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Probabilistic Iterated Function Systems and Probabilistic Systems*

Dedicated to Professor Nguyen Duy Tien on the occasion of his 60^{th} *birthday*

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Abstract. We study the relation between probabilistic iterated function systems introduced in [1] and probabilistic systems introduced in [4]. We prove that the probabilistic iterated function system $\{f_i(x) = \rho x + r_i : i = 0, 1, \ldots, m\}$ with probabilities $p_i \geq 0$, $\sum_{i=0}^{m} p_i = 1$ and the probabilistic system in the sense of [4] induce the same measure and therefore these systems are equivalent to each other.

1. Introduction

Let $\{f_0, f_1, ..., f_m\}$ be an *iterated function system* (IFS for short) of contractive similitudes on $\mathbb{R}^d$ defined by

$$f_i(x) = \rho_i R_i x + b_i, \ 0 \leq i \leq m, \tag{1.1}$$

where for all i, $0 < \rho_i < 1$, $b_i \in \mathbb{R}^d$ and R_i is a $d \times d$ orthogonal matrix.

The unique non - empty compact set $E \subset \mathbb{R}^d$ satisfying the equation

$$E = \bigcup_{i=0}^{m} f_i(E)$$

is called the *attractor* or *invariant set* of the IFS $\{f_0, f_1, \ldots, f_m\}$.

Assume that $\{f_0, f_1, \ldots, f_m\}$ is an IFS on $X \subset \mathbb{R}$ and let $p_0, p_1, \ldots, p_m$ be probabilities, with $0 \leq p_i \leq 1$ for all i and $\sum_{i=0}^{m} p_i = 1$. Following [1], such a

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system is called a *probabilistic iterated function system.* Then we say that the probabilities $\{p_0, p_1, \dots, p_m\}$ are *associated* with the IFS $\{f_0, f_1, \dots, f_m\}$. Let μ be a measure on $X \subset \mathbb{R}$. Then the *support* of μ is the smallest closed subset X on $\mathbb{R}$ such that $\mu(\mathbb{R}\backslash X) = 0$. The support of μ is denoted by supp μ (where we say that measure μ is a probability measure if $\mu(\text{supp }\mu) = 1$).

A probabilistic iterated function system leads to a measure on X as it was shown in the following theorem, see [1].

Theorem 1.1. [1] *Let $\{f_0, f_1, \dots, f_m\}$ be an iterated function system on $X \subset \mathbb{R}$ with associated probabilities $\{p_0, p_1, \dots, p_m\}$. Then there exists a unique Borel probability measure μ_F (that is $\mu_F(X) = 1$) such that*

$$\mu_F(A) = \sum_{i=0}^{m} p_i \mu_F(f_i^{-1}(A)) \tag{1.2}$$

for all Borel sets A. Moreover, supp $\mu_F = E$, *where E is the attractor of the IFS $\{f_i : 0 \leq i \leq m$ and $p_i > 0\}$.*

Following [5], by a *probabilistic system* we mean a sequence $X_0, X_1, \dots$ of independent identically distributed random variables each taking real values $r_0, r_1, \dots, r_m$ with respective probabilities $p_0, p_1, ..., p_m$. The system is said to be *uniformly distributed* if $p_i = \frac{1}{m+1}$ for every $i = 0, 1, ..., m$. For $0 < \rho < 1$, let

$$S = \sum_{n=0}^{\infty} \rho^n X_n$$

and let μ_P be the probability measure induced by S, i.e.

$$\mu_P(A) = \text{prob}\,\{\omega : S(\omega) \in A\}.$$

The measure μ_P is called the *fractal measure* associated with the probabilistic system. In the case of uniform distribution, the fractal measure is denoted simply by μ.

This paper is organized as follows. In Sec. 2 we construct an iterated function system whose attractor is supp μ in the case of uniform distribution, and give another representation of the mentioned support. In Sec. 3 we investigate the relation between probabilistic iterated function system introduced in [1] and probabilistic system introduced in [4]. We prove that the iterated fucntion system $\{f_0, f_1, \dots, f_m\}$ associated with probabilities $p_i \geq 0$, $\sum_{i=0}^{m} p_i = 1$ and the probabilistic system induce the same measure, i.e. $\mu_F(A) = \mu_P(A)$ for all Borel sets $A \subset \mathbb{R}$ and therefore the probabilistic iterated function system is equivalent to the probabilistic system.

2. Support of a Fractal Measure Associated with a Uniformly Distributed Probabilistic System

In this section we consider the fractal measure associated with a uniformly distributed probabilistic system in the case $r_i = i$ for $i = 0, 1, \ldots, m$ and $\rho = \frac{1}{q}$, $m \geq q \geq 2$, q is an integer.

Let $\mathbb{N}$ denote the set of all nonnegative integers. For $m \in \mathbb{N}$ we denote

$$\mathbb{D}_m = \{0, 1, \ldots, m\}, \text{ and } \mathbb{D}_m^n = \{0, 1, \ldots, m\}^n, \text{ where } n \leq \infty.$$

For $q \geq 2$ let

$$S = \sum_{k=0}^{\infty} q^{-k} X_k, \text{ and } S_n = \sum_{k=0}^{n} q^{-k} X_k.$$

There is no confusion if we also use the notation

$$S : \mathbb{D}_m^{\infty} \to \mathbb{R}^+, \text{ and } S_n : \mathbb{D}_m^{n+1} \to \mathbb{R}^+$$

for functions defined by

$$S(x) = \sum_{k=0}^{\infty} q^{-k} x_k \text{ for } x = (x_0, x_1, \ldots) \in \mathbb{D}_m^{\infty},$$

and

$$S_n(x) = \sum_{k=0}^{n} q^{-k} x_k \text{ for } x = (x_0, x_1, ...) \in \mathbb{D}_m^{n+1}.$$

Let μ and μ_n denote the probability measures induced by S and S_n, respectively.

The following lemma was shown in [5].

Lemma 2.1. *Let $s_n(0) < s_n(1) < \ldots < s_n(k_n)$ denote the set of all distinct values of* supp μ_n. *Then we have*

1. $s_n(0) = 0$ *and* $s_{n+1}(k_{n+1}) = s_n(k_n) + mq^{-n-1}$ *for every* $n \in \mathbb{N}$.
2. *The distance between any two consecutive points in* supp μ_n *is* q^{-n}.
3. Supp $\mu_n \subset$ supp μ_{n+1} *for every* $n \in \mathbb{N}$ *and* supp $\mu = \bigcup_{n=0}^{\infty}$ supp μ_n.
4. *The set* supp μ_n *consists of* $k_n = \frac{m(q^{n+1}-1)}{q-1} + 1$ *points runing from* 0 *to* $\frac{m(q^{n+1}-1)}{q^n(q-1)}$.

As an immediate consequence of Lemma 2.1 we have

Corollary 2.2. $F =$ supp μ *is a compact set.*

As we have seen, there are two main problems that arise in connection with IFSs. First, given a fractal E, find an IFS with attractor E or, at least, a close approximation to E. The second (the inverse problem) is to reconstruct the attractor E for a given IFS. In this section we study the first problem, that is to find an IFS with attractor E, the support of the fractal measure μ associated with the uniformly probabilistic system. We have the following theorem.

Theorem 2.3. *The* supp μ *is the attractor of the IFS* $\{f_0, f_1, \ldots, f_m\}$ *defined by*

$$f_0(x) = \frac{x}{q},\ f_1(x) = 1 + \frac{x}{q}, \ldots,\ f_m(x) = m + \frac{x}{q}, \tag{2.2}$$

that is

$$F = \bigcup_{k=0}^{m} f_k(F).$$

Proof. Assume that $x \in F$, then x is of the form

$$x = \sum_{k=0}^{\infty} \frac{x_k}{q^k} = x_0 + \frac{x_1}{q} + \frac{x_2}{q^2} + \ldots.$$

We take

$$x' = x_1 + \frac{x_2}{q} + \frac{x_3}{q^2} + \cdots + \frac{x_n}{q^{n-1}} + \cdots \in F,$$

and assume that $x_0 = k$. Then we have

$$f_k(x') = x_0 + \frac{x_1}{q} + \frac{x_2}{q^2} + \ldots = x.$$

Therefore

$$x \in \bigcup_{k=0}^{m} f_k(F).$$

Conversely, if $x \in \bigcup_{k=0}^{m} f_k(F)$, then there is $k \in \{0, 1, \ldots, m\}$ such that $x \in f_k(F)$. Assume that $x = f_k(x')$, where

$$x' = x_0 + \frac{x_1}{q} + \frac{x_2}{q^2} + \ldots.$$

Hence

$$f_k(x') = k + \frac{x_0}{q} + \frac{x_1}{q^2} + \frac{x_2}{q^3} + \ldots.$$

Putting

$$x'_0 = k,\ x'_1 = x_0,\ x'_2 = x_1, \ldots,\ x'_i = x_{i-1}, \ldots,$$

we get

$$x'_i \in \{0, 1, \ldots, m\},\ 0 \le i \le m.$$

Therefore

$$x = f_k(x') = x'_0 + \frac{x'_1}{q} + \frac{x'_2}{q^2} + \ldots \in F.$$

Consequently

$$F = \bigcup_{k=0}^{m} f_k(F).$$

Since F is a non-empty compact set, the proof is finished. ∎

Let $\mathcal{S}$ denote the class of non-empty compact subsets of $\mathbb{R}$. We define a transformation $f : \mathcal{S} \longrightarrow \mathcal{S}$ by

$$f(A) = \bigcup_{i=0}^{m} f_i(A)$$

for $A \in \mathcal{S}$. It was shown in [1] that, if $A \in \mathcal{S}$ such that $f_i(A) \subset A$ for all i, then

$$F = \bigcap_{k=0}^{\infty} f^k(A).$$

Consequently, by Theorem 2.3, we get

Theorem 2.4. *$F = \bigcap_{k=0}^{\infty} f^k(A)$, where $A = [0, \frac{mq}{q-1}]$.*

Proof. For $x \in A$ and $i = 0, 1, \ldots, m$, we have

$$0 \leq f_i(x) \leq f_m(\frac{mq}{q-1}) = m + \frac{mq}{(q-1)q} = \frac{mq}{q-1}.$$

That is

$$f_i(A) \subset A \text{ for all } i = 0, 1, \ldots, m.$$

Therefore the assertion follows. ■

3. The Equivalence Between Two Probabilistic Systems

In this section we consider the probabilistic system in the general setting and will show that the probabilistic system and the probabilistic iterated function system induce the same measure. Therefore these systems are equivalent to each other.

Let $X_0, X_1, \ldots$ be a sequence of independent identically distributed random variables each taking real values $r_0 < r_1 < \ldots < r_m$ with respective probabilities $p_0, p_1, \ldots, p_m,\ 0 \leq p_i \leq 1$ for all i and $\sum_{i=0}^{m} p_i = 1$. For $0 < \rho < 1$ we put

$$S = \sum_{n=0}^{\infty} \rho^n X_n.$$

Let μ_P be the fractal measure induced by S, that is

$$\mu_P(A) = \text{prob}\{\omega \in \Omega :\ S(\omega) \in A\}, \tag{3.1}$$

where Ω is sampled space. Observe that every $\omega \in \Omega$ can be identified with a sequence $(x_0, x_1, \ldots)$ where $x_i \in \{r_0, r_1, \ldots, r_m\}$. Therefore, we may write $\omega \equiv (x_0, x_1, \ldots)$.

Let μ_P is the fractal measure defined by (3.1). Then supp μ_P is given by the following formula which is a generalization of Theorem 2.3.

Proposition 3.1.

$$\text{supp } \mu_P = \{\sum_{n=0}^{\infty} \rho^n x_n,\ x_n \in D,\ n = 0, 1, \ldots\}, \tag{3.2}$$

where $D = \{r_0, r_1, \ldots, r_m\}$ and $p_i \neq 0$ for all $i = 0, 1, \ldots, m$.

Proof. Let us put

$$G = \{\sum_{n=0}^{\infty} \rho^n x_n :\ x_n \in D,\ n = 0, 1, 2, \ldots\}.$$

Observe that G is closed and $\mu_P(\mathbb{R}\backslash G) = 0$. Therefore

$$\text{supp } \mu_P \subset G. \tag{3.3}$$

To obtain the reverse inclusion, we need the following lemma

Lemma 3.2.

$$\text{supp } \mu_P = \{x \in \mathbb{R} : \ \mu_P(B_\epsilon(x)) > 0 \text{ for all } \epsilon > 0\} \tag{3.4}$$

where $B_\epsilon(x) = (x - \epsilon, x + \epsilon)$.

Proof. Let us put

$$K = \{x \in \mathbb{R} : \ \mu_P(B_\epsilon(x)) > 0 \text{ for all } \epsilon > 0\}.$$

Let $x \in \text{supp } \mu_P$. If $x \notin K$, then there exists $\epsilon_0 > 0$ such that

$$\mu_P(B_{\epsilon_0}(x)) = 0.$$

Then we have

$$\begin{aligned}\mu_P\big(\mathbb{R}\backslash(\text{supp } \mu_P \backslash B_\epsilon(x))\big) &= \mu_P\big(\mathbb{R}\backslash\text{supp } \mu_P \cup B_{\epsilon_0}(x)\big)\\ &\le \mu_P(\mathbb{R}\backslash\text{supp } \mu_P) + \mu_P(B_{\epsilon_0}(x)) = 0.\end{aligned}$$

Since supp μ_P is closed and $B_{\epsilon_0}(x)$ is open, supp $\mu_P \backslash B_{\epsilon_0}(x)$ is closed. Moreover, since supp $\mu_P \backslash B_{\epsilon_0}(x) \subset$ supp μ_P, this contradicts the definition of support. Therefore $x \in K$. Hence

$$\text{supp } \mu_P \subset K.$$

Conversely, let $x \in K$. If $x \notin$ supp μ_P, then there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \cap \text{supp } \mu_P = \emptyset$. Therefore

$$B_{\epsilon_0}(x) \subset \mathbb{R}\backslash\text{supp } \mu_P.$$

It follows that

$$\mu_P(B_{\epsilon_0}(x)) = 0.$$

This is a contradiction to $x \in K$. Hence $x \in$ supp μ_P. The Lemma is thus proved. ∎

We return to the proof of Proposition 3.1. Let $s \in G$, then $s = \sum_{n=0}^{\infty} \rho^n x_n$, $x_n \in D, \ n = 0, 1, \ldots$.

Let $\epsilon > 0$. We take $n \in \mathbb{N}$ such that

$$\frac{\rho^{n+1} r}{1-\rho} < \frac{\epsilon}{2} \quad \text{and} \quad s - \frac{\epsilon}{2} < s_n < s + \frac{\epsilon}{2},$$

where $r = \max\{r_m, r_m - r_0\}$.
Putting

$$B = \{\omega \in \Omega : \ \omega = (x_0, x_1, \ldots, x_n, y_0, y_1, \ldots), \ y_k \in D, \ k = 0, 1, \ldots\},$$

we will show that

$$\omega \in B \text{ implies } S(\omega) \in B_\epsilon(s). \tag{3.5}$$

In fact, for $\omega = (x_0, x_1, \ldots, x_n, y_0, y_1, \ldots) \in B$ we get

$$\begin{aligned} S(\omega) &= s_n + \rho^{n+1}(y_0 + \rho y_1 + \rho^2 y_2 + \dots) \leq s_n + \frac{\rho^{n+1} r_m}{1-\rho} \\ &\leq s_n + \frac{\rho^{n+1} r}{1-\rho} \leq s + \frac{\epsilon}{2} + \frac{\epsilon}{2} = s + \epsilon, \end{aligned} \tag{3.6}$$

where $s_n = x_0 + \rho x_1 + \cdots + \rho^n x_n$.

On the other hand we have

$$s_n = s - \rho^{n+1}(x_{n+1} + \rho x_{n+2} + \dots).$$

Therefore

$$\begin{aligned} S(\omega) &= s_n + \rho^{n+1}(y_0 + \rho y_1 + \rho^2 y_2 + \dots) \\ &= s - \rho^{n+1}((x_{n+1} - y_0) + \rho(x_{n+2} - y_1) + \dots) \\ &\geq s - \frac{\rho^{n+1}(r_m - r_0)}{1-\rho} \geq s - \frac{\rho^{n+1} r}{1-\rho} > s - \epsilon. \end{aligned}$$

Thus, assertion (3.5) follows from the latter and (3.6).

From (3.5) we get

$$\begin{aligned} \mu_P(B_\epsilon(s)) &= \text{prob}\{\omega \in \Omega :\ S(\omega) \in B_\epsilon(s)\} \geq \text{prob}\{\omega \in \Omega :\ \omega \in B\} \\ &= \text{prob}\{(x_0, x_1, \dots, x_n, y_0, y_1, \dots),\ y_k \in D,\ k = 0, 1, \dots\} \\ &= p(x_0)p(x_1)\dots p(x_n)\text{prob}\{(y_0, y_1, \dots),\ y_k \in D,\ k = 0, 1, \dots\} \\ &= p(x_0)p(x_1)\dots p(x_n) > 0 \end{aligned} \tag{3.7}$$

where $p(x_i) = p_j$ if $x_i = r_j,\ i = 0, 1, \dots, n;\ j = 0, 1, \dots, m$.

From the latter and Lemma 3.2 it follows that $x \in \text{supp}\ \mu_P$, that is

$$G \subset \text{supp}\ \mu_P.$$

The last inclusion and (3.3) prove Proposition 3.1. ∎

Let μ_P be the probabilistic measure defined by (3.1). Then the family of contractions $\{f_i :\ 0 \leq i \leq m\}$ defined by

$$f_i(x) = \rho x + r_i, \text{ for } x \in \mathbb{R},\ i = 0, 1, \dots, m \tag{3.8}$$

is an IFS on $X = [\frac{r_0}{1-\rho}, \frac{r_m}{1-\rho}]$ with associated probabilities $p_0, p_1, \dots, p_m$. In its turn, this probabilistic iterated function system induces a new probabilistic measure that denoted by μ_F, defined on Borel sets of $\mathbb{R}$. Namely,

$$\mu_F(A) = \sum_{i=0}^{m} p_i \mu_F(f_i^{-1}(A)) \quad \text{for all Borel sets} \quad A \subset \mathbb{R}. \tag{3.9}$$

There arises a question: Is there any relation between the given measure μ_P and the reconstructed measure μ_F? In this section we prove that $\mu_F = \mu_P$.

We begin first with some auxiliary facts. For $\omega = (x_0, x_1, \dots) \in \Omega,\ x_i \in D,\ i = 0, 1, 2, \dots$, we put $\omega_j = (r_j, x_0, x_1, \dots) \in \Omega,\ j = 0, 1, \dots, m$. Then we have

Proposition 3.3. *For $j = 0, 1, \dots, m,\ S(\omega) \in f_j^{-1}(A)$ if and only if $S(\omega_j) \in A$.*

Proof. Let $S(\omega) = x_0 + \rho x_1 + \rho^2 x_2 + \cdots \in f_j^{-1}(A)$, i.e.

$$f_j(S(\omega)) \in A.$$

Therefore

$$S(\omega_j) = r_j + \rho x_0 + \rho^2 x_1 + \cdots = f_j(S(\omega)) \in A.$$

Conversely, assume $S(\omega_j) \in A$, that is

$$r_j + \rho x_0 + \rho^2 x_1 + \cdots \in A.$$

We have

$$f_j(S(\omega)) = f_j(x_0 + \rho x_1 + \rho^2 x_2 + \dots) = r_j + \rho x_0 + \rho^2 x_1 + \cdots \in A.$$

Hence

$$S(\omega) \in f_j^{-1}(A).$$

The proof of Proposition 3.3 is finished. ∎

Using Proposition 3.4 we obtain the following theorem which is the main result of this section.

Theorem 3.4. *Let μ_P and μ_F be the probabilistic measures defined by* (3.1) *and* (3.9), *respectively. Then we have*

$$\mu_P(A) = \mu_F(A) \text{ for all Borel sets } A \subset \mathbb{R}. \tag{3.10}$$

Proof. First observe that

$$\text{prob}\{\omega_j = (r_j, x_0, x_1, \dots)\} = p_j \text{prob}\{\omega = (x_0, x_1, \dots)\} \tag{3.11}$$

for $j = 0, 1, \dots, m$ and $x_i \in D,\ i = 0, 1, \dots$. We have

$$\{\omega \in \Omega : S(\omega) \in A\} = \{(x_0, x_1, \dots) : x_i \in D,\ i = 0, 1, \dots, S((x_0, x_1, \dots)) \in A\}$$
$$= \bigcup_{j=0}^{m} \{(r_j, x_1, x_2, \dots) :\ x_i \in D,\ i = 1, 2, \dots,\ S(r_j, x_1, x_2, \dots) \in A\}.$$

Therefore, using (3.1), (3.11) and Proposition 3.3 we get

$$\begin{aligned}
\mu_P(A) &= \text{prob}\{\omega \in \Omega : S(\omega) \in A\} \\
&= \sum_{j=0}^{m} \text{prob}\{(r_j, x_1, x_2, \dots) : x_i \in D, i = 1, 2, \dots,\ S(r_j, x_1, x_2, \dots) \in A\} \\
&= \sum_{j=0}^{m} \text{prob}\{(r_j, x_1, x_2, \dots) : x_i \in D, i = 1, 2, \dots,\ S(x_1, x_2, \dots) \in f_j^{-1}(A)\} \\
&= \sum_{j=0}^{m} p_j \text{prob}\{(x_1, x_2, \dots) : x_i \in D, i = 1, 2, \dots,\ S(x_1, x_2, \dots) \in f_j^{-1}(A)\} \\
&= \sum_{j=0}^{m} p_j \text{prob}\{\omega \in \Omega : s(\omega) \in f_j^{-1}(A)\} = \sum_{j=0}^{m} p_j \mu_P(f_j^{-1}(A)).
\end{aligned}$$

That is, the measure μ_P satisfies formula (3.9). Therefore by the uniqueness of measure satisfying that formula, we conclude that

$$\mu_P(A) = \mu_F(A)$$

for all Borel sets $A \subset \mathbb{R}$. The proof of the theorem is finished. ∎

The following result is an immediate consequence of Theorem 3.4.

Corollary 3.5.

$$\text{supp } \mu_P = \text{supp } \mu_F. \tag{3.12}$$

By Theorem 1.1 supp μ_F is the attractor of the IFS defined by (3.8). Therefore, from Proposition 3.1 and Corollary 3.5 we get

Corollary 3.6. *Let* $\{f_0, f_1, \dots f_m\}$ *be an IFS defined by*

$$f_i(x) = \rho x + r_i \textit{ for } i = 0, 1, \dots , m; \; x \in \mathbb{R},$$

where $0 < \rho < 1$. *Then the attractor* E *of the IFS* $\{f_i : \; 0 \le i \le m\}$ *is of the form*

$$E = \{\sum_{n=0}^{\infty} \rho^n x_n : \; x_n \in D, \; n = 0, 1, \dots \},$$

where $D = \{r_0, r_1, \dots , r_m\}$.

Remark 3.7. As we have seen, the probabilistic system consisting of a sequence $X_0, X_1, \dots$ of independent identically distributed random variables each taking real values $r_0 < r_1 < \cdots < r_m$ with respective probabilities $p_0, p_1, \dots , p_m$ induces a probabilistic measure μ_P defined by

$$\mu_P(A) = \text{prob}\{\omega : \; S(\omega) \in A\},$$

where

$$S = \sum_{n=0}^{\infty} \rho^n X_n.$$

We construct an IFS $\{f_i : \; 0 \le i \le m\}$ defined by

$$f_i(x) = \rho x + r_i, \; x \in \mathbb{R}, \; i = 0, 1, \dots , m$$

with associated probabilities $p_0, p_1, \dots , p_m$. The IFS defined above is called the probabilistic iterated function system. In its turn, this system induces a new probabilistic measure μ_F and $\mu_F(A) = \mu_P(A)$ for all Borel sets $A \subset \mathbb{R}$.

Convers7ely, assume that we have a probabilistic IFS defined by

$$f_i(x) = \rho x + r_i, \; x \in \mathbb{R}, \; i = 0, 1, \dots , m$$

with associated probabilities $p_0, p_1, \dots , p_m, \; 0 \le p_i \le 1, \; i = 0, 1, \dots , m$ and $\sum_{i=0}^{m} p_i = 1$. This system induces a probabilistic measure μ_F by the formula

(1.2). Then the sequence $X_0, X_1, \ldots$ of independent identically distributed random variables each taking real values $r_0, r_1, \ldots, r_m$ with respective probabilities $p_0, p_1, \ldots, p_m$. That is, we obtain a probabilistic measure μ_P and it is clear that

$$\mu_P(A) = \mu_F(A) \text{ for all Borel sets } A \subset \mathbb{R}.$$

Thus, we have proved the following theorem.

Theorem 3.8. *The probabilistic system is equivalent to the probabilistic iterated function system.*

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References

1. K. J. Falconer, *Techniques in Fractal Geometry*, John Wiley & Sons, 1997.
2. K. J. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*, John Wiley & Sons, 1993.
3. N. T. Nhu, *Fractal and Non-Fractal Dimensions*, Statistic Seminar, Part 3, New Mexico State University, 2000.
4. N. T. Nhu, Iterated function systems of finite type and the weak separation property, *Proc. Amer. Math. Soc.* **130** (2002) 483–487.
5. Tian-you Hu and Nguyen To Nhu, Local dimensions of fractal measures associated with uniformly distributed probabilistic systems, *Proc. Amer. Math. Soc.* (to appear).