of $\mathcal{B}$ with $B_1 = B$. But if $p = 1$ then by definition $B \in \mathcal{B}_1$. It contradicts the fact that $B \in \mathcal{B} \setminus \mathcal{B}_1$. Therefore $p > 1$. However, since $\mathcal{A}_n \uparrow \mathcal{A}$, there exists $A_1 \in \underset{n}{\cup} \mathcal{A}_n$ such that

$$P(A_1 \Delta B) < \min\left(P(B_j),\ j \le p\right).$$

This implies

$$B \subset A_1 \text{ and } P(A_1 \cap B_j) = 0,\ 2 \le j \le p.$$

Consequently, the set $A \cap A_1 \in \underset{n}{\cup} \mathcal{A}_n$ contains no more elements of $\mathcal{B}$ except B. It means also that $B \in \mathcal{B}_1$. This again contradicts the assumption that $B \in \mathcal{B} \setminus \mathcal{B}_1$. Thus in any case, A should contain an infinite number of elements of $\mathcal{B} \setminus \mathcal{B}_1$. It proves the claim which guarantees that $\operatorname{card}(\mathcal{B} \setminus \mathcal{B}_1) = \infty$. Now let (B_n^0) denote a sequence of all elements of $\mathcal{B} \setminus \mathcal{B}_1$. Since $\mathcal{A}_n \uparrow \mathcal{A}$, there exists $k_1 \in N$ and $D_1 \in \mathcal{A}_{k_1}$ such that

$$P\Big(D_1 \Delta \bigcup_{j \ge 2} B_j^0\Big) < \min\Big(P(B_1^0), \frac{1}{2} \sum_{j \ge 2} P(B_j^0)\Big).$$

Then

$$D_1 \cap B_1^0 = \emptyset,$$

and

$$0 < P(D_1) < \frac{3}{2} \sum_{j \ge 2} P(B_j^0).$$

Furthermore, D_1 contains some element of $(B_j^0,\ j \geq 2)$. Hence by the claim D_1 contains an infinite number of elements of $\mathcal{B} \setminus \mathcal{B}_1$. Equivalently, it contains an infinite number of elements of $(B_j^0,\ j \geq 2)$. Let (B_n^1) denote in turn all the elements of $(B_j^0,\ j \geq 2)$ which are contained in D_1. Then by the same argument as given above, there exists $k_2 > k_1$ and $D_2 \in \mathcal{A}_{k_2}$ such that $D_2 \subset D_1$ and

$$P\Big(D_2 \Delta \bigcup_{j\geq 2} B_j^1\Big) \leq \min\Big(P(B_j^0),\ j \leq 2,\ P(B_1^1),\ \frac{1}{2}\sum_{j\geq 2} P(B_j^1)\Big).$$

Consequently

$$P(D_2 \cap B_j^0) = P(D_2 \cap B_1^1) = 0,\ j \leq 2,$$

and

$$0 < P(D_2) \leq \frac{3}{2}\sum_{j\geq 2} P(B_j^1) \leq \frac{3}{2}\sum_{j\geq 3} P(B_j^0).$$

Moreover, by the same argument used for D_1 one can conclude that D_2 contains some, hence an infinite number of elements of $(B_j^1,\ j \geq 2)$. Let (B_n^2) denote in turn all the elements of $(B_j^1,\ j \geq 2)$ which are contained in D_2. Then it is clear that (B_n^2) is contained in $(B_j^0,\ j \geq 3)$. Thus by induction one can construct a strictly increasing sequence (k_n) of N, a decreasing sequence (D_n), adapted to $(\mathcal{A}_{k_n})$ and a countable family of sequences $(B_j^{n-1},\ j \geq 1),\ n \in N$ such that each sequence $(B_j^n,\ j \geq 1)$ is contained in the sequence $B_j^0,\ j \geq n+1)$ and for every n we have

$$P\Big(D_n \Delta \bigcup_{j\geq 2} B_j^{n-1}\Big) \leq \min\Big(P(B_j^0),\ j \leq n,\ P(B_1^{n-1}),\ \frac{1}{2}\sum_{j\geq 2} P(B_j^{n-1})\Big).$$

Then

$$P(D_n \cap B_j^0) = P(D_n \cap B_1^{n-1}) = 0,\ j \leq n,$$

and

$$0 < P(D_n) \leq \frac{3}{2}\sum_{j\geq 2} P(B_j^{n-1}) \leq \frac{3}{2}\sum_{j\geq n+1} P(B_j^0).$$

Hence, we get also $\lim_n P(D_n) = 0$. Therefore, if we define the sequence (A_n) as usual by

$$A_p = \begin{cases} \Omega, & p < k_1 \\ D_n, & k_n \leq p < k_{n+1},\ n \in N \end{cases}$$

then (A_n) satisfies the assumption of Theorem 2.2 which proves the corollary.

After all the previous corollaries, we are sure that the assumption of Theorem 2.2 is satisfied for almost cases of $\mathcal{A}$ and $\mathcal{A}_n$. Here we are in a good position to express our thanks to the referee for posing to us the following natural but very important questions:

Question 1. Suppose that the assumption of Theorem 2.2 is not satisfied. Whether or not there exists a quasi-mil which is not a sequential mil?

Question 2. What is the best sufficient condition under which every sequential mil is a quasi-mil?

Unfortunately until now we could not find any quasi-mil which is not a sequential mil. So the first question of the referee is still open. However, the following result would be regarded as one of the best answers to the next question.

Theorem 2.6.
a) *Suppose that $\mathcal{A}$ is finitely generated. Then the class of all sequential mils coincides with that of all quasi-mils.*
b) *There exists a sequential mil on an infinitely generated probability space which is not a quasi-mil.*

Proof. Suppose first that $\mathcal{A}$ is finitely generated. Then one can assume without any loss of generality that $\mathcal{A}_1 \equiv \mathcal{A}$. Then for any $q \in N$ and $\tau \in T$ with $\tau \geq q$ we have $X_q(\tau) = X_\tau$. Combining this fact with the following two standard inequalities

$$\sup_{p\leq q,\ s\leq m} |X_q - X_s| \leq \sup_{p\leq q\leq m} |X_q - X_n| + \sup_{p\leq s\leq m} |X_s - X_n|, \text{ a.s}$$

and

$$\sup_{p\leq q\leq m} |X_n - X_q| \leq \sup_{p\leq q,\, s} |X_q - X_s|, \text{ a.s}$$

that are valid for any $p, m, n \in N$ satisfying $p < m < n$, it follows that a sequence (X_n) in $L^1(\mathcal{A})$ is a quasi-mil if and only if it converges a.s. On the other hand for any $p \in N$ and $\tau \in T$ with $\tau \geq p$ we also have

$$\sup_{p\leq q,\ s\leq \tau} |X_q - X_s| \leq \sup_{p\leq q\leq \tau} |X_q - X_\tau| + \sup_{p\leq s\leq \tau} |X_s - X_\tau|, \text{ a.s}$$

and

$$\sup_{p\leq q\leq \tau} |X_\tau - X_q| \leq \sup_{p\leq q,\, s} |X_q - X_s|.$$

Then it is clear that a sequence (X_n) in $L^1(\mathcal{A})$ is a sequential mil if and only if it converges a.s. Thus the first part of the theorem has been proved.

To construct an example as claimed in the next part of the theorem, let $(\Omega, \mathcal{A}, P)$, a_n and Q_n be as given in the construction of Example 2.1. Define $b_0 = 0$ and $b_m = \sum_{j=0}^{m-1} a_j,\ m \geq 1$. Then for every $n \geq 2$, there exists a unique $m \geq 2$ and $1 \leq j \leq a_{m-1}$ such that $n = b_{m-1} + j$. For such a decomposition of n, let $\mathcal{A}_n$ denote the σ-field generated by Q_m and define the real-valued function (X_n) as follows: On the j-th interval $I_j^{(m-1)}$ of Q_{m-1} set

$$X_n = 2^{m-1} \quad \text{or} \quad X_n = -2^{-m-1}, \text{ resp.}$$

on the first or on the second interval, resp. of Q_m which is contained in $I_j^{(m-1)}$ and $X_n = 0$, elsewhere. On any other interval I of Q_{m-1}, set $X_n = 2^m$ on the first interval of Q_{m-1} which is contained in I and $X_n = 0$, elsewhere. It is easily checked that defined in such a way we have $E(|X_n|) = 1$,

$$X_{b_{m-1}}(n) = \begin{cases} 0 & \text{on} \quad I_j^{(m-1)}, \\ 1 & \text{elsewhere}, \end{cases} \tag{a}$$

and

$$P\big(\sup_{b_{m-1}<n\leq b_m} |X_n| \neq 0\big) \leq 2^{1-m}.$$

Then

$$(X_n) \qquad \text{converges to zero, a.s.} \tag{b}$$

Now let define increasing sequence (τ_n) of bounded stopping times by

$$\tau_{m-1} = n = b_{m-1} + j \quad \text{on} \quad I_j^{(m-1)}, \ 1 \leq j \leq a_{m-1}, \ m \geq 2.$$

We claim that defined in such a way (X_n) is a $\{\tau_n\}$-mil, hence a sequential mil. Indeed let $0 < \varepsilon < 1$ be given. Then by (a), for any $p, m \in N$ with $p \leq \tau_{m-1}$ we have $X_p(\tau_{m-1}) = 0$ if $p \leq b_{m-1}$ and $X_{b_{m-1}+1}(\tau_{m-1}) = X_{\tau_{m-1}}$, hence

$$P\big(\sup_{p\leq q\leq \tau_{m-1}} |X_q(\tau_{m-1}) - X_q| > \varepsilon\big) \leq P\big(\sup_{p\leq q\leq b_{m-1}} |X_q| > \varepsilon\big) + 2^{1-m}.$$

This with (b) proves the claim. Further, to see the next properties of (X_n), it is useful to note that $X_{b_{m-1}}(b_{m-1} + 1) = 1$, hence for every $p \in N$ we have $X_p(b_{m-1}+1) = 1$ as well. Consequently, again by (b) (X_n) fails to be a quasi-mil, which proves the theorem. ∎

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