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Almost Sure Convergence of Weighted Quadratic Forms for Martingale Difference Sequences

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Abstract. The paper is concerned with the study of almost sure convergence of weighted quadratic forms for martingale difference sequences of random variables.

1. Introduction

Let $(\Omega, \Im, P)$ be a probability space and $\{\Im_n, n \geq 1\}$ be an increasing sequence of sub-sigma fields of $\Im$ and $\{X_n, n \geq 1\}$ be a sequence of independent real random variables adapted to $\{\Im_n, n \geq 1\}$ such that $E[X_k] = 0$ and $E[X_k^2] = 1$ for $k = 1, 2, ...$ Let (a_{jk}), $j, k = 1, 2, ...$ be a matrix of real numbers (not necessarily symmetric). Varberg (1968) has studied the almost sure convergence of $S_n = \sum_{j,k=1}^{n} a_{jk} X_j X_k$ as stated below.

Theorem A. *If* $\sum_{j,k=1}^{\infty} a_{jk}^2 < \infty$ *and* $\sum_{k=1}^{\infty} |a_{kk}| < \infty$, *then* $S_n = \sum_{j,k=1}^{n} a_{jk} X_j X_k$ *converges almost surely.*

Theorem B. *If there are positive constants* δ *and* ε *such that* $P[|X_k| > \delta] \geq \varepsilon, k = 1, 2, \dots$, *if* $\sum_{k=1}^{\infty} |a_{kk}| < \infty$, *then*

(a) $\sum\limits_{k=1}^{n} \Big[\sum\limits_{j=1}^{k-1} (a_{kj} + a_{jk}) X_j \Big]^2$,

(b) $\sum\limits_{k=1}^{n} \Big[\sum\limits_{j=1}^{k-1} (a_{kj} + a_{jk}) X_j \Big]^2 X_k^2$ *and*

(c) $\sum\limits_{k=1}^{n} \Big[\sum\limits_{j=1}^{k-1} (a_{kj} + a_{jk}) X_j \Big] X_k + \sum\limits_{k=1}^{n} a_{kk} X_k^2$ *converge on equivalent sets, i.e. on sets which differs at most by null sets.*

Definition 1. *An adapted sequence of random variables* $\{X_n, \Im_n, n \geq 1\}$ *is said to be a martingale difference sequence if* $E[X_n \,|\, \Im_{n-1}] = 0$ *a.s. for each* $n \geq 2$.

In this paper we have proved results of the type mentioned in Theorems B and A for martingale difference sequences.

2. Main Results

Theorem 2.1. *Let*

(i) $\{X_k, k \geq 1\}$ *be a martingale difference sequence with* $E[X_k^4 \,|\, \Im_{k-1}] = c < \infty$, $E[X_k^3 \,|\, \Im_{k-1}] = 0$, *and* $E[X_k^2 \,|\, \Im_{k-1}] = 1$ *for all* $k \geq 1$;

(ii) (a_{jk}), $j, k = 1, 2, \ldots$ *be a real symmetric matrix with* $\sum_{j,k=1}^{\infty} a_{jk}^2 < \infty$ *and* $\sum_{k=1}^{\infty} |a_{kk}| < \infty$. *Then* $S_n = \sum_{j,k=1}^{n} a_{jk} X_j X_k$ *converges in quadratic mean and almost surely.*

Remark 2.1. If (a_{jk}) is not symmetric, $a_{jk} = a_j b_k$ and if $\sum_{j,k=1}^{\infty} a_{jk}^2 = M < \infty$, then

$$S_n = \sum_{j,k=1}^{n} a_{jk} X_j X_k = \Big(\sum_{j=1}^{n} a_j X_j\Big)\Big(\sum_{k=1}^{n} b_k X_k\Big)$$

and

$$T_n = S_n - \sum_{k=1}^{n} a_{kk} = S_n - \sum_{k=1}^{n} a_k b_k$$

converge almost surely and we do not need the assumption that $E\big(X_n^3 \,|\, \Im_{n-1}\big) = 0$ and $E\big(X_n^4 \,|\, \Im_{n-1}\big) < \infty$ for a martingale difference sequence $\{X_n, n \geq 1\}$, where $\sum_{j,k=1}^{\infty} a_{jk}^2 < \infty$.

Theorem 2.2. *Let*

(i) $(a_{jk}), j, k{=}1, 2, \ldots$ *be a matrix of real numbers with* $\sum_{j,k=1}^{\infty} a_{jk}^2 < \infty$ *and* $\sum_{k=1}^{\infty} |a_{kk}| < \infty$;

(ii) $\{X_k, k \geq 1\}$ *be a martingale difference sequence with* $E\big[X_k^2 \,|\, \Im_{k-1}\big] = 1$ *for all* $k \geq 1$. *Then* $S_n = \sum_{j,k=1}^{n} a_{jk} X_j X_k$ *converges almost surely.*

Theorem 2.3. *Let*

(i) (a_{jk}), $j, k = 1, 2, \ldots$ *be a matrix of real numbers with* $\sum_{j,k=1}^{\infty} a_{jk}^2 < \infty$ *and* $\sum_{k=1}^{\infty} |a_{kk}| < \infty$;

(ii) $\{X_k, k \geq 1\}$ *be a martingale difference sequence with* $E\big[X_k^2 \,|\, \Im_{k-1}\big] = 1$ *for all* $k \geq 1$ *and*

(iii) $V_k = \sum_{j=1}^{k-1} (a_{kj} + a_{jk}) X_j$. *Then* $\sum_{k=1}^{n} V_k X_k + \sum_{k=1}^{n} a_{kk} X_k^2$, $\sum_{k=1}^{n} V_k^2$, *and* $\sum_{k=1}^{n} V_k^2 X_k^2$ *converge almost surely.*

Proof of Theorem 2.1. Let $T_0 = 0$, $T_n = S_n - E[S_n]$ and $Y_n = T_n - T_{n-1}$ for $n = 1, 2, \ldots$

Now

$$\begin{aligned}
T_n = S_n - E[S_n] &= \sum_{j,k=1}^{n} a_{jk} X_j X_k - E\Big[\sum_{j,k=1}^{n} a_{jk} X_j X_k\Big] \\
&= \sum_{j,k=1}^{n} a_{jk} X_j X_k - E\Big[\sum_{k=1}^{n} a_{kk} X_k^2\Big] - E\Big[\sum_{j\neq k=1}^{n} a_{jk} X_j X_k\Big] \\
&= \sum_{j,k=1}^{n} a_{jk} X_j X_k - \sum_{k=1}^{n} a_{kk} E\big[E(X_k^2|\Im_{k-1})\big] - \sum_{j\neq k=1}^{n} a_{jk} E\big[E(X_j X_k\,|\Im_{\min(j,k)})\big] \\
&= \sum_{j,k=1}^{n} a_{jk} X_j X_k - \sum_{k=1}^{n} a_{kk} - \sum_{j\neq k=1}^{n} a_{jk} E\big[X_{\min(j,k)} E(X_{\max(j,k)}\,|\Im_{\min(j,k)})\big] \\
&= \sum_{j,k=1}^{n} a_{jk} X_j X_k - \sum_{k=1}^{n} a_{kk}.
\end{aligned}$$

Hence

$$S_n = T_n + \sum_{k=1}^{n} a_{kk}.$$

Again

$$\sum_{k=1}^{n} Y_k = \sum_{k=1}^{n} (T_k - T_{k-1}) = T_n.$$

For $n \neq m$,

$$\begin{aligned}
&E[Y_n Y_m] = E[(T_n - T_{n-1})(T_m - T_{m-1})] \\
&= E\Big\{\Big[\sum_{j,k=1}^{n} a_{jk} X_j X_k - \sum_{k=1}^{n} a_{kk} - \sum_{j,k=1}^{n-1} a_{jk} X_j X_k + \sum_{k=1}^{n-1} a_{kk}\Big] \\
&\qquad \Big[\sum_{k,j,k=1}^{m} a_{jk} X_j X_k - \sum_{k=1}^{m} a_{kk} - \sum_{j,k=1}^{n-1} a_{jk} X_j X_k + \sum_{k=1}^{m-1} a_{kk}\Big]\Big\} \\
&= E\Big\{\Big[2X_n \sum_{k=1}^{n-1} a_{nk} X_k + a_{nn}(X_n^2 - 1)\Big]\Big[2X_m \sum_{k=1}^{m-1} a_{mk} X_k + a_{mm}(X_m^2 - 1)\Big]\Big\} \\
&= 4E\Big[X_n X_m \Big(\sum_{k=1}^{n-1} a_{nk} X_k\Big)\Big(\sum_{k=1}^{m-1} a_{mk} X_k\Big)\Big] + 2a_{nn} E\Big[(X_m^2 - 1) X_m \Big(\sum_{k=1}^{m-1} a_{mk} X_k\Big)\Big] \\
&\qquad + 2a_{mm} E\Big[(X_m^2 - 1) X_n \Big(\sum_{k=1}^{n-1} a_{nk} X_k\Big)\Big] + a_{nn} a_{mm} E\big[(X_n^2 - 1)(X_m^2 - 1)\big] \\
&= 4E\Big[E\Big\{X_n X_m \Big(\sum_{k=1}^{n-1} a_{nk} X_k\Big)\Big(\sum_{k=1}^{m-1} a_{mk} X_k\Big)\,|\Im_{n-1}\Big\}\Big] \\
&\qquad + 2a_{nn} E\Big[E\Big\{(X_n^2 - 1) X_m \Big(\sum_{k=1}^{m-1} a_{mk} X_k\Big)\,|\Im_{n-1}\Big\}\Big] \\
&\qquad + 2a_{mm} E\Big[E\Big\{(X_m^2 - 1) X_n \Big(\sum_{k=1}^{n-1} a_{nk} X_k\Big)\,|\Im_{n-1}\Big\}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\quad + a_{nn}a_{mm}E\big[E\big\{(X_n^2-1)(X_m^2-1)\,|\Im_{n-1}\big\}\big]\\
&= 4E\Big[X_m\Big(\sum_{k=1}^{n-1}a_{nk}X_k\Big)\Big(\sum_{k=1}^{m-1}a_{mk}X_k\Big)E\{X_n\,|\Im_{n-1}\}\Big]\\
&\quad + 2a_{nn}E\Big[X_m\Big(\sum_{k=1}^{m-1}a_{mk}X_k\Big)E\big\{(X_n^2-1)\,|\Im_{n-1}\big\}\Big]\\
&\quad + 2a_{mm}E\Big[(X_m^2-1)\Big(\sum_{k=1}^{n-1}a_{nk}X_k\Big)E\big\{X_n\,|\Im_{n-1}\big\}\Big]\\
&\quad + a_{nn}a_{mm}E\big[(X_m^2-1)E\big\{(X_n^2-1)\,|\Im_{n-1}\big\}\big] = 0.
\end{aligned}
$$

(since X_n is a martingale difference sequence and $E[X_n^2\,|\Im_{n-1}] = 1$) Hence Y_n is orthogonal. Again

$$
Y_n = T_n - T_{n-1} = 2X_n\sum_{k=1}^{n-1}a_{nk}X_k + a_{nn}\big(X_n^2-1\big) = K_n + L_n \qquad \text{(say)}
$$

Now

$$
\begin{aligned}
E[K_n^2] &= E\Big[2X_n\sum_{k=1}^{n-1}a_{nk}X_k\Big]^2 = 4E\Big\{E\Big[X_n^2\Big(\sum_{k=1}^{n-1}a_{nk}X_k\Big)^2\,|\Im_{n-1}\Big]\Big\}\\
&= 4E\Big\{\Big(\sum_{k=1}^{n-1}a_{nk}X_k\Big)^2E[X_n^2\,|\Im_{n-1}]\Big\}\\
&= 4E\Big[\Big(\sum_{k=1}^{n-1}a_{nk}X_k\Big)^2\Big] \quad \big(\text{since } E[X_n^2\,|\Im_{n-1}] = 1\big)\\
&= 4E\Big[\sum_{k=1}^{n-1}a_{nk}^2X_k^2 + 2\sum_{j<k=1}^{n-1}a_{nj}a_{nk}X_jX_k\Big]\\
&= 4\Big[\sum_{k=1}^{n-1}a_{nk}^2E(X_k^2) + 2\sum_{j<k=1}^{n-1}a_{nj}a_{nk}E(X_jX_k)\Big]\\
&= 4\Big[\sum_{k=1}^{n-1}a_{nk}^2E\big\{E(X_k^2\,|\Im_{k-1})\big\} + 2\sum_{j<k=1}^{n-1}a_{nj}a_{nk}E\big\{E(X_jX_k\,|\Im_j)\big\}\Big]\\
&= 4\sum_{k=1}^{n-1}a_{nk}^2
\end{aligned}
$$

(since $E(X_k\,|\Im_j) = 0$, X_k being a martingale difference sequence) and

$$
\begin{aligned}
E[L_n^2] &= E\big[a_{nn}^2(X_n^2-1)^2\big] = E\big[a_{nn}^2(X_n^4-2X_n^2+1)\big]\\
&= a_{nn}^2\big[E\big\{E(X_n^4\,|\Im_{n-1})\big\} - 2E\big\{E(X_n^2\,|\Im_{n-1})\big\} + 1\big]\\
&= a_{nn}^2[c-2+1] = a_{nn}^2[c-1].
\end{aligned}
$$

And lastly

$$\begin{aligned}E[K_nL_n] &= E\Big[2X_n\Big(\sum_{k=1}^{n-1}a_{nk}X_k\Big)a_{nn}(X_n^2-1)\Big]\\ &= 2a_{nn}E\Big\{E\Big[\Big(\sum_{k=1}^{n-1}a_{nk}X_k\Big)(X_n^3-X_n)\,|\Im_{n-1}\Big]\Big\}\\ &= 2a_{nn}E\Big\{\Big(\sum_{k=1}^{n-1}a_{nk}X_k\Big)E\big[(X_n^3-X_n)\,|\Im_{n-1}\big]\Big\}=0\end{aligned}$$

(since $E[X_n^3\,|\Im_{n-1}]=E[X_n\,|\Im_{n-1}]=0$). So

$$E[Y_n^2]=E[(K_n+L_n)^2]=E[K_n^2]+E[L_n^2]+2E[K_nL_n]=4\sum_{k=1}^{n-1}a_{nk}^2+a_{nn}^2[c-1].$$

We know that $\sum_{k=1}^{n}Y_k=T_n$ converges in quadratic mean if and only if $\sum_{n=1}^{\infty}E[Y_n^2]<\infty$ as $\{Y_n\}$ is orthogonal.

But

$$\begin{aligned}\sum_{n=1}^{\infty}E[Y_n^2] &= \sum_{n=1}^{\infty}\Big[4\sum_{k=1}^{n-1}a_{nk}^2+a_{nn}^2[c-1]\Big]=2\sum_{n=1}^{\infty}\Big[2\sum_{k=1}^{n-1}a_{nk}^2+a_{nn}^2\Big]\\ &\quad+[c-3]\sum_{n=1}^{\infty}a_{nn}^2=2\sum_{j,k=1}^{\infty}a_{jk}^2+[c-3]\sum_{n=1}^{\infty}a_{nn}^2<\infty.\end{aligned}$$

So T_n converges in quadratic mean.

We have $S_n=T_n+\sum_{k=1}^{n}a_{kk}$.

Hence S_n converges in quadratic mean since $\sum_{k=1}^{\infty}a_{kk}<\infty$. To prove the almost sure convergence we have

$$\begin{aligned}T_n=S_n-E[S_n] &= \sum_{j,k=1}^{n}a_{jk}X_jX_k-\sum_{k=1}^{n}a_{kk}\\ &= \sum_{j,k=1}^{n-1}a_{jk}X_jX_k-\sum_{k=1}^{n-1}a_{kk}+2\sum_{k=1}^{n-1}a_{nk}X_kX_n+a_{nn}(X_n^2-1)\end{aligned}$$

So

$$\begin{aligned}&E[T_n\,|\Im_{n-1}]=E\Big[\Big(\sum_{j,k=1}^{n-1}a_{jk}X_jX_k\Big)\,|\Im_{n-1}\Big]\\ &-\sum_{k=1}^{n-1}a_{kk}+2E\Big[\Big(\sum_{k=1}^{n-1}a_{nk}X_kX_n\Big)\,|\Im_{n-1}\Big]+a_{nn}E\big[(X_n^2-1)\,|\Im_{n-1}\big]\\ &=\sum_{j,k=1}^{n-1}a_{jk}X_jX_k-\sum_{k=1}^{n-1}a_{kk}+2\sum_{k=1}^{n-1}a_{nk}X_kE\big[X_n|\Im_{n-1}\big]+a_{nn}\big[E(X_n^2|\Im_{n-1})-1\big]\\ &=\sum_{j,k=1}^{n-1}a_{jk}X_jX_k-\sum_{k=1}^{n-1}a_{kk}=T_{n-1}.\end{aligned}$$

Thus it implies that $\{T_n\}$ is a martingale. Now

$$E[|T_n|] \le \big\{E[T_n^2]\big\}^{1/2} = \Big\{\sum_{k=1}^{n} E(Y_k^2)\Big\}^{1/2} \quad \big(\text{since } \ E(Y_jY_k)=0 \ \text{ for } \ j\ne k\big)$$
$$\le \Big\{\sum_{k=1}^{\infty} E(Y_k^2)\Big\}^{1/2} = \Big\{2\sum_{j,k=1}^{\infty} a_{jk}^2 + (c-3)\sum_{k=1}^{\infty} a_{kk}^2\Big\}^{1/2} < \infty.$$

Hence T_n converges almost surely and so S_n converges almost surely since $S_n = T_n + \sum_{k=1}^{n} a_{kk}$ and $\sum_{k=1}^{\infty} a_{kk} < \infty$.

Proof of Theorem 2.2. We have

$$S_n = \sum_{j,k=1}^{n} a_{jk}X_jX_k$$
$$= \sum_{j=1}^{n} X_j \sum_{k=1}^{j-1} a_{jk}X_k + \sum_{k=1}^{n} X_k \sum_{j=1}^{k-1} a_{jk}X_j + \sum_{k=1}^{n} a_{kk}X_k^2 = K_n + L_n + M_n \text{ (say)}$$

Now

$$E[K_n\,|\Im_{n-1}] = \sum_{j=1}^{n-1} X_j \sum_{k=1}^{j-1} a_{jk}X_k + E\Big[X_n \sum_{k=1}^{n-1} a_{jk}X_k\,|\Im_{n-1}\Big]$$
$$= K_{n-1} + \Big(\sum_{k=1}^{n-1} a_{nk}X_k\Big)E[X_n\,|\Im_{n-1}] = K_{n-1}.$$

So $\{K_n\}$ is a martingale. Again

$$E[K_n^2] = E\Big[\sum_{j=1}^{n} X_j^2\Big(\sum_{k=1}^{j-1} a_{jk}X_k\Big)^2\Big]$$
$$+ E\Big[\sum_{j\ne k}\sum_{l=1}^{j-1}\sum_{m=1}^{k-1} a_{jl}a_{km}X_jX_kX_eX_m\Big]$$
$$= E\Big\{E\Big[\sum_{j=1}^{n} X_j^2\Big(\sum_{k=1}^{j-1} a_{jk}X_k\Big)^2\,|\Im_{j-1}\Big]\Big\}$$
$$+ \sum_{j\ne k}\sum_{l=1}^{j-1}\sum_{m=1}^{k-1} a_{jl}a_{km}E[X_jX_kX_eX_m]$$
$$= E\Big\{\sum_{j=1}^{n}\Big(\sum_{k=1}^{j-1} a_{jk}X_k\Big)^2\Big\}$$

(since for $l < j,k;\ m < j, k \ne k,\ E(X_jX_kX_lX_m) = 0$, and $E[X_j^2\,|\Im_{j-1}] = 1$)

$$= \sum_{j=1}^{n} E\Big(\sum_{k=1}^{j-1} a_{jk}^2 X_k^2 + 2\sum_{k<l=1}^{j-1} a_{jk}a_{jl}X_kX_l\Big) = \sum_{j=1}^{n}\sum_{k=1}^{j-1} a_{jk}^2 < \infty.$$

Hence by martingale convergence theorem K_n converges almost surely. Similarly, it can be shown that L_n is also a martingale with respect to $\Im_n$ and $E[L_n^2] < \infty$. So L_n also converges almost surely. Lastly

$$M_n = \sum_{k=1}^{n} a_{kk}X_k^2 = \sum_{k=1}^{n} a_{kk}(X_k^2 - 1) + \sum_{k=1}^{n} a_{kk}.$$

Here $\sum_{k=1}^{n} a_{kk}(X_k^2 - 1)$ is a martingale and

$$E\Big[\Big|\sum_{k=1}^{n} a_{kk}(X_k^2 - 1)\Big|\Big] \le \sum_{k=1}^{n} |a_{kk}| E\big[|(X_k^2 - 1)|\big] \le 2\sum_{k=1}^{n} |a_{kk}| < \infty.$$

Thus $\sum_{k=1}^{n} a_{kk}(X_k^2 - 1)$ converges almost surely.

Again

$$\sum_{k=1}^{n} a_{kk} \le \Big|\sum_{k=1}^{n} a_{kk}\Big| \le \sum_{k=1}^{n} |a_{kk}| \le \sum_{k=1}^{\infty} |a_{kk}| < \infty.$$

So M_n converges almost surely. Hence the result follows.

Proof of Theorem 2.3

(a) Let $S_n = \sum_{k=1}^{n} V_k X_k + \sum_{k=1}^{n} a_{kk}X_k^2 = P_n + M_n$ (say)

Now

$$E[P_n \,|\Im_{n-1}] = \sum_{k=1}^{n-1} V_k X_k + E[V_n X_n \,|\Im_{n-1}] = P_{n-1} + V_n E[X_n \,|\Im_{n-1}] = P_{n-1},$$

(since X_n is a martingale difference sequence).

So P_n is a martingale. Again

$$\begin{aligned} E[P_n^2] &= E\Big[\sum_{k=1}^{n}\Big\{\sum_{j=1}^{k-1}(a_{kj} + a_{jk})X_j\Big\}^2 X_k^2\Big] \\ &\quad + E\Big[\sum_{k\ne i}^{n}\sum_{j=1}^{k-1}\sum_{l=1}^{i-1}(a_{kj} + a_{jk})(a_{li} + a_{il})X_iX_jX_kX_l\Big] \\ &= \sum_{k=1}^{n} E\Big[E\Big(\Big\{\sum_{j=1}^{k-1}(a_{kj} + a_{jk})X_j\Big\}^2 X_k^2 \,|\Im_{k-1}\Big)\Big] \\ &\quad + \sum_{k\ne i}^{n}\sum_{j=1}^{k-1}\sum_{l=1}^{i-1}(a_{kj} + a_{jk})(a_{li} + a_{il})E[X_iX_jX_kX_l] \\ &= \sum_{k=1}^{n} E\Big[\Big\{\sum_{j=1}^{k-1}(a_{kj} + a_{jk})X_j\Big\}^2\Big] \end{aligned}$$

$$= \sum_{k=1}^{n} E\Big[\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2 X_j^2 + 2\sum_{j<l=1}^{k-1}(a_{kj}+a_{jk})(a_{kl}+a_{lk})X_jX_l\Big]$$
$$= \sum_{k=1}^{n}\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2 < \infty,$$

(using the fact that $E[X_iX_jX_kX_l]$=0 as proved in Theorem 2.2 and $E[X_k^2|\Im_{k-1}]$=1).

So P_n converges almost surely using the martingale convergence theorem. Again

$$M_n = \sum_{k=1}^{n} a_{kk}X_k^2 = \sum_{k=1}^{n} a_{kk}(X_k^2-1) + \sum_{k=1}^{n} a_{kk}.$$

Here $\sum_{k=1}^{n} a_{kk}(X_k^2-1)$ is a martingale and $E\Big[\Big|\sum_{k=1}^{n} a_{kk}(X_k^2-1)\Big|\Big] < \infty$ as shown in Theorem 2.2. Thus $\sum_{k=1}^{n} a_{kk}(X_k^2-1)$ converges almost surely by the martingale convergence theorem.

Again

$$\sum_{k=1}^{n} a_{kk} \le \Big|\sum_{k=1}^{n} a_{kk}\Big| \le \sum_{k=1}^{n}|a_{kk}| \le \sum_{k=1}^{\infty}|a_{kk}| < \infty.$$

So M_n converges almost surely and hence S_n converges almost surely.

(b)
$$\sum_{k=1}^{n} V_k^2 = \sum_{k=1}^{n}\Big[\sum_{j=1}^{k-1}(a_{kj}+a_{jk})X_j\Big]^2$$
$$= \sum_{k=1}^{n}\Big[\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2X_j^2 + \sum_{j\neq l=1}^{k-1}(a_{kj}+a_{jk})(a_{kl}+a_{lk})X_jX_l\Big]$$
$$= K_n + L_n \quad \text{(say)}$$

Now

$$K_n = \sum_{k=1}^{n}\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2(X_j^2-1) + \sum_{k=1}^{n}\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2.$$

Proceeding as in the proof of (a), $\sum_{k=1}^{n}\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2(X_j^2-1)$ is a martingale with $E\Big[\Big|\sum_{k=1}^{n}\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2(X_j^2-1)\Big|\Big] < \infty$. So $\sum_{k=1}^{n}\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2(X_j^2-1)$ converges almost surely.

Again

$$\sum_{k=1}^{n}\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2 < \infty \quad \text{as} \quad n\to\infty.$$

So K_n converges almost surely. Now

$$
\begin{aligned}
L_n &= \sum_{k=1}^{n} \sum_{j\neq l=1}^{k-1} (a_{kj}+a_{jk})(a_{kl}+a_{lk})X_jX_l \\
&\leq \Big[\sum_{k=1}^{n}\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2X_j^2\Big]^{1/2}\Big[\sum_{k=1}^{n}\sum_{l=1}^{k-1}(a_{kl}+a_{lk})^2X_l^2\Big]^{1/2} \quad \text{for} \quad j\neq l.
\end{aligned}
$$

Hence L_n converges almost surely and so $\sum_{k=1}^n V_k^2$ converges almost surely.
(c) $\sum_{k=1}^n V_k^2X_k^2 = \sum_{k=1}^n V_k^2(X_k^2-1) + \sum_{k=1}^n V_k^2 = Q_n + R_n$ (say).
Now

$$
\begin{aligned}
E[Q_n\,|\Im_{n-1}] &= E\Big[\sum_{k=1}^{n} V_k^2(X_k^2-1)\,|\Im_{n-1}\Big] \\
&= \sum_{k=1}^{n-1} V_k^2(X_k^2-1) + E[V_n^2(X_n^2-1)\,|\Im_{n-1}] \\
&= Q_{n-1} + V_n^2[E(X_n^2\,|\Im_{n-1})-1] = Q_{n-1},
\end{aligned}
$$

so Q_n is a martingale.

Now

$$
\begin{aligned}
E[|Q_n|] &= E\Big[\Big|\sum_{k=1}^{n} V_k^2(X_k^2-1)\Big|\Big] \leq \sum_{k=1}^{n} E[|V_k^2|\,|(X_k^2-1)|] \\
&\leq \sum_{k=1}^{n} E\big\{E\big[|V_k^2|\big(|X_k^2|+1\big)\,|\Im_{k-1}\big]\big\} = 2\sum_{k=1}^{n} E[V_k^2] \\
&= 2\sum_{k=1}^{n} E\Big[\sum_{j=1}^{k-1}(a_{kj}+a_{jk})X_j\Big]^2 = 2\sum_{k=1}^{n} E\Big[\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2X_j^2 \\
&\qquad + 2\sum_{j<l=1}^{k-1}(a_{kj}+a_{jk})(a_{kl}+a_{lk})X_jX_l\Big] \\
&= 2\sum_{k=1}^{n}\sum_{j=1}^{k-1}(a_{kj}+a_{jk})^2E\big\{E\big[X_j^2\,|\Im_{j-1}\big] \\
&\qquad + 4\sum_{k=1}^{n}\sum_{j<l=1}^{k-1}(a_{kj}+a_{jk})(a_{kl}+a_{lk})E\big\{E\big[X_jX_l\,|\Im_j\big]\big\} < \infty.
\end{aligned}
$$

So Q_n converges almost surely.

We have already shown that $R_n = \sum_{k=1}^n V_k^2$ converges almost surely. So $\sum_{k=1}^n V_k^2X_k^2$ converges almost surely.

References

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